2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)

Planar Graphs have Bounded Queue-Number

Vida Dujmović*, Gwenaël Joret†, Piotr Micek‡, Pat Morin§, Torsten Ueckerdt¶, David R. Wood‖

* School of Computer Science and Electrical Eng., University of Ottawa, Canada (vida.dujmovic@uottawa.ca)
† Département d'Informatique, Université Libre de Bruxelles, Brussels, Belgium (gjoret@ulb.ac.be)
‡ Faculty of Mathematics and Computer Science, Jagiellonian University, Poland (piotr.micek@tcs.uj.edu.pl)
§ School of Computer Science, Carleton University, Ottawa, Canada (morin@scs.carleton.ca)
¶ Institute of Theoretical Informatics, Karlsruhe Institute of Technology, Germany (torsten.ueckerdt@kit.edu)
‖ School of Mathematics, Monash University, Melbourne, Australia (david.wood@monash.edu)

***Abstract*—We show that planar graphs have bounded queue-number, thus proving a conjecture of Heath, Leighton and Rosenberg from 1992. The key to the proof is a new structural tool called layered partitions, and the result that every planar graph has a vertex-partition and a layering, such that each part has a bounded number of vertices in each layer, and the quotient graph has bounded treewidth. This result generalises for graphs of bounded Euler genus. Moreover, we prove that every graph in a minor-closed class has such a layered partition if and only if the class excludes some apex graph. Building on this work and using the graph minor structure theorem, we prove that every proper minor-closed class of graphs has bounded queue-number. Layered partitions can be interpreted in terms of strong products. We show that every planar graph is a subgraph of the strong product of a path with some graph of bounded treewidth. Similar statements hold for all proper minor-closed classes.**

***Keywords*-graph theory, queue layout, queue-number, planar graph, treewidth, layered partition, strong product, Euler genus, graph minor, graph drawing**

I. Introduction

Stacks and queues are fundamental data structures in computer science. But what is more powerful, a stack or a queue? In 1992, Heath, Leighton and Rosenberg [1] developed a graph-theoretic formulation of this question, where they defined the graph parameters stack-number and queue-number which respectively measure the power of stacks and queues to represent a given graph. Intuitively speaking, if some class of graphs has bounded stack-number and unbounded queue-number, then we would consider stacks to be more powerful than queues for that class (and vice versa). It is known that the stack-number of a graph may be much larger than the queue-number. For example, Heath, Leighton and Rosenberg [1] proved that the n-vertex ternary Hamming graph has queue-number at most $O(\log n)$ and stack-number at least $\Omega(n^{1/9-\epsilon})$. Nevertheless, it is open whether every graph has stack-number bounded by a function of its queue-number, or whether every graph has queue-number bounded by a function of its stack-number [1,2].

Planar graphs are the simplest class of graphs where it is unknown whether both stack and queue-number are bounded. In particular, Buss and Shor [3] first proved that planar graphs have bounded stack-number; the best known upper bound is 4 due to Yannakakis [4]. However, for the last 27 years of research on this topic, the most important open question in this field has been whether planar graphs have bounded queue-number. This question was first proposed by Heath, Leighton and Rosenberg [1] who conjectured that planar graphs have bounded queue-number. This paper proves this conjecture. Moreover, we generalise this result for graphs of bounded Euler genus, and for every proper minor-closed class of graphs.[1]

First we define the stack-number and queue-number of a graph G. Let $V(G)$ and $E(G)$ respectively denote the vertex and edge set of G. Consider disjoint edges $vw, xy \in E(G)$ in a linear ordering $\preccurlyeq$ of $V(G)$. Without loss of generality, $v \prec w$ and $x \prec y$ and $v \prec x$. Then vw and xy are said to *cross* if $v \prec x \prec w \prec y$ and are said to *nest* if $v \prec x \prec y \prec w$. A *stack* (with respect to $\preccurlyeq$) is a set of pairwise non-crossing edges, and a *queue* (with respect to $\preccurlyeq$) is a set of pairwise non-nested edges. Stacks resemble the stack data structure in the following sense. In a stack, traverse the vertex ordering left-to-right. When visiting vertex v, because of the non-crossing property, if $x_1, \dots, x_d$ are the neighbours of v to the left of v in left-to-right order, then the edges $x_d v, x_{d-1} v, \dots, x_1 v$ will be on

Research of Dujmović and Morin is supported by NSERC and the Ontario Ministry of Research and Innovation. Research of Joret is supported by an ARC grant from the Wallonia-Brussels Federation of Belgium. Research of Micek is partially supported by the Polish National Science Center grant (SONATA BIS 5; UMO-2015/18/E/ST6/00299). Research of Wood is supported by the Australian Research Council.

[1] The *Euler genus* of the orientable surface with h handles is $2h$. The *Euler genus* of the non-orientable surface with c cross-caps is c. The *Euler genus* of a graph G is the minimum integer k such that G embeds in a surface of Euler genus k. Of course, a graph is planar if and only if it has Euler genus 0; see [5] for more about graph embeddings in surfaces. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. A class $\mathcal{G}$ of graphs is *minor-closed* if for every graph $G \in \mathcal{G}$, every minor of G is in $\mathcal{G}$. A minor-closed class is *proper* if it is not the class of all graphs. For example, for fixed $g \geqslant 0$, the class of graphs with Euler genus at most g is a proper minor-closed class.

2575-8454/19/$31.00 ©2019 IEEE
DOI 10.1109/FOCS.2019.00056

top of the stack in this order. Pop these edges off the stack. Then if $y_1, \dots, y_{d'}$ are the neighbours of v to the right of v in left-to-right order, then push $vy_{d'}, vy_{d'-1}, \dots, vy_1$ onto the stack in this order. In this way, a stack of edges with respect to a linear ordering resembles a stack data structure. Analogously, the non-nesting condition in the definition of a queue implies that a queue of edges with respect to a linear ordering resembles a queue data structure.

For an integer $k \geqslant 0$, a *k-stack layout* of a graph G consists of a linear ordering $\preccurlyeq$ of $V(G)$ and a partition $E_1, E_2, \dots, E_k$ of $E(G)$ into stacks with respect to $\preccurlyeq$. Similarly, a *k-queue layout* of G consists of a linear ordering $\preccurlyeq$ of $V(G)$ and a partition $E_1, E_2, \dots, E_k$ of $E(G)$ into queues with respect to $\preccurlyeq$. The *stack-number* of G, denoted by $\mathrm{sn}(G)$, is the minimum integer k such that G has a k-stack layout. The *queue-number* of a graph G, denoted by $\mathrm{qn}(G)$, is the minimum integer k such that G has a k-queue layout. Note that k-stack layouts are equivalent to k-page book embeddings, and stack-number is also called page-number, book thickness, or fixed outer-thickness.

As mentioned above, Heath, Leighton and Rosenberg [1] conjectured that planar graphs have bounded queue-number. This conjecture has remained open despite much research on queue layouts [1,2,6–16]. We now review progress on this conjecture.

Pemmaraju [8] studied queue layouts and wrote that he "suspects" that a particular planar graph with n vertices has queue-number $\Theta(\log n)$. The example he proposed had treewidth 3; see Section II-B for the definition of treewidth. Dujmović, Morin and Wood [10] proved that graphs of bounded treewidth have bounded queue-number. So Pemmaraju's example in fact has bounded queue-number.

The first $o(n)$ bound on the queue-number of planar graphs with n vertices was proved by Heath, Leighton, and Rosenberg [1], who observed that every graph with m edges has a $O(\sqrt{m})$-queue layout using a random vertex ordering. Thus every planar graph with n vertices has queue-number $O(\sqrt{n})$, which can also be proved using the Lipton-Tarjan separator theorem. Di Battista, Frati and Pach [16] proved the first breakthrough on this topic, by showing that every planar graph with n vertices has queue-number $O(\log^2 n)$. Dujmović [17] improved this bound to $O(\log n)$ with a simpler proof. Building on this work, Dujmović, Morin and Wood [13] established (poly-)logarithmic bounds for more general classes of graphs. For example, they proved that every graph with n vertices and Euler genus g has queue-number $O(g + \log n)$, and that every graph with n vertices excluding a fixed minor has queue-number $\log^{O(1)} n$.

Recently, Bekos, Förster, Gronemann, Mchedlidze, Montecchiani, Raftopoulou and Ueckerdt [15] proved a second breakthrough result, by showing that planar graphs with bounded maximum degree have bounded queue-number. In particular, every planar graph with maximum degree Δ has queue-number at most $O(\Delta^6)$. Subsequently, Dujmović, Morin and Wood [18] proved that the algorithm of Bekos et al. [15] in fact produces a $O(\Delta^2)$-queue layout. This was the state of the art prior to the current work.[2]

A. Main Results

The fundamental contribution of this paper is to prove the conjecture of Heath, Leighton and Rosenberg [1] that planar graphs have bounded queue-number.

Theorem 1. *The queue-number of planar graphs is bounded.*

The best upper bound that we obtain for the queue-number of planar graphs is 49.

We extend Theorem 1 by showing that graphs with bounded Euler genus have bounded queue-number.

Theorem 2. *Every graph with Euler genus g has queue-number at most $O(g)$.*

We generalise further to show the following:

Theorem 3. *Every proper minor-closed class of graphs has bounded queue-number.*

These results are obtained through the introduction of a new tool, *layered partitions*, that have applications well beyond queue layouts. Loosely speaking, a layered partition of a graph G consists of a partition $\mathcal{P}$ of $V(G)$ along with a layering of G, such that each part in $\mathcal{P}$ has a bounded number of vertices in each layer (called the *layered width*), and the quotient graph $G/\mathcal{P}$ has certain desirable properties, typically bounded treewidth. Layered partitions are the key tool for proving the above theorems. Subsequent to the initial release of this paper, layered partitions and the results in this paper have been used to solve other problems [20–23]. For example, our results for layered partitions were used by Dujmović, Esperet, Joret, Walczak and Wood [23] to prove that planar graphs have bounded nonrepetitive chromatic number, thus solving a well-known open problem. As above, this result generalises for any proper minor-closed class.

II. Tools

In this abbreviated version of the paper, we focus on the proof for planar graphs. Omitted proofs can be found in the full version [24]. Undefined terms and notation can be found in Diestel's text [25]. Throughout the paper, we use the notation $\overrightarrow{X}$ to refer to a particular linear ordering of a set X.

A. Layerings

The following well-known definitions are key concepts in our proofs, and that of several other papers on queue layouts [10,11,13,15,18]. A *layering* of a graph G is an

[2] Wang [19] claimed to prove that planar graphs have bounded queue-number, but despite several attempts, we have not been able to understand the claimed proof.

ordered partition $(V_0, V_1, \dots)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leqslant 1$. If $i = j$ then vw is an *intra-level* edge. If $|i - j| = 1$ then vw is an *inter-level* edge. If r is a vertex in a connected graph G and $V_i := \{v \in V(G) : \mathrm{dist}_G(r, v) = i\}$ for all $i \geqslant 0$, then $(V_0, V_1, \dots)$ is called a ***BFS layering*** of G. Associated with a BFS layering is a ***BFS spanning tree*** T obtained by choosing, for each vertex $v \in V_i$ with $i \geqslant 1$, a neighbour w in V_{i-1}, and adding the edge vw to T. Thus $\mathrm{dist}_T(r, v) = \mathrm{dist}_G(r, v)$ for each vertex v of G. We consider T to be rooted at r. These notions extend to disconnected graphs. If $G_1, \dots, G_c$ are the components of G, and r_j is a vertex in G_j for $j \in \{1, \dots, c\}$, and $V_i := \bigcup_{j=1}^{c} \{v \in V(G_j) : \mathrm{dist}_{G_j}(r_j, v) = i\}$ for all $i \geqslant 0$, then $(V_0, V_1, \dots)$ is called a ***BFS layering*** of G.

B. Treewidth and Layered Treewidth

First we introduce the notion of H-decomposition and tree-decomposition. For graphs H and G, an *H-decomposition* of G consists of a collection $(B_x \subseteq V(G) : x \in V(H))$ of subsets of $V(G)$, called *bags*, indexed by the vertices of H, and with the following properties:

- for every vertex v of G, the set $\{x \in V(H) : v \in B_x\}$ induces a non-empty connected subgraph of H, and
- for every edge vw of G, there is a vertex $x \in V(H)$ for which $v, w \in B_x$.

The *width* of such an H-decomposition is $\max\{|B_x| : x \in V(H)\} - 1$. The elements of $V(H)$ are called *nodes*, while the elements of $V(G)$ are called *vertices*.

A *tree-decomposition* is a T-decomposition for some tree T. The *treewidth* of a graph G is the minimum width of a tree-decomposition of G. Treewidth measures how similar a given graph is to a tree. It is particularly important in structural and algorithmic graph theory; see [26,27].

As mentioned in Section I, Dujmović et al. [10] first proved that graphs of bounded treewidth have bounded queue-number. Their bound on the queue-number was doubly exponential in the treewidth. Wiechert [6] improved this bound to singly exponential.

Lemma 4 ([6])**.** *Every graph with treewidth k has queue-number at most $2^k - 1$.*

Graphs with bounded treewidth provide important examples of minor-closed classes. However, planar graphs have unbounded treewidth. For example, the $n \times n$ planar grid graph has treewidth n. So the above results do not resolve the question of whether planar graphs have bounded queue-number.

Dujmović et al. [13] and Shahrokhi [28] independently introduced the following concept. The *layered treewidth* of a graph G is the minimum integer k such that G has a tree-decomposition $(B_x : x \in V(T))$ and a layering $(V_0, V_1, \dots)$ such that $|B_x \cap V_i| \leqslant k$ for every bag B_x and layer V_i. Applications of layered treewidth include graph colouring [13,30], graph drawing [13,31], book embeddings [32], and intersection graph theory [28]. The related notion of layered pathwidth has also been studied [29,31]. Most relevant to this paper, Dujmović et al. [13] proved that every graph with n vertices and layered treewidth k has queue-number at most $O(k \log n)$. They then proved that planar graphs have layered treewidth at most 3, that graphs of Euler genus g have layered treewidth at most $2g+3$, and more generally that a minor-closed class has bounded layered treewidth if and only if it excludes some apex graph.[3] This implies $O(\log n)$ bounds on the queue-number for all these graphs, and was the basis for the $\log^{O(1)} n$ bound for proper minor-closed classes mentioned in Section I.

C. Partitions and Layered Partitions

The following definitions are central notions in this paper. A *vertex-partition*, or simply *partition*, of a graph G is a set $\mathcal{P}$ of non-empty sets of vertices in G such that each vertex of G is in exactly one element of $\mathcal{P}$. Each element of $\mathcal{P}$ is called a *part*. The *quotient* (sometimes called the *touching pattern*) of $\mathcal{P}$ is the graph, denoted by $G/\mathcal{P}$, with vertex set $\mathcal{P}$ where distinct parts $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in A is adjacent in G to some vertex in B.

A partition of G is *connected* if the subgraph induced by each part is connected. In this case, the quotient is the minor of G obtained by contracting each part into a single vertex. Our results for queue layouts do not depend on the connectivity of partitions. But we consider it to be of independent interest that many of the partitions constructed in this paper are connected. Then the quotient is a minor of the original graph.

A partition $\mathcal{P}$ of a graph G is called an *H-partition* if H is a graph that contains a spanning subgraph isomorphic to the quotient $G/\mathcal{P}$. Alternatively, an *H-partition* of a graph G is a partition $(A_x : x \in V(H))$ of $V(G)$ indexed by the vertices of H, such that for every edge $vw \in E(G)$, if $v \in A_x$ and $w \in A_y$ then $x = y$ (and vw is called an *intra-bag* edge) or $xy \in E(H)$ (and vw is called an *inter-bag* edge). The *width* of such an H-partition is $\max\{|A_x| : x \in V(H)\}$. Note that a layering is equivalent to a path-partition.

A *tree-partition* is a T-partition for some tree T. Tree-partitions are well studied with several applications. For example, every graph with treewidth k and maximum degree Δ has a tree-partition of width $O(k\Delta)$; see [33,34]. This easily leads to a $O(k\Delta)$ upper bound on the queue-number [10]. However, dependence on Δ seems unavoidable when studying tree-partitions [33], so we instead consider H-partitions where H has bounded treewidth greater than 1.

A key innovation of this paper is to consider a layered variant of partitions (analogous to layered treewidth being a layered variant of treewidth). The *layered width* of a partition

[3] A graph G is *apex* if $G - v$ is planar for some vertex v.

$\mathcal{P}$ of a graph G is the minimum integer ℓ such that for some layering $(V_0, V_1, \dots)$ of G, each part in $\mathcal{P}$ has at most ℓ vertices in each layer V_i.

Throughout this paper we consider partitions with bounded layered width such that the quotient has bounded treewidth. We therefore introduce the following definition. A class $\mathcal{G}$ of graphs is said to *admit bounded layered partitions* if there exist $k, \ell \in \mathbb{N}$ such that every graph $G \in \mathcal{G}$ has a partition $\mathcal{P}$ with layered width at most ℓ such that $G/\mathcal{P}$ has treewidth at most k. We first show that this property immediately implies bounded layered treewidth.

Lemma 5. *If a graph G has an H-partition with layered width at most ℓ such that H has treewidth at most k, then G has layered treewidth at most $(k+1)\ell$.*

Proof: Let $(B_x : x \in V(T))$ be a tree-decomposition of H with bags of size at most $k+1$. Replace each instance of a vertex v of H in a bag B_x by the part corresponding to v in the H-partition. Keep the same layering of G. Since $|B_x| \leqslant k+1$, we obtain a tree-decomposition of G with layered width at most $(k+1)\ell$. ■

Lemma 5 means that any property that holds for graph classes with bounded layered treewidth also holds for graph classes that admit bounded layered partitions. For example, Norin proved that every n-vertex graph with layered treewidth at most k has treewidth less than $2\sqrt{kn}$ (see [13]). With Lemma 5, this implies that if an n-vertex graph G has a partition with layered width ℓ such that the quotient graph has treewidth at most k, then G has treewidth at most $2\sqrt{(k+1)\ell n}$. This in turn leads to $O(\sqrt{n})$ balanced separator theorems for such graphs.

Lemma 5 suggests that having a partition of bounded layered width, whose quotient has bounded treewidth, seems to be a more stringent requirement than having bounded layered treewidth. Indeed the former structure leads to $O(1)$ bounds on the queue-number, instead of $O(\log n)$ bounds obtained via layered treewidth. That said, it is open whether graphs of bounded layered treewidth have bounded queue-number. It is even possible that graphs of bounded layered treewidth admit bounded layered partitions.

Before continuing, we show that if one does not care about the exact treewidth bound, then it suffices to consider partitions with layered width 1.

Lemma 6. *If a graph G has an H-partition of layered width ℓ with respect to a layering $(V_0, V_1, \dots)$, for some graph H of treewidth at most k, then G has an H'-partition of layered width 1 with respect to the same layering, for some graph H' of treewidth at most $(k+1)\ell - 1$.*

Proof: Let $(A_v : v \in V(H))$ be an H-partition of G of layered width ℓ with respect to $(V_0, V_1, \dots)$, for some graph H of treewidth at most k. Let $(B_x : x \in V(T))$ be a tree-decomposition of H with width at most k. Let H' be the graph obtained from H by replacing each vertex v of H by an ℓ-clique X_v and replacing each edge vw of H by a complete bipartite graph $K_{\ell,\ell}$ between X_v and X_w. For each $x \in V(T)$, let $B'_x := \cup\{X_v : v \in B_x\}$. Observe that $(B'_x : x \in V(T))$ is a tree-decomposition of H' of width at most $(k+1)\ell - 1$. For each vertex v of H, and layer V_i, there are at most ℓ vertices in $A_v \cap V_i$. Assign each vertex in $A_v \cap V_i$ to a distinct element of X_v. We obtain an H'-partition of G with layered width 1, and the treewidth of H is at most $(k+1)\ell - 1$. ■

D. Queue Layouts via Layered Partitions

The next lemma is at the heart of all our results about queue layouts.

Lemma 7. *For all graphs H and G, if H has a k-queue layout and G has an H-partition of layered width ℓ, then G has a $(3\ell k + \lfloor \frac{3}{2}\ell \rfloor)$-queue layout. In particular,*

$$\operatorname{qn}(G) \leqslant 3\ell \operatorname{qn}(H) + \lfloor \tfrac{3}{2}\ell \rfloor.$$

We postpone the proof of Lemma 7 until Section VI. Lemmas 4 and 7 imply that a graph class that admits bounded layered partitions has bounded queue-number. In particular:

Corollary 8. *If a graph G has a partition $\mathcal{P}$ of layered width ℓ such that $G/\mathcal{P}$ has treewidth at most k, then G has queue-number at most $3\ell(2^k - 1) + \lfloor \frac{3}{2}\ell \rfloor$.*

III. Planar Graphs

Our proof that planar graphs have bounded queue-number employs Corollary 8. Thus our goal is to show that planar graphs admit bounded layered partitions, which is achieved in the following key contribution of the paper.

Theorem 9. *Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $G/\mathcal{P}$ has treewidth at most 8. Moreover, there is such a partition for every BFS layering of G.*

This theorem and Corollary 8 imply that planar graphs have bounded queue-number (Theorem 1) with an upper bound of $3(2^8 - 1) + \lfloor \frac{3}{2} 3 \rfloor = 766$.

We now set out to prove Theorem 9. The proof is inspired by the following elegant result of Pilipczuk and Siebertz [35]: Every planar graph G has a partition $\mathcal{P}$ into geodesics such that $G/\mathcal{P}$ has treewidth at most 8. Here, a *geodesic* is a path of minimum length between its endpoints. We consider the following particular type of geodesic. If T is a tree rooted at a vertex r, then a non-empty path $(x_0, \dots, x_p)$ in T is *vertical* if for some $d \geqslant 0$ for all $i \in \{0, \dots, p\}$ we have $\operatorname{dist}_T(x_i, r) = d + i$. The vertex x_1 is called the *upper endpoint* of the path and x_p is its *lower endpoint*. Note that every vertical path in a BFS spanning tree is a geodesic. Thus the next theorem strengthens the result of Pilipczuk and Siebertz [35].

Theorem 10. *Let T be a rooted spanning tree in a connected planar graph G. Then G has a partition $\mathcal{P}$ into vertical paths in T such that $G/\mathcal{P}$ has treewidth at most* 8.

Proof of Theorem 9 assuming Theorem 10: We may assume that G is connected (since if each component of G has the desired partition, then so does G). Let T be a BFS spanning tree of G. By Theorem 10, G has a partition $\mathcal{P}$ into vertical paths in T such that $G/\mathcal{P}$ has treewidth at most 8. Each path in $\mathcal{P}$ is connected and has at most one vertex in each BFS layer corresponding to T. Hence $\mathcal{P}$ is connected and has layered width 1. ∎

The proof of Theorem 10 is an inductive proof of a stronger statement given in Lemma 11 below. A *plane graph* is a graph embedded in the plane with no crossings. A *near-triangulation* is a plane graph, where the outer-face is a simple cycle, and every internal face is a triangle. For a cycle C, we write $C = [P_1, \dots, P_k]$ if $P_1, \dots, P_k$ are pairwise disjoint non-empty paths in C, and the endpoints of each path P_i can be labelled x_i and y_i so that $y_i x_{i+1} \in E(C)$ for $i \in \{1, \dots, k\}$, where x_{k+1} means x_1. This implies that $V(C) = \bigcup_{i=1}^{k} V(P_i)$.

Lemma 11. *Let G^+ be a plane triangulation, let T be a spanning tree of G^+ rooted at some vertex r on the outer-face of G^+, and let $P_1, \dots, P_k$ for some $k \in \{1, 2, \dots, 6\}$, be pairwise disjoint vertical paths in T such that $F = [P_1, \dots, P_k]$ is a cycle in G^+. Let G be the near-triangulation consisting of all the edges and vertices of G^+ contained in F and the interior of F.*

Then G has a connected partition $\mathcal{P}$ into paths in G that are vertical in T, such that $P_1, \dots, P_k \in \mathcal{P}$ and the quotient $H := G/\mathcal{P}$ has a tree-decomposition in which every bag has size at most 9 and some bag contains all the vertices of H corresponding to $P_1, \dots, P_k$.

Proof of Theorem 10 assuming Lemma 11: The result is trivial if $|V(G)| < 3$. Now assume $|V(G)| \geqslant 3$. Let r be the root of T. Let G^+ be a plane triangulation containing G as a spanning subgraph with r on the outer-face of G. The three vertices on the outer-face of G are vertical (singleton) paths in T. Thus G^+ satisfies the assumptions of Lemma 11, which implies that G^+ has a partition $\mathcal{P}$ into vertical paths in T such that $G^+/\mathcal{P}$ has treewidth at most 8. Note that $G/\mathcal{P}$ is a subgraph of $G^+/\mathcal{P}$. Hence $G/\mathcal{P}$ has treewidth at most 8. ∎

Our proof of Lemma 11 employs the following well-known variation of Sperner's Lemma (see [36]):

Lemma 12 (Sperner's Lemma)**.** *Let G be a near-triangulation whose vertices are (possibly improperly) coloured* 1, 2, 3*, with the outer-face $F = [P_1, P_2, P_3]$ where each vertex in P_i is coloured i. Then G contains an internal face whose vertices are coloured* 1, 2, 3.

Proof of Lemma 11: The proof is by induction on $n = |V(G)|$. If $n = 3$, then G is a 3-cycle and $k \leqslant 3$. The partition into vertical paths is $\mathcal{P} = \{P_1, \dots, P_k\}$. The tree-decomposition of H consists of a single bag that contains the $k \leqslant 3$ vertices corresponding to $P_1, \dots, P_k$.

For $n > 3$ we wish to make use of Sperner's Lemma on some 3-colouring of the vertices of G. We begin by colouring the vertices of F, as illustrated in Figure 1. There are three cases to consider:

1) If $k = 1$ then, since F is a cycle, P_1 has at least three vertices, so $P_1 = [v, P_1', w]$ for two distinct vertices v and w. We set $R_1 := v$, $R_2 := P_1'$ and $R_3 := w$.
2) If $k = 2$ then we may assume without loss of generality that P_1 has at least two vertices so $P_1 = [v, P_1']$. We set $R_1 := v$, $R_2 := P_1'$ and $R_3 := P_2$.
3) If $k \in \{3, 4, 5, 6\}$ then we group consecutive paths by taking $R_1 := [P_1, \dots, P_{\lfloor k/3 \rfloor}]$, $R_2 := [P_{\lfloor k/3 \rfloor + 1}, \dots, P_{\lfloor 2k/3 \rfloor}]$ and $R_3 := [P_{\lfloor 2k/3 \rfloor + 1}, \dots, P_k]$. Note that in this case each R_i consists of one or two of $P_1, \dots, P_k$.

For $i \in \{1, 2, 3\}$, colour each vertex in R_i by i. Now, for each remaining vertex v in G, consider the path P_v from v to the root of T. Since r is on the outer-face of G^+, P_v contains a vertex of F. If the first vertex of P_v that belongs to F is in R_i then assign the colour i to v. In this way we obtain a 3-colouring of the vertices of G that satisfies the conditions of Sperner's Lemma. Therefore there exists a triangular face $\tau = v_1 v_2 v_3$ of G whose vertices are coloured 1, 2, 3 respectively.

For each $i \in \{1, 2, 3\}$, let Q_i be the path in T from v_i to the first ancestor v_i' of v_i in T that is contained in F. Observe that Q_1, Q_2, and Q_3 are disjoint since Q_i consists only of vertices coloured i. Note that Q_i may consist of the single vertex $v_i = v_i'$. Let Q_i' be Q_i minus its final vertex v_i'. Imagine for a moment that the cycle F is oriented clockwise, which defines an orientation of R_1, R_2 and R_3. Let R_i^- be the subpath of R_i that contains v_i' and all vertices that precede it, and let R_i^+ be the subpath of R_i that contains v_i' and all vertices that succeed it.

Consider the subgraph of G that consists of the edges and vertices of F, the edges and vertices of τ, and the edges and vertices of $Q_1 \cup Q_2 \cup Q_3$. This graph has an outer-face, an inner face τ, and up to three more inner faces F_1, F_2, F_3 where $F_i = [Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$, where we use the convention that $Q_4 = Q_1$ and $R_4 = R_1$. Note that F_i may be *degenerate* in the sense that $[Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$ may consist only of a single edge $v_i v_{i+1}$.

Consider any non-degenerate $F_i = [Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$. Note that these four paths are pairwise disjoint, and thus F_i is a cycle. If Q_i' and Q_{i+1}' are non-empty, then each is a vertical path in T. Furthermore, each of R_i^- and R_{i+1}^+ consists of at most two vertical paths in T. Thus, F_i is the concatenation of at most six vertical paths in T. Let G_i be the near-triangulation consisting of all the edges and vertices of G^+ contained in F_i and the interior of F_i. Observe that G_i

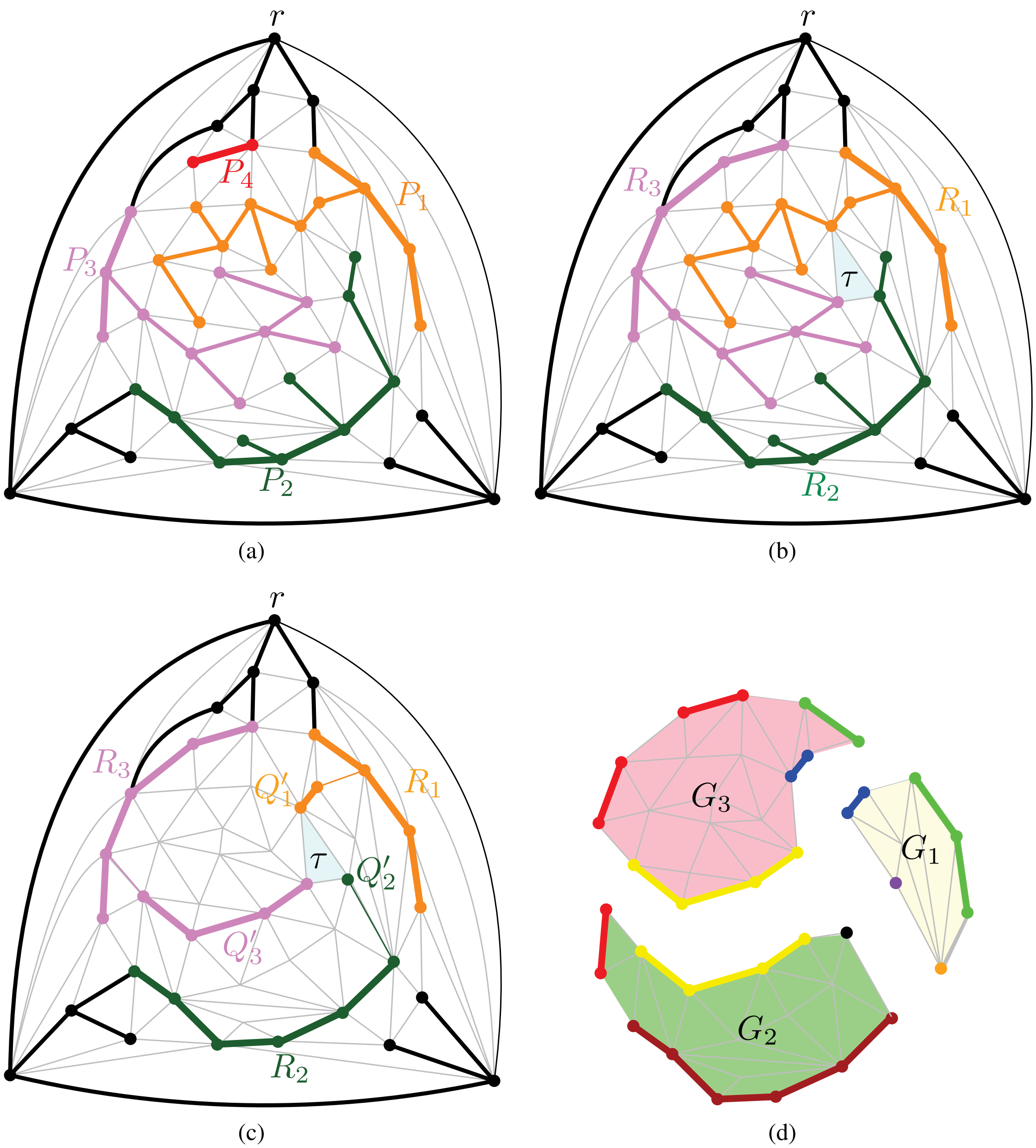

Figure 1. The inductive proof of Lemma 11: (a) the spanning tree T and the paths $P_1, \dots, P_4$; (b) the paths R_1, R_2, R_3, and the Sperner triangle τ; (c) the paths Q'_1, Q'_2 and Q'_3; (d) the near-triangulations G_1, G_2, and G_3, with the vertical paths of T on F_1, F_2, and F_3.

contains v_i and v_{i+1} but not the third vertex of τ. Therefore F_i satisfies the conditions of the lemma and has fewer than n vertices. So we may apply induction on F_i to obtain a partition $\mathcal{P}_i$ of G_i into vertical paths in T, such that $H_i := G_i/\mathcal{P}_i$ has a tree-decomposition $(B^i_x : x \in V(J_i))$ in which every bag has size at most 9, and some bag $B^i_{u_i}$ contains the vertices of H_i corresponding to the at most six vertical paths that form F_i. We do this for each non-degenerate F_i.

We now construct the desired partition $\mathcal{P}$ of G. Initialise $\mathcal{P} := \{P_1, \dots, P_k\}$. Then add each non-empty Q'_i to $\mathcal{P}$. Now for each non-degenerate F_i, each path in $\mathcal{P}_i$ is either an *external* path (that is, fully contained in F_i) or is an *internal path* with none of its vertices in F_i. Add all the internal paths of $\mathcal{P}_i$ to $\mathcal{P}$. By construction, $\mathcal{P}$ partitions $V(G)$ into vertical paths in T and $\mathcal{P}$ contains $P_1, \dots, P_k$.

Let $H := G/\mathcal{P}$. Next we exhibit the desired tree-decomposition $(B_x : x \in V(J))$ of H. Let J be the tree obtained from the disjoint union of J_i, taken over the $i \in \{1, 2, 3\}$ such that F_i is non-degenarate, by adding one new node u adjacent to each u_i. (Recall that u_i is the node of J_i for which the bag $B^i_{u_i}$ contains the vertices of H_i corresponding to the paths that form F_i.) Let the bag B_u contain all the vertices of H corresponding to $P_1, \dots, P_k, Q'_1, Q'_2, Q'_3$. For each non-degenerate F_i, and for each node $x \in V(J_i)$, initialise $B_x := B^i_x$. Recall that vertices of H_i correspond to contracted paths in $\mathcal{P}_i$. Each internal path in $\mathcal{P}_i$ also lies in $\mathcal{P}$. Each external path P in $\mathcal{P}_i$ is a subpath of P_j for some $j \in \{1, \dots, k\}$ or is one of the paths among Q'_1, Q'_2, Q'_3. For each such path P, for every $x \in V(J)$, in bag B_x, replace each instance of the vertex of H_i corresponding to P by the vertex of H corresponding to the path among $P_1, \dots, P_k, Q'_1, \dots, Q'_3$ that contains P. This completes the description of $(B_x : x \in V(J))$. By construction, $|B_x| \leqslant 9$ for every $x \in V(J)$.

First we show that for each vertex a in H, the set $X := \{x \in V(J) : a \in B_x\}$ forms a subtree of J. If a corresponds to a path distinct from $P_1, \dots, P_k, Q'_1, Q'_2, Q'_3$ then X is fully contained in J_i for some $i \in \{1, 2, 3\}$. Thus, by induction X is non-empty and connected in J_i, so it is in J. If a corresponds to P which is one of the paths among $P_1, \dots, P_k, Q'_1, Q'_2, Q'_3$ then $u \in X$ and whenever X contains a vertex of J_i it is because some external path of $\mathcal{P}_i$ was replaced by P. In particular, we would have $u_i \in X$ in that case. Again by induction each $X \cap J_i$ is connected and since $uu_i \in E(T)$, we conclude that X induces a (connected) subtree of J.

Finally we show that, for every edge ab of H, there is a bag B_x that contains a and b. If a and b are both obtained by contracting any of $P_1, \dots, P_k, Q'_1, Q'_2, Q'_3$, then a and b both appear in B_u. If a and b are both in H_i for some $i \in \{1, 2, 3\}$, then some bag B^i_x contains both a and b. Finally, when a is obtained by contracting a path P_a in $G_i - V(F_i)$ and b is obtained by contracting a path P_b not in G_i, then the cycle F_i separates P_a from P_b so the edge ab is not present in H. This concludes the proof that $(B_x : x \in V(J))$ is the desired tree-decomposition of H. ∎

A. Reducing the Bound

We now set out to reduce the constant in Theorem 1 from 766 to 49. This is achieved by proving the following variant of Theorem 9.

Theorem 13. *Every planar graph G has a partition $\mathcal{P}$ with layered width 3 such that $G/\mathcal{P}$ is planar with treewidth at most 3. Moreover, there is such a partition for every BFS layering of G.*

Theorem 13 and Lemma 7, and a result of Alam, Bekos, Gronemann, Kaufmann and Pupyrev [14], who proved that every planar graph with treewidth at most 3 has queue-number at most 5, imply that planar graphs have bounded queue-number (Theorem 1) with an upper bound of $3 \cdot 3 \cdot 5 + \lfloor \frac{3}{2} \cdot 3 \rfloor = 49$.

Note that Theorem 13 is stronger than Theorem 9 in that the treewidth bound is smaller, whereas Theorem 9 is stronger than Theorem 13 in that the partition is connected and the layered width is smaller. Also note that Theorem 13 is tight in terms of the treewidth of H: For every ℓ, there exists a planar graph G such that, if G has a partition $\mathcal{P}$ of layered width ℓ, then $G/\mathcal{P}$ has treewidth at least 3, see the full version of the paper [24].

While Theorem 10 partitions the vertices of a planar graph into vertical paths, to prove Theorem 13 we instead partition the vertices of a triangulation G^+ into parts each of which is a union of up to three vertical paths. Formally, in a spanning tree T of a graph G, a *tripod* consists of up to three pairwise disjoint vertical paths in T whose lower endpoints form a clique in G. Theorem 13 quickly follows from the next result which we prove in the full version of the paper [24].

Theorem 14. *Let T be a rooted spanning tree in a plane triangulation G. Then G has a partition $\mathcal{P}$ into tripods in T such that $G/\mathcal{P}$ has treewidth at most 3.*

IV. Bounded-Genus Graphs

As was the case for planar graphs, our proof that bounded genus graphs have bounded queue-number employs Corollary 8. Thus the goal of this section is to show that our construction of bounded layered partitions for planar graphs can be generalised for graphs of bounded Euler genus.

Theorem 15. *Every graph G of Euler genus g has a connected partition $\mathcal{P}$ with layered width at most $\max\{2g, 1\}$ such that $G/\mathcal{P}$ has treewidth at most 9. Moreover, there is such a partition for every BFS layering of G.*

This theorem and Corollary 8 imply that graphs of Euler genus g have bounded queue-number (Theorem 2) with an upper bound of $3 \cdot 2g \cdot (2^9 - 1) + \lfloor \frac{3}{2} 2g \rfloor = O(g)$.

Note that Theorem 15 is best possible in the following sense. Suppose that every graph G of Euler genus g has a partition $\mathcal{P}$ with layered width at most ℓ such that $G/\mathcal{P}$ has treewidth at most k. By Lemma 5, G has layered treewidth $O(k\ell)$. Dujmović et al. [13] showed that the maximum layered treewidth of graphs with Euler genus g is $\Theta(g)$. Thus $k\ell \geqslant \Omega(g)$.

The next lemma is the key to the proof of Theorem 15. Many similar results are known in the literature (for example, [37, Lemma 8] or [5, Section 4.2.4]), but none prove exactly what we need.

Lemma 16. *Let G be a connected graph with Euler genus g. For every BFS spanning tree T of G with corresponding BFS layering $(V_0, V_1, \dots)$, there is a subgraph $Z \subseteq G$ with at most $2g$ vertices in each layer V_i, such that Z is connected and $G - V(Z)$ is planar. Moreover, there is a connected planar graph G^+ containing $G - V(Z)$ as a subgraph, and there is a BFS spanning tree T^+ of G^+ with corresponding BFS layering $(W_0, W_1, \dots)$ of G^+, such that $W_i \cap (V(G) \setminus V(Z)) = V_i \setminus V(Z)$ for all $i \geqslant 0$, and $P \cap (V(G) \setminus V(Z))$ is a vertical path in T for every vertical path P in T^+.*

Proof of Theorem 15 assuming Lemma 16: We may assume that G is connected (since if each component of G has the desired partition, then so does G). Let T be a BFS spanning tree of G with corresponding BFS layering $(V_0, V_1, \dots)$. By Lemma 16, there is a subgraph $Z \subseteq G$ with at most $2g$ vertices in each layer V_i, a connected planar graph G^+ containing $G - V(Z)$ as a subgraph, and a BFS spanning tree T^+ of G^+ with corresponding BFS layering $(W_0, W_1, \dots)$, such that $W_i \cap V(G) \setminus V(Z) = V_i \setminus V(Z)$ for all $i \geqslant 0$, and $P \cap V(G) \setminus V(Z)$ is a vertical path in T for every vertical path P in T^+.

By Theorem 10, G^+ has a partition $\mathcal{P}^+$ into vertical paths in T^+ such that $G^+/\mathcal{P}^+$ has treewidth at most 8. Let $\mathcal{P} := \{P \cap V(G) \setminus V(Z) : P \in \mathcal{P}^+\} \cup \{V(Z)\}$. Thus $\mathcal{P}$ is a partition of G. Since $P \cap V(G) \setminus V(Z)$ is a vertical path in T and Z is a connected subgraph of G, $\mathcal{P}$ is a connected partition. Note that the quotient $G/\mathcal{P}$ is obtained from a subgraph of $G^+/\mathcal{P}^+$ by adding one vertex corresponding to Z. Thus $G/\mathcal{P}$ has treewidth at most 9. Since $P \cap V(G) \setminus V(Z)$ is a vertical path in T, it has at most one vertex in each layer V_i. Thus each part of $\mathcal{P}$ has at most $\max\{2g, 1\}$ vertices in each layer V_i. Hence $\mathcal{P}$ has layered width at most $\max\{2g, 1\}$. ∎

The same proof in conjunction with Theorem 14 instead of Theorem 10 shows the following.

Theorem 17. *Every graph of Euler genus g has a partition $\mathcal{P}$ with layered width at most $\max\{2g, 3\}$ such that $G/\mathcal{P}$ has treewidth at most 4.*

Note that Theorem 17 is stronger than Theorem 15 in that the treewidth bound is smaller, whereas Theorem 15 is stronger than Theorem 17 in that the partition is connected (and the layered width is smaller for $g \in \{0, 1\}$). Both Theorems 15 and 17 (with Lemma 7) imply that graphs with Euler genus g have $O(g)$ queue-number, but better constants are obtained by a more direct argument that uses Lemma 16 and Theorem 1 to circumvent the use of Theorem 15 and prove that the queue-number of graphs with Euler genus g is at most $4g + 49$.

V. Excluded Minors

This section first introduces the graph minor structure theorem of Robertson and Seymour, which shows that every graph in a proper minor-closed class can be constructed using four ingredients: graphs on surfaces, vortices, apex vertices, and clique-sums. We then use this theorem to prove that every proper minor-closed class has bounded queue-number (Theorem 3).

Let G_0 be a graph embedded in a surface Σ. Let F be a facial cycle of G_0 (thought of as a subgraph of G_0). An *F-vortex* is an F-decomposition $(B_x \subseteq V(H) : x \in V(F))$ of a graph H such that $V(G_0 \cap H) = V(F)$ and $x \in B_x$ for each $x \in V(F)$. For $g, p, a, k \geqslant 0$, a graph G is (g, p, k, a)-*almost-embeddable* if for some set $A \subseteq V(G)$ with $|A| \leqslant a$, there are graphs $G_0, G_1, \dots, G_s$ for some $s \in \{0, \dots, p\}$ such that:

- $G - A = G_0 \cup G_1 \cup \dots \cup G_s$,
- $G_1, \dots, G_s$ are pairwise vertex-disjoint;
- G_0 is embedded in a surface of Euler genus at most g,
- there are s pairwise vertex-disjoint facial cycles $F_1, \dots, F_s$ of G_0, and
- for $i \in \{1, \dots, s\}$, there is an F_i-vortex $(B_x \subseteq V(G_i) : x \in V(F_i))$ of G_i of width at most k.

The vertices in A are called *apex* vertices. They can be adjacent to any vertex in G.

A graph is *k-almost-embeddable* if it is (k, k, k, k)-almost-embeddable.

Let $C_1 = \{v_1, \dots, v_k\}$ be a k-clique in a graph G_1. Let $C_2 = \{w_1, \dots, w_k\}$ be a k-clique in a graph G_2. Let G be the graph obtained from the disjoint union of G_1 and G_2 by identifying v_i and w_i for $i \in \{1, \dots, k\}$, and possibly deleting some edges in C_1 ($= C_2$). Then G is a *clique-sum* of G_1 and G_2.

The following graph minor structure theorem by Robertson and Seymour [38] is at the heart of graph minor theory.

Theorem 18 ([38]). *For every proper minor-closed class $\mathcal{G}$, there is a constant k such that every graph in $\mathcal{G}$ is obtained by clique-sums of k-almost-embeddable graphs.*

We now show that graphs that satisfy the ingredients of the graph minor structure theorem have bounded queue-number. Building on Theorem 15, we prove the following result in the case of no apex vertices.

Lemma 19. *Every $(g, p, k, 0)$-almost embeddable graph G has a connected partition $\mathcal{P}$ with layered width at most $\max\{2g + 4p - 4, 1\}$ such that $G/\mathcal{P}$ has treewidth at most $11k + 10$.*

Lemmas 7 and 19 imply the following result, where the edges incident to each apex vertex are put in their own queue:

Lemma 20. *Every k-almost embeddable graph has queue-number less than $9k \cdot 2^{11(k+1)}$.*

Theorem 3, which says that every proper minor-closed class has bounded queue-number, follows from Lemma 20 and some general-purpose machinery of Dujmović et al. [13] for performing clique-sums.

A. Characterisation

Bounded layered partitions are the key structure in this paper. So it is natural to ask which minor-closed classes admit bounded layered partitions. The following definition leads to the answer to this question. A graph G is *strongly (g, p, k, a)-almost-embeddable* if it is (g, p, k, a)-almost-embeddable and (using the notation in the definition of (g, p, k, a)-almost-embeddable) there is no edge between an apex vertex and a vertex in $G_0 - (G_1 \cup \dots \cup G_s)$. That is, each apex vertex is only adjacent to other apex vertices or vertices in the vortices. A graph is *strongly k-almost-embeddable* if it is strongly (k,k,k,k)-almost-embeddable.

The following is the main result of this section. See [13,39,40] for the definition of (linear) local treewidth.

Theorem 21. *The following are equivalent for a minor-closed class of graphs $\mathcal{G}$:*
(1) there exists $k,\ell \in \mathbb{N}$ such that every graph $G \in \mathcal{G}$ has a partition $\mathcal{P}$ with layered width at most ℓ, such that $G/\mathcal{P}$ has treewidth at most k.
(2) there exists $k \in \mathbb{N}$ such that every graph $G \in \mathcal{G}$ has a partition $\mathcal{P}$ with layered width at most 1, such that $G/\mathcal{P}$ has treewidth at most k.
(3) there exists $k \in \mathbb{N}$ such that every graph in $\mathcal{G}$ has layered treewidth at most k,
(4) $\mathcal{G}$ has linear local treewidth,
(5) $\mathcal{G}$ has bounded local treewidth,
(6) there exists an apex graph not in $\mathcal{G}$,
(7) there exists $k \in \mathbb{N}$ such that every graph in $\mathcal{G}$ is obtained from clique-sums of strongly k-almost-embeddable graphs.

Proof: Lemma 6 says that (1) implies (2). Lemma 5 says that (2) implies (3). Dujmović et al. [13] proved that (3) implies (4), which implies (5) by definition. Eppstein [39] proved that (5) and (6) are equivalent; see [41] for an alternative proof. Dvořák and Thomas [42] proved that (6) implies (7). Building on Lemma 19, we prove that every graph obtained from clique-sums of strongly k-almost embeddable graphs has a partition of layered width $12k$ such that the quotient has treewidth at most $20k+10$. This says that (7) implies (1). ■

Note that Demaine and Hajiaghayi [40] previously proved that (3) and (4) are equivalent. Also note that the assumption of a minor-closed class in Theorem 21 is essential: Dujmović, Eppstein and Wood [43] proved that the $n \times n \times n$ grid G_n has bounded local treewidth but has unbounded, indeed $\Omega(n)$, layered treewidth. By Lemma 5, if G_n has a partition with layered width ℓ such that the quotient has treewidth at most k, then $k\ell \geqslant \Omega(n)$. That said, it is open whether (1), (2) and (3) are equivalent in a subgraph-closed class.

VI. Strong Products

This section provides an alternative and helpful perspective on layered partitions. The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex set $V(A) \times V(B)$, where distinct vertices $(v,x),(w,y) \in V(A) \times V(B)$ are adjacent if:

- $v = w$ and $xy \in E(B)$, or
- $x = y$ and $vw \in E(A)$, or
- $vw \in E(A)$ and $xy \in E(B)$.

The next observation follows immediately from the definitions.

Observation 22. *For every graph H, a graph G has an H-partition of layered width at most ℓ if and only if G is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some path P.*

Note that a general result about the queue-number of strong products by Wood [44] implies that $\operatorname{qn}(H \boxtimes P) \leqslant 3\operatorname{qn}(H)+1$. It is easily shown that $\operatorname{qn}(Q \boxtimes K_\ell) \leqslant \ell \cdot \operatorname{qn}(Q) + \lfloor \frac{\ell}{2} \rfloor$. Together these results say that $\operatorname{qn}(H \boxtimes P \boxtimes K_\ell) \leqslant \ell(3\operatorname{qn}(H)+1) + \lfloor \frac{\ell}{2} \rfloor$, which proves Lemma 7.

The results in this section show that every graph in certain minor-closed classes is a subgraph of a particular graph product, such as a subgraph of $H \boxtimes P$ for some bounded treewidth graph H and path P. First note that Observation 22 and Theorems 9 and 13 imply the following result conjectured by Wood [45].[4]

Theorem 23. *Every planar graph is a subgraph of:*
(a) $H \boxtimes P$ for some graph H with treewidth at most 8 and some path P.
(b) $H \boxtimes P \boxtimes K_3$ for some graph H with treewidth at most 3 and some path P.

Observation 22 and Theorems 15 and 17 imply the following generalisation of Theorem 23 for graphs of bounded Euler genus.

Theorem 24. *Every graph of Euler genus g is a subgraph of:*
(a) $H \boxtimes P \boxtimes K_{\max\{2g,1\}}$ for some graph H with treewidth at most 9 and for some path P.
(b) $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some graph H with treewidth at most 4 and for some path P.

Lemma 19 and Observation 22 imply the following further generalisation, where $A + B$ is the complete join of graphs A and B.

Theorem 25. *Every (g,p,k,a)-almost embeddable graph is a subgraph of $(H \boxtimes P \boxtimes K_{\max\{2g+4p,1\}}) + K_a$ for some graph H with treewidth at most $11k+10$ and some path P.*

Theorems 18 and 25 imply the following result for any proper minor-closed class.

Theorem 26. *For every proper minor-closed class $\mathcal{G}$ there are integers k and a such that every graph $G \in \mathcal{G}$ can be obtained by clique-sums of graphs $G_1,\dots,G_n$ such that for $i \in \{1,\dots,n\}$, for some graph H_i with treewidth at most k and some path P_i, we have $G_i \subseteq (H_i \boxtimes P_i) + K_a$.*

Note that it is easily seen that in all of the above results, the graph H and the path P have at most $|V(G)|$ vertices.

[4] To be precise, Wood [45] conjectured that for every planar graph G there are graphs X and Y, such that both X and Y have bounded treewidth, Y has bounded maximum degree, and G is a minor of $X \boxtimes Y$, such that the preimage of each vertex of G has bounded radius in $X \boxtimes Y$. Theorem 23(a) is stronger than this conjecture since it has a subgraph rather than a shallow minor, and Y is a path.

We can interpret these results as saying that strong products and complete joins form universal graphs for the above classes. For all n and k there is a graph $H_{n,k}$ with treewidth k that contains every graph with n vertices and treewidth k as a subgraph (take the disjoint union of all such graphs). The proof of Theorem 23 then shows that $H_{n,8} \boxtimes P_n$ contains every planar graph with n vertices. There is a substantial literature on universal graphs for planar graphs and other classes. For example, Babai, Chung, Erdős, Graham and Spencer [46] constructed a graph on $O(n^{3/2})$ edges that contains every planar graph on n vertices as a subgraph. While $H_{n,8} \boxtimes P_n$ contains much more than $O(n^{3/2})$ edges, it has the advantage of being highly structured and with bounded average degree. Taking this argument one step further, there is an infinite graph $\mathcal{T}_k$ with treewidth k that contains every (finite) graph with treewidth k as a subgraph. Similarly, the infinite path $\mathcal{Q}$ contains every (finite) path as a subgraph. Thus our results imply that $\mathcal{T}_8 \boxtimes \mathcal{Q}$ contains every planar graph. Analogous statements can be made for the other classes above.

VII. Non-Minor-Closed Classes

This section gives three examples of non-minor-closed classes of graphs that have bounded queue-number. The following elementary lemma will be helpful.

Lemma 27. *Let G_0 be a graph with a k-queue layout. Fix integers $c \geqslant 1$ and $\Delta \geqslant 2$. Let G be the graph with $V(G) := V(G_0)$ where $vw \in E(G)$ whenever there is a vw-path P in G_0 of length at most c, such that every internal vertex on P has degree at most Δ. Then*

$$\operatorname{qn}(G) < 2(2k(\Delta+1))^{c+1}.$$

Our result for graphs of bounded Euler genus generalises to allow for a bounded number of crossings per edge. A graph is *(g,k)-planar* if it has a drawing in a surface of Euler genus g with at most k crossings per edge and with no three edges crossing at the same point. A $(0,k)$-planar graph is called *k-planar*; see [47] for a survey about 1-planar graphs. Even in the simplest case, there are 1-planar graphs that contain arbitrarily large complete graph minors [43]. Nevertheless, such graphs have bounded queue-number. An easy application of Lemma 27 shows the following lemma, which can also be concluded from a result of Dujmović and Wood [2] in conjunction with Theorem 2.

Proposition 28. *Every (g,k)-planar graph G has queue-number at most $2(40g+490)^{k+2}$.*

Map graphs are defined as follows. Start with a graph G_0 embedded in a surface of Euler genus g, with each face labelled a 'nation' or a 'lake', where each vertex of G_0 is incident with at most d nations. Let G be the graph whose vertices are the nations of G_0, where two vertices are adjacent in G if the corresponding faces in G_0 share a vertex. Then G is called a *(g,d)-map graph*. A $(0,d)$-map graph is called a (plane) *d-map graph*; such graphs have been extensively studied [48,49]. The $(g,3)$-map graphs are precisely the graphs of Euler genus at most g (see [43]). So (g,d)-map graphs provide a natural generalisation of graphs embedded in a surface. An easy application of Lemma 27 shows the following:

Proposition 29. *Every (g,d)-map graph G has queue-number at most $2\big(8g+98)(d+1)\big)^3$.*

A *string graph* is the intersection graph of a set of curves in the plane with no three curves meeting at a single point [50–54]. For an integer $k \geqslant 2$, if each curve is in at most k intersections with other curves, then the corresponding string graph is called a *k-string graph*. A *(g,k)-string* graph is defined analogously for curves on a surface of Euler genus at most g. An easy application of Lemma 27 shows the following:

Proposition 30. *Every (g,k)-string graph has queue-number at most $2(40g+490)^{2k+1}$.*

VIII. Applications and Connections

In this section, we discuss implications of our results such as resolving open problems about 3-dimensional graph drawings.

A. Track Layouts

Track layout are a type of graph layout closely related to queue layouts. A *vertex k-colouring* of a graph G is a partition $\{V_1,\dots,V_k\}$ of $V(G)$ into independent sets; that is, for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. A *track* in G is an independent set equipped with a linear ordering. A partition $\{\overrightarrow{V_1},\dots,\overrightarrow{V_k}\}$ of $V(G)$ into k tracks is a *k-track layout* if for distinct $i,j \in \{1,\dots,k\}$ no two edges of G cross between $\overrightarrow{V_i}$ and $\overrightarrow{V_j}$. That is, for all distinct edges $vw, xy \in E(G)$ with $v,x \in V_i$ and $w,y \in V_j$, if $v \prec x$ in $\overrightarrow{V_i}$ then $w \preccurlyeq y$ in $\overrightarrow{V_j}$. The minimum k such that G has a k-track layout is called the *track-number* of G, denoted by $\operatorname{tn}(G)$. The following lemmas show that queue-number and track-number are tied.

Lemma 31 ([10])**.** *For every graph G,* $\operatorname{qn}(G) \leqslant \operatorname{tn}(G) - 1$.

Lemma 32 ([12])**.** *There is a function f such that* $\operatorname{tn}(G) \leqslant f(\operatorname{qn}(G))$ *for every graph G. In particular, every graph with queue-number at most k has track-number at most*

$$4k \cdot 4^{k(2k-1)(4k-1)}.$$

The following lemma often gives better bounds on the track-number than Lemma 32. A proper graph colouring is *acyclic* if every cycle gets at least three colours. The *acyclic chromatic number* of a graph G is the minimum integer c such that G has an acyclic c-colouring.

Lemma 33 ([10]). *Every graph G with acyclic chromatic number at most c and queue-number at most k has track-number at most $c(2k)^{c-1}$.*

Borodin [55] proved that planar graphs have acyclic chromatic number at most 5, which with Lemma 33 and Theorem 1 implies:

Theorem 34. *Every planar graph has track-number at most* $5(2 \cdot 49)^4 = 461,184,080$.

Note that the best lower bound on the track-number of planar graphs is 7, due to Dujmović et al. [12].

Heawood [56] and Alon, Mohar and Sanders [57] respectively proved that every graph with Euler genus g has chromatic number $O(g^{1/2})$ and acyclic chromatic number $O(g^{4/7})$. Lemma 33 and Theorem 2 then imply:

Theorem 35. *Every graph with Euler genus g has track-number at most $g^{O(g^{4/7})}$.*

For proper minor-closed classes, Lemma 32 and Theorem 3 imply:

Theorem 36. *Every proper minor-closed class has bounded track-number.*

It follows from Lemma 33 and the work of Van den Heuvel and Wood [30] connecting layered treewidth and r-strong colouring number, along with bounds on layered treewidth for (g, k)-planar graphs, (g, d)-map graphs and (g, k)-string graphs [43] that such graphs have bounded track-number.

B. Three-Dimensional Graph Drawing

Further motivation for studying queue and track layouts is their connection with 3-dimensional graph drawing. A *3-dimensional grid drawing* of a graph G represents the vertices of G by distinct grid points in $\mathbb{Z}^3$ and represents each edge of G by the open segment between its endpoints so that no two edges intersect. The *volume* of a 3-dimensional grid drawing is the number of grid points in the smallest axis-aligned grid-box that encloses the drawing. For example, Cohen, Eades, Lin and Ruskey [58] proved that the complete graph K_n has a 3-dimensional grid drawing with volume $O(n^3)$ and this bound is optimal. Pach, Thiele and Tóth [59] proved that every graph with bounded chromatic number has a 3-dimensional grid drawing with volume $O(n^2)$, and this bound is optimal for $K_{n/2,n/2}$.

Track layouts and 3-dimensional graph drawings are connected by the following lemma.

Lemma 37 ([10,60]). *If a c-colourable n-vertex graph G has a t-track layout, then G has 3-dimensional grid drawings with $O(t^2 n)$ volume and with $O(c^7 tn)$ volume. Conversely, if a graph G has a 3-dimensional grid drawing with $A \times B \times C$ bounding box, then G has track-number at most $2AB$.*

Lemma 37 is the foundation for all of the following results. Dujmović and Wood [60] proved that every graph with bounded maximum degree has a 3-dimensional grid drawing with volume $O(n^{3/2})$, and the same bound holds for graphs from a proper minor-closed class. In fact, every graph with bounded degeneracy has a 3-dimensional grid drawing with $O(n^{3/2})$ volume [61]. Dujmovi'c et al. [10] proved that every graph with bounded treewidth has a 3-dimensional grid drawing with volume $O(n)$.

Prior to this work, whether planar graphs have 3-dimensional grid drawings with $O(n)$ volume was a major open problem, due to Felsner, Liotta, and Wismath [62]. The previous best known bound on the volume of 3-dimensional grid drawings of planar graphs was $O(n \log n)$ by Dujmović [17]. Lemma 37 and Theorem 34 together resolve the open problem of Felsner et al. [62].

Theorem 38. *Every planar graph with n vertices has a 3-dimensional grid drawing with $O(n)$ volume.*

Lemma 37 and Theorems 35 and 36 imply the following strengthenings of Theorem 38.

Theorem 39. *Every graph with Euler genus g and n vertices has a 3-dimensional grid drawing with $g^{O(g^{4/7})}n$ volume.*

Theorem 40. *For every proper minor-closed class $\mathcal{G}$, every graph in $\mathcal{G}$ with n vertices has a 3-dimensional grid drawing with $O(n)$ volume.*

As shown in Section VIII-A, (g, k)-planar graphs, (g, d)-map graphs and (g, k)-string graphs have bounded track-number (for fixed g, k, d). By Lemma 37, such graphs have 3-dimensional grid drawings with $O(n)$ volume.

IX. Open Problems

1) What is the maximum queue-number of planar graphs? We can tweak our proof of Theorem 1 to show that every planar graph has queue-number at most 48, but it seems new ideas are required to obtain a significant improvement. The best lower bound on the maximum queue-number of planar graphs is 4, due to Alam et al. [14].
   More generally, does every graph with Euler genus g have $o(g)$ queue-number? Complete graphs provide a $\Theta(\sqrt{g})$ lower bound. Note that every graph with Euler genus g has $O(\sqrt{g})$ stack-number [63].
2) Is there a polynomial function f such that every graph with treewidth k has queue-number at most $f(k)$? The best lower and upper bounds on $f(k)$ are $k + 1$ and $2^k - 1$, both due to Wiechert [6].
3) As discussed in Section I it is open whether there is a function f such that $\mathrm{sn}(G) \leqslant f(\mathrm{qn}(G))$ for every graph G. Heath, Leighton and Rosenberg [1] proved that every 1-queue graph has stack-number at most 2. Dujmović and Wood [2] showed that there is such a function f

if and only if every 2-queue graph has bounded stack-number.

Similarly, it is open whether there is a function f such that $\mathrm{qn}(G) \leqslant f(\mathrm{sn}(G))$ for every graph G. Heath, Leighton and Rosenberg [1] proved that every 1-stack graph has queue-number at most 2. Since 2-stack graphs are planar, this paper solves the first open case of this question. Dujmović and Wood [2] showed that there is such a function f if and only if every 3-stack graph has bounded queue-number.

4) Is there a proof of Theorem 3 that does not use the graph minor structure theorem and with more reasonable bounds?
5) Queue layouts naturally extend to posets. The *cover graph* G_P of a poset P is the undirected graph with vertex set P, where $vw \in E(G)$ if $v <_P w$ and $v <_P x <_P w$ for no $x \in P$ (or $w <_P v$ and $w <_P x <_P v$ for no $x \in P$). Thus the cover graph encodes relations in P that are not implied by transitivity. A k*-queue layout* of a poset P consists of a linear extension $\preccurlyeq$ of P and a partition $E_1, E_2, \dots, E_k$ of $E(G_P)$ into queues with respect to $\preccurlyeq$. The *queue-number* of a poset P is the minimum integer k such that P has a k-queue layout. Heath and Pemmaraju [65] conjecture that the queue-number of a planar poset is at most its height (the maximum number of pairwise comparable elements). This was disproved by Knauer, Micek and Ueckerdt [66] who presented a poset of height 2 and queue-number 4. Theorem 1 and results of Knauer, Micek and Ueckerdt imply that planar posets of height h have queue-number $O(h)$; see Theorem 6 in [66]. Heath and Pemmaraju [65] also conjecture that every poset of width w (the maximum number of pairwise incomparable elements) has queue-number at most w. The best known upper bounds are $O(w^2)$ for general posets and $3w - 2$ for planar posets [66].
6) It is natural to ask for the largest class of graphs with bounded queue-number. First note that Theorem 3 cannot be extended to the setting of an excluded topological minor, since graphs with bounded degree have arbitrarily high queue-number [1,64]. However, it is possible that every class of graphs with strongly sub-linear separators has bounded queue-number. Here a class $\mathcal{G}$ of graphs has *strongly sub-linear separators* if $\mathcal{G}$ is closed under taking subgraphs, and there exists constants $c, \beta > 0$, such that every n-vertex graph in $\mathcal{G}$ has a balanced separator of order $cn^{1-\beta}$. Already the $\beta = \frac{1}{2}$ case looks challenging, since this would imply Theorem 3.
7) Do the results in the present paper have algorithmic applications? Consider the method of Baker [67] for designing polynomial-time approximation schemes for problems on planar graphs. This method partitions the graph into BFS layers, such that the problem can be solved optimally on each layer (since the induced subgraph has bounded treewidth), and then combines the solutions from each layer. Our results (Theorem 9) give a more precise description of the layered structure of planar graphs (and other more general classes). It is conceivable that this extra structural information is useful when designing algorithms.

Note that all our proofs lead to polynomial-time algorithms for computing the desired decomposition and queue layout. Pilipczuk and Siebertz [35] claim $O(n^2)$ time complexity for their decomposition. The same is true for Lemma 11: Given the colours of the vertices on F, we can walk down the BFS tree T in linear time and colour every vertex. Another linear-time enumeration of the faces contained in F finds the trichromatic triangle. It is easily seen that Lemma 16 has polynomial time complexity (given the embedding). Polynomial-time algorithms for our other results follow based on the linear-time algorithm of Mohar [68] to test if a given graph has Euler genus at most any fixed number g, and the polynomial-time algorithm of Demaine, Hajiaghayi and Kawarabayashi [69] for computing the decomposition in the graph minor structure theorem (Theorem 18).

Acknowledgements

This research was completed at the 7th Annual Workshop on Geometry and Graphs held at Bellairs Research Institute in March 2019. Thanks to the other workshop participants for creating a productive working atmosphere.

References

[1] L. S. Heath, F. T. Leighton, and A. L. Rosenberg, "Comparing queues and stacks as mechanisms for laying out graphs," *SIAM J. Discrete Math.*, vol. 5, no. 3, pp. 398–412, 1992.
[2] V. Dujmović and D. R. Wood, "Stacks, queues and tracks: Layouts of graph subdivisions," *Discrete Math. Theor. Comput. Sci.*, vol. 7, pp. 155–202, 2005. http://dmtcs.episciences.org/346
[3] J. F. Buss and P. Shor, "On the pagenumber of planar graphs," in *Proc. 16th ACM Symp. on Theory of Computing* (STOC '84). ACM, 1984, pp. 98–100.
[4] M. Yannakakis, "Embedding planar graphs in four pages," *J. Comput. System Sci.*, vol. 38, no. 1, pp. 36–67, 1989.
[5] B. Mohar and C. Thomassen, *Graphs on surfaces*. Johns Hopkins University Press, 2001.
[6] V. Wiechert, "On the queue-number of graphs with bounded tree-width," *Electron. J. Combin.*, vol. 24, no. 1, p. 1.65, 2017. https://www.combinatorics.org/v24i1p65
[7] L. S. Heath and A. L. Rosenberg, "Laying out graphs using queues," *SIAM J. Comput.*, vol. 21, no. 5, pp. 927–958, 1992.
[8] S. V. Pemmaraju, "Exploring the powers of stacks and queues via graph layouts," Ph.D. dissertation, Virginia Polytechnic Institute and State University, U.S.A., 1992.

[9] S. Rengarajan and C. E. Veni Madhavan, "Stack and queue number of 2-trees," in *Proc. 1st Annual International Conf. on Computing and Combinatorics (COCOON '95)*, D.-Z. Du and M. Li, Eds., Lecture Notes in Comput. Sci., vol. 959. Springer, 1995, pp. 203–212.

[10] V. Dujmović, P. Morin, and D. R. Wood, "Layout of graphs with bounded tree-width," *SIAM J. Comput.*, vol. 34, no. 3, pp. 553–579, 2005.

[11] V. Dujmović and D. R. Wood, "On linear layouts of graphs," *Discrete Math. Theor. Comput. Sci.*, vol. 6, no. 2, pp. 339–358, 2004. http://dmtcs.episciences.org/317

[12] V. Dujmović, A. Pór, and D. R. Wood, "Track layouts of graphs," *Discrete Math. Theor. Comput. Sci.*, vol. 6, no. 2, pp. 497–522, 2004. http://dmtcs.episciences.org/315

[13] V. Dujmović, P. Morin, and D. R. Wood, "Layered separators in minor-closed graph classes with applications," *J. Combin. Theory Ser. B*, vol. 127, pp. 111–147, 2017.

[14] J. M. Alam, M. A. Bekos, M. Gronemann, M. Kaufmann, and S. Pupyrev, "Queue layouts of planar 3-trees," in *Proc. 26th International Symposium on Graph Drawing and Network Visualization (GD 2018)*, T. C. Biedl and A. Kerren, Eds., Lecture Notes in Computer Science, vol. 11282. Springer, 2018, pp. 213–226.

[15] M. A. Bekos, H. Förster, M. Gronemann, T. Mchedlidze, F. Montecchiani, C. N. Raftopoulou, and T. Ueckerdt, "Planar graphs of bounded degree have constant queue number," in *Proc 51st Annual Symposium on Theory of Computing* (STOC '19), 2019, pp. 176–184, arXiv: 1811.00816.

[16] G. Di Battista, F. Frati, and J. Pach, "On the queue number of planar graphs," *SIAM J. Comput.*, vol. 42, no. 6, pp. 2243–2285, 2013.

[17] V. Dujmović, "Graph layouts via layered separators," *J. Combin. Theory Series B.*, vol. 110, pp. 79–89, 2015.

[18] V. Dujmović, P. Morin, and D. R. Wood, "Queue layouts of graphs with bounded degree and bounded genus," 2019, arXiv: 1901.05594.

[19] J. Wang, "Layouts for plane graphs on constant number of tracks," 2017, arXiv: 1708.02114.

[20] Z. Dvořák and J.-S. Sereni, "On fractional fragility rates of graph classes," 2019, arXiv: 1907.12634.

[21] M. Bonamy, C. Gavoille, and M. Pilipczuk, "Shorter labeling schemes for planar graphs," 2019, arXiv: 1908.03341.

[22] M. Dębski, S. Felsner, P. Micek, and F. Schröder, "Improved bounds for centered colorings," 2019, arXiv: 1907.04586.

[23] V. Dujmović, L. Esperet, G. Joret, B. Walczak, and D. R. Wood, "Planar graphs have bounded nonrepetitive chromatic number," 2019, arXiv: 1904.05269.

[24] V. Dujmović, G. Joret, P. Micek, P. Morin, T. Ueckerdt, and D. R. Wood, "Planar graphs have bounded queue-number," 2019, arXiv: 1904.04791.

[25] R. Diestel, *Graph theory*, 4th ed., Graduate Texts in Mathematics. Springer, vol. 173, 2010.

[26] D. J. Harvey and D. R. Wood, "Parameters tied to treewidth," *J. Graph Theory*, vol. 84, no. 4, pp. 364–385, 2017.

[27] B. A. Reed, "Tree width and tangles: a new connectivity measure and some applications," in *Surveys in combinatorics*, Cambridge Univ. Press, 1997, London Math. Soc. Lecture Note Ser., vol. 241, pp. 87–162.

[28] F. Shahrokhi, "New representation results for planar graphs," in *29th European Workshop on Computational Geometry* (EuroCG 2013), 2013, pp. 177–180, arXiv: 1502.06175.

[29] V. Dujmović, D. Eppstein, G. Joret, P. Morin and D. R. Wood, "Minor-closed graph classes with bounded layered pathwidth," 2018. https://arxiv.org/abs/1810.08314.

[30] J. van den Heuvel and D. R. Wood, "Improper colourings inspired by Hadwiger's conjecture," J. London Math. Soc., vol. 98, no. 1, pp. 129–148, 2018. See arXiv: 1704.06536.

[31] M. J. Bannister, W. E. Devanny, V. Dujmović, D. Eppstein, and D. R. Wood, "Track layouts, layered path decompositions, and leveled planarity," *Algorithmica*, vol. 81, no. 4, pp. 1561–1583, 2019.

[32] V. Dujmović and F. Frati, "Stack and queue layouts via layered separators," *J. Graph Algorithms Appl.*, vol. 22, no. 1, pp. 89–99, 2018.

[33] D. R. Wood, "On tree-partition-width," *European J. Combin.*, vol. 30, no. 5, pp. 1245–1253, 2009.

[34] G. Ding and B. Oporowski, "Some results on tree decomposition of graphs," *J. Graph Theory*, vol. 20, no. 4, pp. 481–499, 1995.

[35] M. Pilipczuk and S. Siebertz, "Polynomial bounds for centered colorings on proper minor-closed graph classes," in *Proc. 30th Annual ACM-SIAM Symposium on Discrete Algorithms*, 2019, pp. 1501–1520, arXiv: 1807.03683.

[36] M. Aigner and G. M. Ziegler, *Proofs from The Book*, 4th ed. Springer, 2010.

[37] S. Cabello, É. C. de Verdière, and F. Lazarus, "Algorithms for the edge-width of an embedded graph," *Comput. Geom.*, vol. 45, no. 5-6, pp. 215–224, 2012.

[38] N. Robertson and P. Seymour, "Graph minors. XVI. Excluding a non-planar graph," *J. Combin. Theory Ser. B*, vol. 89, no. 1, pp. 43–76, 2003.

[39] D. Eppstein, "Diameter and treewidth in minor-closed graph families," *Algorithmica*, vol. 27, no. 3-4, pp. 275–291, 2000.

[40] E. D. Demaine and M. Hajiaghayi, "Equivalence of local treewidth and linear local treewidth and its algorithmic applications," in *Proc. 15th Annual ACM-SIAM Symposium on Discrete Algorithms* (SODA '04). SIAM, 2004, pp. 840–849.

[41] ——, "Diameter and treewidth in minor-closed graph families, revisited," *Algorithmica*, vol. 40, no. 3, pp. 211–215, 2004.

[42] Z. Dvořák and R. Thomas, "List-coloring apex-minor-free graphs," 2014, arXiv: 1401.1399.

[43] V. Dujmović, D. Eppstein, and D. R. Wood, "Structure of graphs with locally restricted crossings," *SIAM J. Discrete Math.*, vol. 31, no. 2, pp. 805–824, 2017.

[44] D. R. Wood, "Queue layouts of graph products and powers," *Discrete Math. Theor. Comput. Sci.*, vol. 7, no. 1, pp. 255–268, 2005. http://dmtcs.episciences.org/352

[45] ——, "The structure of cartesian products," 2008. https://www.birs.ca/workshops/2008/08w5079/report08w5079.pdf

[46] L. Babai, F. R. K. Chung, P. Erdős, R. L. Graham, and J. H. Spencer, "On graphs which contain all sparse graphs," in *Theory and practice of combinatorics*, 1982, North-Holland Math. Stud., vol. 60, pp. 21–26.

[47] S. G. Kobourov, G. Liotta, and F. Montecchiani, "An annotated bibliography on 1-planarity," *Comput. Sci. Rev.*, vol. 25, pp. 49–67, 2017.

[48] F. V. Fomin, D. Lokshtanov, and S. Saurabh, "Bidimensionality and geometric graphs," in *Proc. 23rd Annual ACM-SIAM Symposium on Discrete Algorithms*, 2012, pp. 1563–1575.

[49] Z.-Z. Chen, M. Grigni, and C. H. Papadimitriou, "Map graphs," *J. Assoc. Comput. Mach.*, vol. 49, no. 2, pp. 127–138, 2002.

[50] J. Pach and G. Tóth, "Recognizing string graphs is decidable," *Discrete Comput. Geom.*, vol. 28, no. 4, pp. 593–606, 2002.

[51] M. Schaefer and D. Štefankovič, “Decidability of string graphs,” *J. Comput. System Sci.*, vol. 68, no. 2, pp. 319–334, 2004.

[52] M. Schaefer, E. Sedgwick, and D. Štefankovič, “Recognizing string graphs in NP,” *J. Comput. System Sci.*, vol. 67, no. 2, pp. 365–380, 2003.

[53] J. Fox and J. Pach, “A separator theorem for string graphs and its applications,” *Combin. Probab. Comput.*, vol. 19, no. 3, pp. 371–390, 2010.

[54] ——, “Applications of a new separator theorem for string graphs,” *Combin. Probab. Comput.*, vol. 23, no. 1, pp. 66–74, 2014.

[55] O. V. Borodin, “On acyclic colorings of planar graphs,” *Discrete Mathematics*, vol. 25, no. 3, pp. 211–236, 1979.

[56] P. J. Heawood, “Map colour theorem,” *Quart. J. Pure Appl. Math.*, vol. 24, pp. 332–338, 1890.

[57] N. Alon, B. Mohar, and D. P. Sanders, “On acyclic colorings of graphs on surfaces,” *Israel J. Math.*, vol. 94, pp. 273–283, 1996.

[58] R. F. Cohen, P. Eades, T. Lin, and F. Ruskey, “Three-dimensional graph drawing,” *Algorithmica*, vol. 17, no. 2, pp. 199–208, 1996.

[59] J. Pach, T. Thiele, and G. Tóth, “Three-dimensional grid drawings of graphs,” in *Advances in discrete and computational geometry*, B. Chazelle, J. E. Goodman, and R. Pollack, Eds. Amer. Math. Soc., 1999, Contemporary Mathematics, vol. 223, pp. 251–255.

[60] V. Dujmović and D. R. Wood, “Three-dimensional grid drawings with sub-quadratic volume,” in *Towards a Theory of Geometric Graphs*, ser. Contemporary Mathematics, J. Pach, Ed. Amer. Math. Soc., 2004, vol. 342, pp. 55–66.

[61] ——, “Upward three-dimensional grid drawings of graphs,” *Order*, vol. 23, no. 1, pp. 1–20, 2006.

[62] S. Felsner, G. Liotta, and S. K. Wismath, “Straight-line drawings on restricted integer grids in two and three dimensions,” in *Proc. 9th International Symp. on Graph Drawing* (GD ’01), P. Mutzel, M. Jünger, and S. Leipert, Eds., Lecture Notes in Comput. Sci., vol. 2265. Springer, 2002, pp. 328–342.

[63] S. M. Malitz, “Genus g graphs have pagenumber $O(\sqrt{g})$,” *J. Algorithms*, vol. 17, no. 1, pp. 85–109, 1994.

[64] D. R. Wood, “Bounded-degree graphs have arbitrarily large queue-number,” *Discrete Math. Theor. Comput. Sci.*, vol. 10, no. 1, pp. 27–34, 2008. http://dmtcs.episciences.org/434

[65] K. S. Heath, and S. V. Pemmaraju, “Stack and queue layouts of posets,” *SIAM J. Discrete Math.*, vol. 10, no. 4, pp. 599–625, 1997.

[66] K. Knauer, P. Micek, and T. Ueckerdt, “The queue-number of posets of bounded width or height,” in *Proc. Graph Drawing and Network Visualization* (GD ’18), T. Biedl, A. Kerren, Eds., *Lecture Notes in Comput. Sci.*, vol. 11282, Springer, 2018, pp. 200–212.

[67] B. S. Baker, “Approximation algorithms for NP-complete problems on planar graphs,” *J. Assoc. Comput. Mach.*, vol. 41, no. 1, pp. 153–180, 1994.

[68] B. Mohar, “A linear time algorithm for embedding graphs in an arbitrary surface,” *SIAM J. Discrete Math.*, vol. 12, no. 1, pp. 6–26, 1999.

[69] E. D. Demaine, M. Hajiaghayi, and K. Kawarabayashi, “Algorithmic graph minor theory: Decomposition, approximation, and coloring,” in *Proc. 46th Annual IEEE Symposium on Foundations of Computer Science* (FOCS ’05). IEEE, 2005, pp. 637–646.