

A Proof of the Kahn–Kalai Conjecture

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Abstract—Proving the “expectation-threshold” conjecture of Kahn and Kalai, we show that for any increasing property $\mathcal{F}$ on a finite set X ,

$$p_c(\mathcal{F}) = O(q(\mathcal{F}) \log \ell(\mathcal{F})),$$

where $p_c(\mathcal{F})$ and $q(\mathcal{F})$ are the threshold and “expectation threshold” of $\mathcal{F}$, and $\ell(\mathcal{F})$ is the maximum size of a minimal member of $\mathcal{F}$.

Index Terms—monotone Boolean functions, thresholds, expectation thresholds, random graphs

I. INTRODUCTION

Given a finite set X , write 2^X for the power set of X . For $p \in [0, 1]$, let μ_p be the product measure on 2^X given by $\mu_p(A) = p^{|A|}(1-p)^{|X \setminus A|}$. In this paper we always denote $\mathcal{F} \subseteq 2^X$ an *increasing property*, meaning that if $B \supseteq A \in \mathcal{F}$, then $B \in \mathcal{F}$ (so an increasing property can also be viewed as a monotone Boolean function). It is a well-known fact that $\mu_p(\mathcal{F}) := \sum_{A \in \mathcal{F}} \mu_p(A)$ is strictly increasing in p for any $\mathcal{F} \neq \emptyset, 2^X$. The *threshold*, $p_c(\mathcal{F})$, is then the unique p for which $\mu_p(\mathcal{F}) = 1/2$. In this paper, we resolve a conjecture of Kahn and Kalai [13], reiterated by Talagrand [21], relating the threshold and the “expectation-threshold” (definition is below).

Following [21], we say $\mathcal{F}$ is *p-small* if there is $\mathcal{G} \subseteq 2^X$ such that

$$\mathcal{F} \subseteq \langle \mathcal{G} \rangle := \bigcup_{S \in \mathcal{G}} \{T : T \supseteq S\} \quad (1)$$

and

$$\sum_{S \in \mathcal{G}} p^{|S|} \leq 1/2. \quad (2)$$

We say that $\mathcal{G}$ is a *cover* of $\mathcal{F}$ if (1) holds. The *expectation-threshold* of $\mathcal{F}$, $q(\mathcal{F})$, is defined to be the maximum p such that $\mathcal{F}$ is *p-small*. Observe that $q(\mathcal{F})$ is a trivial lower bound on $p_c(\mathcal{F})$, since

$$\mu_p(\mathcal{F}) \leq \mu_p(\langle \mathcal{G} \rangle) \leq \sum_{S \in \mathcal{G}} p^{|S|}. \quad (3)$$

Note that, with X_p the random variable whose distribution is μ_p , the right-hand side of (3) is $\mathbb{E}[|\{S \in \mathcal{G} : S \subseteq X_p\}|]$.

Given an increasing property $\mathcal{F}$, write $\ell(\mathcal{F})$ for the size of a largest minimal element of $\mathcal{F}$. Our main theorem resolves the aforementioned expectation-threshold conjecture of Kahn and Kalai [13].

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Theorem 1.1 (The Kahn–Kalai Conjecture). *There is a universal constant K such that for every finite set X and increasing property $\mathcal{F} \subseteq 2^X$,*

$$p_c(\mathcal{F}) \leq Kq(\mathcal{F}) \log \ell(\mathcal{F}).$$

Roughly speaking, Theorem 1.1 says that for any increasing property, the threshold is always close to its trivial lower bound given by the expectation threshold.

Thresholds have been a central subject in the study of random discrete structures since its initiation by Erdős and Rényi [5], [6]; the study of which has flourished in random graph theory, computer science [14], [21], and statistical physics [10]. The definition of the threshold above is finer than the original Erdős–Rényi notion, according to which $p^* = p^*(n)$ is a threshold for $\mathcal{F} = \mathcal{F}_n$ if $\mu_p(\mathcal{F}) \rightarrow 0$ when $p \ll p^*$ and $\mu_p(\mathcal{F}) \rightarrow 1$ when $p \gg p^*$. That $p_c(\mathcal{F})$ is always an Erdős–Rényi threshold follows from [4], in which it was observed that *every* increasing $\mathcal{F}$ admits a threshold in the Erdős–Rényi sense. While much work has been done identifying thresholds for specific properties (see [3], [11]), another intensively studied direction in the study of thresholds is “sharpness” of thresholds: we refer interested readers to [8], [9].

To emphasize the strength of Theorem 1.1, we point out that, in [13], Kahn and Kalai write that “It would probably be more sensible to conjecture that it is not true.” For example, Theorem 1.1 easily implies some very hard results on the location of thresholds, for example, the appearance of perfect matchings in random r -uniform hypergraphs [12], and the appearance of a given bounded degree spanning tree in a random graph [16]. For more discussion on the significance and applications of this theorem, we refer the readers to [7], in which a weaker (“fractional”) version of Theorem 1.1 is proved.

Part of our proof is inspired by the algorithm and analysis in breakthrough work on the sunflower conjecture [1], and previous works on the fractional version of the theorem [7], [15], though our implementation is significantly different from the ideas in those papers. In particular, our analysis completely avoids the use of “spread,” which was a key ingredient in the aforementioned papers. We also remove the need to separate into pathological and non-pathological cases as in [7], [15]. The key object that we introduce is the “minimum fragment.” Using the minimum fragment, our proof technique can be

easily adapted to give short direct proofs of the current best bound on the sunflower or robust sunflower problem in [2], [19] and [1], as well as the proof of the main lemmas of [7] and [15]. We omit the full details of the derivation in this extended abstract.

The use of minimum fragments is inspired by our work [18], where we resolve some conjectures and questions of Talagrand on suprema of certain positive stochastic processes. In particular, we prove Talagrand’s selector process conjecture (raised in [21], [20] and [22]) and answer a question of Talagrand about suprema of positive empirical processes (raised in [20] and [23]). These give the first known analogs of Talagrand’s celebrated results on suprema of Gaussian processes in the setting of general positive empirical processes and selector processes.

Reformulation. In Section II we prove Theorem I.2 below, which implies Theorem I.1.

A *hypergraph* on X is a collection $\mathcal{H}$ of subsets of X , and a member of $\mathcal{H}$ is called an *edge* of $\mathcal{H}$. We say $\mathcal{H}$ is ℓ -*bounded* if each of its edges has size at most ℓ . Recall that $\langle \mathcal{H} \rangle = \bigcup_{S \in \mathcal{H}} \{T : T \supseteq S\}$. Note that we can extend the definition of p -small to $\mathcal{H}$ without any modification. For an integer m , we use an m -*subset* of X for a subset of X of size m , and X_m for a uniformly random m -subset of X .

Theorem I.2. *There is a universal constant L such that for any ℓ -bounded hypergraph $\mathcal{H}$ on X that is **not** p -small, a uniformly random $((Lp \log \ell)|X|)$ -element subset of X belongs to $\langle \mathcal{H} \rangle$ with probability $1 - o_{\ell \rightarrow \infty}(1)$.*

Remark I.3. *It is easy to see from the proof that one can obtain a quantitative bound for the $o_{\ell \rightarrow \infty}(1)$ term of the form $(\log \ell)^{-c}$ for $c > 0$.*

Derivation of Theorem I.1 from Theorem I.2. Let $\mathcal{F}$ be as in Theorem I.1. We assume Theorem I.2 and derive that if $q > q(\mathcal{F})$ then, with $p = Kq \log \ell(\mathcal{F})$ (K is a universal constant to be determined), we have $\mathbb{P}(X_p \in \mathcal{F}) > 1/2$. Here, we recall that for $0 \leq p \leq 1$, we denote by X_p the random variable with distribution μ_p .

Let $\mathcal{H}$ be the set of minimal elements of $\mathcal{F}$ (so $\langle \mathcal{H} \rangle = \mathcal{F}$). Then $\mathcal{H}$ is $\ell(\mathcal{F})$ -bounded and not q -small (since $q > q(\mathcal{F})$). Let C be a (universal) constant for which, with $\ell = C\ell(\mathcal{F})$, the exceptional probability in Theorem I.2 is less than $1/4$.

Now, with $m = (Lq \log \ell)|X|$ and $p' = 2m/|X|$,

$$\begin{aligned} \mathbb{P}(X_{p'} \in \langle \mathcal{H} \rangle) &\geq \mathbb{P}(|X_{p'}| \geq m) \mathbb{P}(X_m \in \langle \mathcal{H} \rangle) \\ &\geq (3/4) \mathbb{P}(|X_{p'}| \geq m) > 1/2, \end{aligned}$$

where the last inequality follows from standard concentration bound, upon noting that $\mathcal{H}$ is not q -small implies $|X|q > 1/2$ and hence $m > (L \log \ell)/2$. Finally, letting K be such that $p \geq p'$ concludes the derivation. $\square$

Notations and Conventions. All logarithms are base 2 unless specified otherwise. We have not made an attempt to optimize the absolute constants.

II. PROOF OF THEOREM I.2

Before going through the proof in detail, we first give an informal overview of our strategy. A hypergraph $\mathcal{H}$ is p -small if $\mathcal{H}$ admits a “cheap” cover, where being cheap refers to the condition in (2). Our proof uses a randomized iterative process. Starting with $\mathcal{H}_0 = \mathcal{H}$, at the i th step for $i \geq 1$, we assume that we have some hypergraph $\mathcal{H}_{i-1}$ produced from the $(i-1)$ th step for which $|H_{i-1}| \leq 0.9^{i-1}\ell$ for all $H_{i-1} \in \mathcal{H}_{i-1}$, and consider W_i which is a random subset of $X \setminus (\bigcup_{1 \leq j < i} W_j)$ of size $w_i \approx Lpn$ (we will see later in the proof that for the purpose of producing a tail bound going to 0 with ℓ , it is helpful to choose slightly larger w_i towards the last iterations). We will show that there is a sub-hypergraph $\mathcal{G}_i$ of $\mathcal{H}_{i-1}$ that admits a cover $\mathcal{U}_i$ which has small cost with high probability, as well as a hypergraph $\mathcal{H}_i$ with the following key properties:

$$\mathcal{H}_{i-1} \setminus \mathcal{G}_i \subseteq \langle \mathcal{H}_i \rangle, \text{ and} \quad (4)$$

$$\left(\bigcup_{j \leq i} W_j \right) \cup H_i \in \langle \mathcal{H} \rangle \text{ for all } H_i \in \mathcal{H}_i, \text{ and} \quad (5)$$

$$|H_i| \leq 0.9^i \ell \text{ for all } H_i \in \mathcal{H}_i. \quad (6)$$

The choice of $\mathcal{G}_i, \mathcal{U}_i$ and $\mathcal{H}_i$ depend on W_i . In the $(i+1)$ th step we repeat our process with the hypergraph $\mathcal{H}_i$. Property (4) reduces the task of finding a cover of the “leftover” $\mathcal{H}_{i-1} \setminus \mathcal{G}_i$ to finding a cover of $\mathcal{H}_i$. Property (6) implies that for $\gamma > \log_{0.9}(1/\ell)$, either $\mathcal{H}_\gamma$ is empty, or $\mathcal{H}_\gamma$ only contains the empty set. In the former case, one can check that $\bigcup_i \mathcal{U}_i$ is a cover of $\mathcal{H}$; and in the latter case, one has from (5) that $W = \bigcup_i W_i \in \langle \mathcal{H} \rangle$. Since $\mathcal{H}$ does not admit a cheap cover (as it does not satisfy (2)), and the cover $\bigcup_i \mathcal{U}_i$ has small cost with high probability, we conclude that $W \in \langle \mathcal{H} \rangle$ with high probability. Theorem I.2 then follows.

In Section II-A we describe our construction of the cheap cover $\mathcal{U} = \mathcal{U}_i$ (in each step), and in Section II-B we analyze the iteration, concluding the proof.

A. Constructing a cover

We use n for $|X|$. Let $L \geq 1024$ and $\mathcal{H}$ be ℓ -bounded. In the following argument, we always assume that $S, S', \hat{S} \in \mathcal{H}$ and $W \in \binom{X}{w}$, where $w := Lpn$ (as usual, $\binom{X}{w}$ is the collection of w -subsets of X).

The following notion of a *minimum fragment* is key in our proof. Given S and W , we say that $T = T(S, W)$ is a minimum (S, W) -fragment if T is the set of the smallest size (possibly empty) such that T can be written as $S' \setminus W$ for some $S' \in \mathcal{H}$ with $S' \subseteq W \cup S$ (breaking ties arbitrarily). We use $t = t(S, W)$ for $|T(S, W)|$.

Given $W, \mathcal{G} = \mathcal{G}(W)$ is the collection of S whose minimum fragment with respect to W is “large;” formally,

$$\mathcal{G}(W) := \{S \in \mathcal{H} : t(S, W) \geq 0.9\ell\}.$$

Then we define $\mathcal{U}(W)$, a cover of $\mathcal{G}(W)$, as

$$\mathcal{U}(W) := \{T(S, W) : S \in \mathcal{G}(W)\}$$

(the fact that $\mathcal{U}(W)$ covers $\mathcal{G}(W)$ follows since each minimum fragment $T(S, W)$ is a subset of S).

Note that the edges in $\mathcal{H} \setminus \mathcal{G}(W)$ are not necessarily covered by $\mathcal{U}(W)$. We define

$$\mathcal{H}' = \mathcal{H}'(W) = \{T(S, W) : S \in \mathcal{H} \setminus \mathcal{G}(W)\}; \quad (7)$$

this $\mathcal{H}'$, which is 0.9ℓ -bounded, will be the host hypergraph in the next iteration step (see (12)). Note that $\mathcal{H} \setminus \mathcal{G}(W) \subseteq \langle \mathcal{H}' \rangle$ (thus satisfying (4)), so in particular,

$$\text{a cover of } \mathcal{H}' \text{ also covers } \mathcal{H} \setminus \mathcal{G}(W). \quad (8)$$

The following lemma is our key lemma, which says that the cover $\mathcal{U}(W)$ of $\mathcal{G}(W)$ has small cost with high probability (over the randomness of W).

Lemma II.1. *For W uniformly chosen from $\binom{X}{w}$, with probability at least $1 - L^{-0.1\ell}$,*

$$\sum_{U \in \mathcal{U}(W)} p^{|U|} < L^{-0.5\ell}.$$

Note that Lemma II.1 follows immediately from Markov's inequality upon having the following bound

$$\sum_{W \in \binom{X}{w}} \sum_{U \in \mathcal{U}(W)} p^{|U|} < \binom{n}{w} L^{-0.6\ell}, \quad (9)$$

since then we have

$$\mathbb{E} \left[\sum_{U \in \mathcal{U}(W)} p^{|U|} \right] < L^{-0.6\ell}$$

(where the expectation is over the choice of W). The proof of this key estimate (9) can be found in the full version of the paper [17].

B. Iteration

Recall that $n = |X|, \ell \rightarrow \infty$, and $L \geq 1024$. In the following definitions, $i = 1, 2, \dots, \lfloor \log_{0.9}(1/\ell) \rfloor + 1 =: \gamma$. Let $\ell_i = 0.9^i \ell$ and note that

$$0 < \ell_\gamma < 1. \quad (10)$$

Let $X_0 = X$ and W_i be uniform from $\binom{X_{i-1}}{w_i}$, where $X_i = X_{i-1} \setminus W_i$ and $w_i = L_i p n$ with

$$L_i = \begin{cases} L & \text{if } i < \gamma - \sqrt{\log_{0.9}(1/\ell)} \\ L\sqrt{\log \ell} & \text{if } \gamma - \sqrt{\log_{0.9}(1/\ell)} \leq i \leq \gamma. \end{cases}$$

At the end, $W := \bigcup_{i=1}^\gamma W_i$ is a uniformly random $(CLp \log \ell)n$ -subset of X where $C' \leq C$ for some absolute constant $C' > 0$. Note that there is an absolute constant $c > 0$ for which

$$\ell_i > \exp(c\sqrt{\log \ell}) \quad \forall i < \gamma - \sqrt{\log_{0.9}(1/\ell)}. \quad (11)$$

By iteratively applying our argument in Section II-A, we produce a sequence $\{\mathcal{H}_i\}$ with $\mathcal{H}_0 = \mathcal{H}$ and

$$\mathcal{H}_i = \mathcal{H}'_{i-1} \quad (12)$$

(see (7) to recall the definition of $\mathcal{H}'$). Note that each $\mathcal{H}_i$ is ℓ_i -bounded, and associated to each set W_i in step i , we

have $\mathcal{G}_i = \mathcal{G}_i(W_i) \subseteq \mathcal{H}_{i-1}$ and a cover $\mathcal{U}_i = \mathcal{U}_i(W_i)$ of $\mathcal{G}_i$. Properties (4), (6), and inductively Property (5), can be easily verified from our construction of $\mathcal{G}_i, \mathcal{U}_i$ and $\mathcal{H}_i$.

For $i = 1, 2, \dots, \gamma$, call W_i *successful* if

$$\sum_{U \in \mathcal{U}(W_i)} p^{|U|} < L_i^{-0.5\ell_i}. \quad (13)$$

Note that $\ell_\gamma < 1$, hence either $\mathcal{H}_\gamma = \emptyset$ or $\mathcal{H}_\gamma = \{\emptyset\}$. We say that our process *terminates successfully* if W_i are successful for all $i \leq \gamma$ and $\mathcal{H}_\gamma = \emptyset$. In this case, as shown in the next proposition and the following discussion, the process terminates successfully with a cover of $\mathcal{H}$ that has a small cost.

Proposition II.2. *If the process terminates successfully, then $\mathcal{U} := \bigcup_{i \leq \gamma} \mathcal{U}(W_i)$ covers $\mathcal{H}$.*

Proof. For $S \in \mathcal{H}$, let $S = S_0, S_1, S_2, \dots$ ($S_i \in \mathcal{H}_i$) be the evolution of S in the iteration process, i.e., $S_i := T(S_{i-1}, W_i)$. In each iteration, either $S_i \in \mathcal{G}_i$ and is covered by $\mathcal{U}_i$, or otherwise, $S_{i+1} \in \mathcal{H}_{i+1}$. Since $\mathcal{H}_\gamma = \emptyset$, there exists $j < \gamma$ for which $S_j \in \mathcal{G}_j$. Hence, S is covered by $\mathcal{U}$. $\square$

If we have a successful termination, the cost for the cover $\mathcal{U}$ is

$$\begin{aligned} \sum_{U \in \mathcal{U}} p^{|U|} &\stackrel{(13)}{<} \sum_{i \leq \gamma} L_i^{-0.5\ell_i} \\ &\stackrel{(11)}{\leq} 2L^{-0.5 \exp(c\sqrt{\log \ell})} + O((L\sqrt{\log \ell})^{-c'}) \\ &\stackrel{(*)}{\leq} 1/2, \end{aligned}$$

where $c' > 0$ is some constant, and $(*)$ holds for ℓ sufficiently large. This, combined with Proposition II.2, implies that $\mathcal{H}$ is p -small. Thus, if $\mathcal{H}$ is not p -small, then

$$\mathbb{P} \left(\{\mathcal{H}_\gamma = \emptyset\} \wedge \left(\bigwedge_{i \leq \gamma} \{W_i \text{ successful}\} \right) \right) = 0. \quad (14)$$

Proposition II.3. *If $\mathcal{H}_\gamma = \{\emptyset\}$, then $W \in \langle \mathcal{H} \rangle$.*

Proof. Suppose that $\mathcal{H}_\gamma$ contains the empty set. By Property (5), $W_1 \cup \dots \cup W_\gamma \cup \emptyset \in \langle \mathcal{H} \rangle$, and hence $W \in \langle \mathcal{H} \rangle$. $\square$

Letting $\mathcal{E}_i$ be the event that (13) fails, using Proposition II.3 we have

$$\begin{aligned} &\mathbb{P} \left(\{\mathcal{H}_\gamma = \{\emptyset\}\} \vee \left(\bigvee_{i \leq \gamma} \{W_i \text{ not successful}\} \right) \right) \\ &\leq \sum_{i \leq \gamma} \mathbb{P}(\mathcal{E}_i) + \mathbb{P}(W \in \langle \mathcal{H} \rangle) \\ &\leq \sum_{i \leq \gamma} L_i^{-0.1\ell_i} + \mathbb{P}(W \in \langle \mathcal{H} \rangle). \end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{i \leq \gamma} L_i^{-0.1\ell_i} \\
&= \sum_{i < \gamma - \sqrt{\log_{0.9}(1/\ell)}} L_i^{-0.1\ell_i} + \sum_{\gamma - \sqrt{\log_{0.9}(1/\ell)} \leq i \leq \gamma} L_i^{-0.1\ell_i} \\
&\stackrel{(10),(11)}{\leq} 2L^{-0.1 \exp(c\sqrt{\log \ell})} + O((L\sqrt{\log \ell})^{-c''}) \\
&= o_\ell(1)
\end{aligned} \tag{15}$$

for some constant $c'' > 0$. Together with (14), this implies that for $\mathcal{H}$ not p -small,

$$\mathbb{P} \left(\{\mathcal{H}_\gamma = \{\emptyset\}\} \vee \left(\bigvee_{i \leq \gamma} \{W_i \text{ not successful}\} \right) \right) = 1,$$

and so

$$\mathbb{P}(W \in \langle \mathcal{H} \rangle) \geq 1 - o_\ell(1),$$

which is the desired conclusion.

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