

# Approximating Constraint Satisfaction Problems on High-Dimensional Expanders

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*Abstract*—We consider the problem of approximately solving constraint satisfaction problems with arity  $k > 2$  ( $k$ -CSPs) on instances satisfying certain expansion properties, when viewed as hypergraphs. Random instances of  $k$ -CSPs, which are also highly expanding, are well-known to be hard to approximate using known algorithmic techniques (and are widely believed to be hard to approximate in polynomial time). However, we show that this is not necessarily the case for instances where the hypergraph is a *high-dimensional expander*.

We consider the spectral definition of high-dimensional expansion used by Dinur and Kaufman [FOCS 2017] to construct certain primitives related to PCPs. They measure the expansion in terms of a parameter  $\gamma$  which is the analogue of the second singular value for expanding graphs. Extending the results by Barak, Raghavendra and Steurer [FOCS 2011] for 2-CSPs, we show that if an instance of **MAX  $k$ -CSP** over alphabet  $[q]$  is a high-dimensional expander with parameter  $\gamma$ , then it is possible to approximate the maximum fraction of satisfiable constraints up to an additive error  $\varepsilon$  using  $q^{O(k)} \cdot (k/\varepsilon)^{O(1)}$  levels of the sum-of-squares SDP hierarchy, provided  $\gamma \leq \varepsilon^{O(1)} \cdot (1/(kq))^{O(k)}$ .

Based on our analysis, we also suggest a notion of threshold-rank for hypergraphs, which can be used to extend the results for approximating 2-CSPs on low threshold-rank graphs. We show that if an instance of **MAX  $k$ -CSP** has threshold rank  $r$  for a threshold  $\tau = (\varepsilon/k)^{O(1)} \cdot (1/q)^{O(k)}$ , then it is possible to approximately solve the instance up to additive error  $\varepsilon$ , using  $r \cdot q^{O(k)} \cdot (k/\varepsilon)^{O(1)}$  levels of the sum-of-squares hierarchy. As in the case of graphs, high-dimensional expanders (with sufficiently small  $\gamma$ ) have threshold rank 1 according to our definition.

*Keywords*-CSP; HDX; SOS;

## I. INTRODUCTION

We consider the problem of approximately solving constraint satisfaction problems (CSPs) on instances satisfying certain expansion properties. The role of expansion in understanding the approximability of CSPs with two variables in each constraint (2-CSPs) has been extensively studied and has led to several

results, which can also be viewed as no-go results for PCP constructions (since PCPs are hard instances of CSPs). It was shown by Arora et al. [AKK<sup>+</sup>08] (and strengthened by Makarychev and Makarychev [MM11]) that the Unique Games problem is easily approximable on expanding instances, thus proving that the Unique Games Conjecture of Khot [Kho02] cannot be true for expanding instances. Their results were extended to all 2-CSPs and several partitioning problems in works by Barak, Raghavendra and Steurer [BRS11], Guruswami and Sinop [GS11], and Oveis Gharan and Trevisan [OGT15] under much weaker notions of expansion.

We consider the following question:

*When are expanding instances of  $k$ -CSPs easy for  $k > 2$ ?*

At first glance, the question does not make much sense, since random instances of  $k$ -CSPs (which are also highly expanding) are known to be hard for various models of computation (see [KMOW17] for an excellent survey). However, while the kind of expansion exhibited by random instances of CSPs is useful for constructing codes, it is not sufficient for constructing primitives for PCPs, such as locally testable codes [BSHR05]. On the other hand, objects such as high-dimensional expanders, which possess a form of “structured multi-scale expansion” have been useful in constructing derandomized direct-product and direct-sum tests (which can be viewed as locally testable distance amplification codes) [DK17], lattices with large distance [KM18], list-decodable direct product codes [DHK<sup>+</sup>18], and are thought to be intimately connected with PCPs [DK17]. Thus, from the PCP perspective, it is more relevant to ask if this form of expansion can be used to efficiently approximate constraint satisfaction problems.

**Connections to coding theory.** Algorithmic results related to expanding CSPs are also relevant for the

problem of *decoding* locally testable codes. Consider a code  $C$  constructed via  $k$ -local operations (such as  $k$ -fold direct-sum) on a base code  $C_0$  with smaller distance. Then, a codeword in  $C$  is simply an instance of a CSP, where each bit places a constraint on  $k$  bits (which is  $k$ -XOR in case of direct sum) of the relevant codeword in  $C_0$ . The task of decoding a noisy codeword is then equivalent to finding an assignment in  $C_0$ , satisfying the maximum number of constraints for the above instance. Thus, algorithms for solving CSPs on expanding instances may lead to new decoding algorithms for codes obtained by applying local operations to a base code. In fact, the list decoding algorithm for direct-product codes by Dinur et al. [DHK<sup>+</sup>18] also relied on algorithmic results for expanding unique games. Since all constructions of locally testable codes need to have at least some weak expansion [DK12], it is interesting to understand what notions of expansion are amenable to algorithmic techniques.

**High-dimensional expanders and our results.** A  $d$ -dimensional expander is a downward-closed hypergraph (simplicial complex), say  $X$ , with edges of size at most  $d + 1$ , such that for every hyperedge  $\alpha \in X$  (with  $|\alpha| \leq d - 1$ ), a certain “neighborhood graph”  $G(X_\alpha)$  is a spectral expander<sup>1</sup>. Here, the graph  $G(X_\alpha)$  is defined to have the vertex set  $\{i \mid \alpha \cup \{i\} \in X\}$  and edge-set  $\{i, j \mid \alpha \cup \{i, j\} \in X\}$ . If the (normalized) second singular value of each of the neighborhood graph is bounded by  $\gamma$ ,  $X$  is said to be a  $\gamma$ -high-dimensional expander ( $\gamma$ -HDX).

Note that (the downward closure of) a random sparse  $(d + 1)$ -uniform hypergraph, say with  $n$  vertices and  $c \cdot n$  edges, is very unlikely to be a  $d$ -dimensional expander. With high probability, no two hyperedges share more than one vertex and thus for any  $i \in [n]$ , the neighborhood graph  $G_i$  is simply a disjoint union of cliques of size  $d$ , which is very far from an expander. While random hypergraphs do not yield high-dimensional expanders, such objects are indeed known to exist via (surprising) algebraic constructions [LSV05b], [LSV05a], [KO18a], [CTZ18] and are known to have several interesting properties and applications [KKL16], [DHK<sup>+</sup>18], [KM17], [KO18b], [DDFH18], [DK17], [PRT16].

Expander graphs can simply be thought of as the one-dimensional case of the above definition. The results of Barak, Raghavendra and Steurer [BRS11] for

<sup>1</sup>While there are several definitions of high-dimensional expanders, we consider the one by Dinur and Kaufman [DK17], which is most closely related to spectral expansion, and was also the one shown to be related to PCP applications. Our results also work for a weaker but more technical definition by Dikstein et al. [DDFH18], which we defer till later.

2-CSPs yield that if the constraint graph of a 2-CSP instance (with size  $n$  and alphabet size  $q$ ) is a sufficiently good (one dimensional) spectral expander, then one can efficiently find solutions satisfying  $\text{OPT} - \epsilon$  fraction of constraints, where  $\text{OPT}$  denotes the maximum fraction of constraints satisfiable by any assignment. Their algorithm is based on  $(q/\epsilon)^{O(1)}$  levels of the Sum-of-Squares (SoS) SDP hierarchy, and the expansion requirement on the constraint graph is that the (normalized) second singular value should be at most  $(\epsilon/q)^{O(1)}$ . We show a similar result for  $k$ -CSPs when the corresponding simplicial complex  $X_{\mathcal{J}}$ , which is obtained by including one hyperedge for each constraint and taking a downward closure, is a sufficiently good  $(k - 1)$ -dimensional expander.

**Theorem I.1 (Informal).** *Let  $\mathcal{J}$  be an instance of MAX  $k$ -CSP on  $n$  variables taking values over an alphabet of size  $q$ , and let  $\epsilon > 0$ . Let the simplicial complex  $X_{\mathcal{J}}$  be a  $\gamma$ -HDX with  $\gamma = \epsilon^{O(1)} \cdot (1/(kq))^{O(k)}$ . Then, there is an algorithm based on  $(k/\epsilon)^{O(1)} \cdot q^{O(k)}$  levels of the Sum-of-Squares hierarchy, which produces an assignment satisfying  $\text{OPT} - \epsilon$  fraction of the constraints.*

**Remark I.2.** *While the level- $t$  relaxation for MAX  $k$ -CSP can be solved in time  $(nq)^{O(t)}$  [RW17], the rounding algorithms used by [BRS11] and our work do not need the full power of this relaxation. Instead, they are captured by the “local rounding” framework of Guruswami and Sinop [GS12] who show how to implement a local rounding algorithm based on  $t$  levels of the SoS hierarchy, in time  $q^{O(t)} \cdot n^{O(1)}$  (where  $q$  denotes the alphabet size).*

A complete version of our results with detailed proofs is given in [AJT19].

**Our techniques.** We start by using essentially the same argument for analyzing the SoS hierarchy as was used by [BRS11] (specialized to the case of expanders). They viewed the SoS solution as giving a joint distribution on each pair of variables forming a constraint, and proved that for sufficiently expanding graphs, these distributions can be made close to product distributions, by conditioning on a small number of variables (which governs the number of levels required). Similarly, we consider the conditions under which joint distributions on  $k$ -tuples corresponding to constraints can be made close to product distributions. Since the [BRS11] argument shows how to split a joint distribution into two marginals, we can use it to recursively split a set of size  $k$  into two smaller ones (one can think of all splitting operations as forming a binary tree with  $k$  leaves).

However, our arguments differ in the kind of expansion required to perform the above splitting op-

erations. In the case of 2-CSPs, one splits along the edges of the constraint graph, and thus we only need the expansion of the constraint graph (which is part of the assumption). However, in the case of  $k$ -CSPs, we may split a set of size  $(\ell_1 + \ell_2)$  into disjoint sets of size  $\ell_1$  and  $\ell_2$ . This requires understanding the expansion of the following family of (weighted) bipartite graphs arising from the complex  $X_{\mathcal{J}}$ : The vertices in the graph are sets of variables of size  $\ell_1$  and  $\ell_2$  that occur in some constraint, and the weight of an edge  $\{a_1, a_2\}$  for  $a_1 \cap a_2 = \emptyset$ , is proportional to the probability that a random constraint contains  $a_1 \sqcup a_2$ . Note that this graph may be weighted even if the  $k$ -CSP instance  $\mathcal{J}$  is unweighted.

We view the above graphs as random walks, which we call “swap walks” on the hyperedges (faces) in the complex  $X$ . While several random walks on high-dimensional expanders have been shown to have rapid mixing [KM17], [KO18b], [DK17], [LLP17], we need a stronger condition. To apply the argument from [BRS11], we not only need that the second singular value is bounded away from one, but require it to be an arbitrarily small constant (as a function of  $\varepsilon$ ,  $k$  and  $q$ ). We show that this is indeed ensured by the condition that  $a_1 \cap a_2 = \emptyset$ , and obtain a bound of  $k^{O(k)} \cdot \gamma$  on the second singular value. This bound, which constitutes much of the technical work in the paper, is obtained by first expressing these walks in terms of more canonical walks, and then using the beautiful machinery of harmonic analysis on expanding posets by Dikstein et al. [DDFH18] to understand their spectra.

The swap walks analyzed above represent natural random walks on simplicial complexes, and their properties may be of independent interest for other applications. Just as the high-dimensional expanders are viewed as “derandomized” versions of the complete complex (containing all sets of size at most  $k$ ), one can view the swap walks as derandomized versions of (bipartite) Kneser graphs, which have vertex sets  $\binom{[n]}{\ell_1}$  and  $\binom{[n]}{\ell_2}$ , and edges  $(a, b)$  iff  $a \cap b = \emptyset$ . We provide a more detailed and technical overview in Section III after discussing the relevant preliminaries in Section II.

**High-dimensional threshold rank.** The correlation breaking method in [BRS11] can be applied as long as the graph has low threshold rank i.e., the number of singular values above a threshold  $\tau = (\varepsilon/q)^{O(1)}$  is bounded. Similarly, the analysis described above can be applied, as long as all the swap walks which arise when splitting the  $k$ -tuples have bounded threshold rank. This suggests a notion of high-dimensional threshold rank for hypergraphs (discussed in Section VII), which can be defined in terms of the thresh-

old ranks of the relevant swap walks. We remark that it is easy to show that dense hypergraphs (with  $\Omega(n^k)$  hyperedges) have small-threshold rank according to this notion, and thus it can be used to recover known algorithms for approximating  $k$ -CSPs on dense instances [FK96] (as was true for threshold rank in graphs).

**Other related work.** While we extend the approach taken by [BRS11] for 2-CSPs, somewhat different approaches were considered by Guruswami and Sinop [GS11], and Oveis-Gharan and Trevisan [OGT15]. The work by Guruswami and Sinop relied on the expansion of the label extended graph, and used an analysis based on low-dimensional approximations of the SDP solution. Oveis-Gharan and Trevisan used low-threshold rank assumptions to obtain a regularity lemma, which was then used to approximate the CSP. For the case of  $k$ -CSPs, the Sherali-Adams hierarchy can be used to solve instances with bounded treewidth [WJ04] and approximately dense instances [YZ14], [MR17]. Brandao and Harrow [BH13] also extended the results by [BRS11] for 2-CSPs to the case of 2-local Hamiltonians. We show that their ideas can also be used to prove a similar extension of our results to  $k$ -local Hamiltonians on high-dimensional expanders.

In case of high-dimensional expanders, in addition to canonical walks described here, a “non-lazy” version of these walks (moving from  $s$  to  $t$  only if  $s \neq t$ ) was also considered by Kaufman and Oppenheim [KO18b], Anari et al. [ALGV18] and Dikstein et al. [DDFH18]. The swap walks studied in this paper were also considered independently in a very recent work of Dikstein and Dinur [DD19] (under the name “complement walks”).

In a recent follow-up work [AJQ<sup>+</sup>19], the algorithms developed here were also used to obtain new unique and list decoding algorithms for direct sum and direct product codes, obtained by “lifting” a base code  $C_0$  via  $k$ -local operations to amplify distance. This work also showed that the hypergraphs obtained by considering collections of length- $k$  walks on an expanding graph also satisfy (a slight variant of) splittability, and admit similar algorithms.

## II. PRELIMINARIES AND NOTATION

### A. Linear Algebra

Recall that for an operator  $A : V \rightarrow W$  between two finite-dimensional inner product spaces  $V$  and  $W$ , the operator norm can be written as

$$\|A\|_{\text{op}} = \sup_{f, g \neq 0} \frac{\langle Af, g \rangle}{\|f\| \|g\|}.$$

Also, for such an  $A$  the adjoint  $A^\dagger : W \rightarrow V$  is defined as the (unique) operator satisfying  $\langle Af, g \rangle = \langle f, A^\dagger g \rangle$  for all  $f \in V, g \in W$ . For  $A : V \rightarrow W$ , we take  $\|A\|_{\text{op}} = \sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_r(A) > 0$  to be its singular values in descending order. Note that for  $A : V \rightarrow V$ ,  $\sigma_2(A)$  denotes its second largest eigenvalue in absolute value.

### B. High-Dimensional Expanders

A high-dimensional expander (HDX) is a particular kind of downward-closed hypergraph (simplicial complex) satisfying an expansion requirement. We elaborate on these properties and define well known natural walks on HDXs below.

#### 1) Simplicial Complexes:

**Definition II.1.** A simplicial complex  $X$  with ground set  $[n]$  is a downward-closed collection of subsets of  $[n]$  i.e., for all sets  $\mathfrak{s} \in X$  and  $\mathfrak{t} \subseteq \mathfrak{s}$ , we also have  $\mathfrak{t} \in X$ . The sets in  $X$  are also referred to as **faces** of  $X$ .

We use the notation  $X(i)$  to denote the collection of all faces  $\mathfrak{s}$  in  $X$  with  $|\mathfrak{s}| = i$ . When faces are of cardinality at most  $d$ , we also use the notation  $X(\leq d)$  to denote all the faces of  $X$ . By convention, we take  $X(0) := \{\emptyset\}$ .

A simplicial complex  $X(\leq d)$  is said to be a **pure simplicial complex** if every face of  $X$  is contained in some face of size  $d$ . Note that in a pure simplicial complex  $X(\leq d)$ , the top slice  $X(d)$  completely determines the complex.

Note that it is more common to associate a geometric representation to simplicial complexes, with faces of cardinality  $i$  being referred to as faces of *dimension*  $i - 1$  (and the collection being denoted by  $X(i - 1)$  instead of  $X(i)$ ). However, since we will only be treating these as hypergraphs, we prefer to index faces by their cardinality, to improve readability of related expressions.

An important simplicial complex is the complete complex.

**Definition II.2** (Complete Complex  $\Delta_d(n)$ ). We denote by  $\Delta_d(n)$  the complete complex with faces of size at most  $d$  i.e.,  $\Delta_d(n) := \{\mathfrak{s} \subseteq [n] \mid |\mathfrak{s}| \leq d\}$ .

2) Walks and Measures on Simplicial Complexes: Let  $C^k$  denote the space of real valued functions on  $X(k)$  i.e.,

$$C^k := \{f \mid f : X(k) \rightarrow \mathbb{R}\} \cong \mathbb{R}^{X(k)}.$$

We describe natural walks on simplicial complexes considered in [DK17], [DDFH18], [KO18b], as stochastic operators, which map functions in  $C^i$  to  $C^{i+1}$  and vice-versa.

To define the stochastic operators associated with the walks, we first need to describe a set of probability measures which serve as the stationary measures for these random walks. For a pure simplicial complex  $X(\leq d)$ , we define a collection of probability measures  $(\Pi_1, \dots, \Pi_d)$ , with  $\Pi_i$  giving a distribution on faces in the slice  $X(i)$ .

**Definition II.3** (Probability measures  $(\Pi_1, \dots, \Pi_d)$ ). Let  $X(\leq d)$  be a pure simplicial complex and let  $\Pi_d$  be an arbitrary probability measure on  $X(d)$ . We define a coupled array of random variables  $(\mathfrak{s}^{(d)}, \dots, \mathfrak{s}^{(1)})$  as follows: sample  $\mathfrak{s}^{(d)} \sim \Pi_d$  and (recursively) for each  $i \in [d]$ , take  $\mathfrak{s}^{(i-1)}$  to be a uniformly random subset of  $\mathfrak{s}^{(i)}$ , of size  $i - 1$ .

The distributions  $\Pi_{d-1}, \dots, \Pi_1$  are then defined to be the marginal distributions of the random variables  $\mathfrak{s}^{(d-1)}, \dots, \mathfrak{s}^{(1)}$  as defined above.

The following is immediate from the definition above.

**Proposition II.4.** Let  $\mathfrak{a} \in X(\ell)$  be an arbitrary face. For all  $j \geq 0$ , one has

$$\sum_{\substack{\mathfrak{b} \in X(\ell+j): \\ \mathfrak{b} \supseteq \mathfrak{a}}} \Pi_{\ell+j}(\mathfrak{b}) = \binom{\ell+j}{j} \cdot \Pi_\ell(\mathfrak{a}).$$

For all  $k$ , we define the inner product of functions  $f, g \in C^k$ , according to associated measure  $\Pi_k$

$$\langle f, g \rangle = \mathbb{E}_{\mathfrak{s} \sim \Pi_k} [f(\mathfrak{s})g(\mathfrak{s})] = \sum_{\mathfrak{s} \in X(k)} f(\mathfrak{s})g(\mathfrak{s}) \cdot \Pi_k(\mathfrak{s}).$$

We now define the up and down operators  $U_i : C^i \rightarrow C^{i+1}$  and  $D_{i+1} : C^{i+1} \rightarrow C^i$  as

$$\begin{aligned} [U_i g](\mathfrak{s}) &= \mathbb{E}_{\substack{\mathfrak{s}' \in X(i), \\ \mathfrak{s}' \subseteq \mathfrak{s}}} [g(\mathfrak{s}')] = \frac{1}{i+1} \cdot \sum_{x \in \mathfrak{s}} g(\mathfrak{s} \setminus \{x\}) \\ [D_{i+1} g](\mathfrak{s}) &= \mathbb{E}_{\mathfrak{s}' \sim \Pi_{i+1} \mid \mathfrak{s}' \supseteq \mathfrak{s}} [g(\mathfrak{s}')], \\ &= \frac{1}{i+1} \cdot \sum_{x \notin \mathfrak{s}} g(\mathfrak{s} \sqcup \{x\}) \cdot \frac{\Pi_{i+1}(\mathfrak{s} \sqcup \{x\})}{\Pi_i(\mathfrak{s})}. \end{aligned}$$

An important consequence of the above definition is that  $U_i$  and  $D_{i+1}$  are adjoints with respect to the inner products of  $C^i$  and  $C^{i+1}$ .

**Fact II.5.**  $U_i = D_{i+1}^\dagger$ , i.e.,  $\langle U_i f, g \rangle = \langle f, D_{i+1} g \rangle$  for every  $f \in C^i$  and  $g \in C^{i+1}$ .

Note that the operators can be thought of as defining random walks in a simplicial complex  $X(\leq d)$ . The operator  $U_i$  moves *down* from a face  $\mathfrak{s} \in X(i+1)$  to a face  $\mathfrak{s}' \in X(i)$ , but lifts a function  $g \in C^i$  *up* to a function  $U_i g \in C^{i+1}$ . Similarly, the operator  $D_{i+1}$  can be thought of as defining a random walk which moves *up*

from  $\mathfrak{s} \in X(i)$  to  $\mathfrak{s}' \in X(i+1)$ . It is easy to verify that these walks respectively map the measure  $\Pi_{i+1}$  to  $\Pi_i$ , and  $\Pi_i$  to  $\Pi_{i+1}$ .

3) *High-Dimensional Expansion*: We recall the notion of high-dimensional expansion (defined via local spectral expansion) considered by [DK17]. We first need a few pieces of notation.

For a complex  $X(\leq d)$  and  $\mathfrak{s} \in X(i)$  for some  $i \in [d]$ , we denote by  $X_{\mathfrak{s}}$  the link complex

$$X_{\mathfrak{s}} := \{\mathfrak{t} \setminus \mathfrak{s} \mid \mathfrak{s} \subseteq \mathfrak{t} \in X\}.$$

When  $|\mathfrak{s}| \leq d-2$ , we also associate a natural weighted graph  $G(X_{\mathfrak{s}})$  to a link  $X_{\mathfrak{s}}$ , with vertex set  $X_{\mathfrak{s}}(1)$  and edge-set  $X_{\mathfrak{s}}(2)$ . The edge-weights are taken to be proportional to the measure  $\Pi_{\mathfrak{s}}$  on the complex  $X_{\mathfrak{s}}$ , which is in turn proportional to the measure  $\Pi_{|\mathfrak{s}|+2}$  on  $X$ . The graph  $G(X_{\mathfrak{s}})$  is referred to as the *skeleton* of  $X_{\mathfrak{s}}$ . Dinur and Kaufman [DK17] define high-dimensional expansion in terms of spectral expansion of the skeletons of the links.

**Definition II.6** ( $\gamma$ -HDX from [DK17]). *A simplicial complex  $X(\leq d)$  is said to be  $\gamma$ -High Dimensional Expander ( $\gamma$ -HDX) if for every  $0 \leq i \leq d-2$  and for every  $\mathfrak{s} \in X(i)$ , the graph  $G(X_{\mathfrak{s}})$  satisfies  $\sigma_2(G(X_{\mathfrak{s}})) \leq \gamma$ , where  $\sigma_2(G(X_{\mathfrak{s}}))$  denotes the second singular value of the (normalized) adjacency matrix of  $G(X_{\mathfrak{s}})$ .*

### C. Constraint Satisfaction Problems (CSPs)

A  $k$ -CSP instance  $\mathfrak{J} = (H, \mathcal{C}, w)$  with alphabet size  $q$  consists of a  $k$ -uniform hypergraph, a set of constraints

$$\mathcal{C} = \{\mathcal{C}_{\mathfrak{a}} \subseteq [q]^{\mathfrak{a}} : \mathfrak{a} \in H\},$$

and a non-negative weight function  $w \in \mathbb{R}_+^H$  on the constraints, satisfying  $\sum_{\mathfrak{a} \in H} w(\mathfrak{a}) = 1$ .

A constraint  $\mathcal{C}_{\mathfrak{a}}$  is said to be satisfied by an assignment  $\eta$  if we have  $\eta|_{\mathfrak{a}} \in \mathcal{C}_{\mathfrak{a}}$  i.e., the restriction of  $\eta$  on  $\mathfrak{a}$  is contained in  $\mathcal{C}_{\mathfrak{a}}$ . We write,  $\text{SAT}_{\mathfrak{J}}(\eta)$  for the (weighted fraction of the constraints) satisfied by the assignment  $\eta$  i.e.,

$$\text{SAT}_{\mathfrak{J}}(\eta) = \sum_{\mathfrak{a} \in H} w(\mathfrak{a}) \cdot \mathbf{1}[\eta|_{\mathfrak{a}} \in \mathcal{C}_{\mathfrak{a}}] = \mathbb{E}_{\mathfrak{a} \sim w} \mathbf{1}[\eta|_{\mathfrak{a}} \in \mathcal{C}_{\mathfrak{a}}].$$

We denote by  $\text{OPT}(\mathfrak{J})$  the maximum of  $\text{SAT}_{\mathfrak{J}}(\eta)$  over all  $\eta \in [q]^{V(H)}$ .

Any  $k$ -uniform hypergraph  $H$  can be associated with a pure simplicial complex in a canonical way by just setting  $X_{\mathfrak{J}} = \{\mathfrak{b} : \exists \mathfrak{a} \in H \text{ and } \mathfrak{a} \supseteq \mathfrak{b}\}$  – notice that  $X_{\mathfrak{J}}(k) = H$ . We will refer to this complex as the *constraint complex* of the instance  $\mathfrak{J}$ . The probability

distribution  $\Pi_k$  on  $X_{\mathfrak{J}}$  will be derived from the weights function  $w$  of the constraint, i.e

$$\Pi_k(\mathfrak{a}) = w(\mathfrak{a}) \quad \forall \mathfrak{a} \in X_{\mathfrak{J}}(k) = H.$$

### D. Sum-of-Squares Relaxations and $t$ -local PSD Ensembles

The Sum-of-Squares (SoS) hierarchy gives a sequence of increasingly tight semidefinite programming relaxations for several optimization problems, including CSPs. Since we will use relatively few facts about the SoS hierarchy, already developed in the analysis of Barak, Raghavendra and Steurer [BRS11], we will adapt their notation of  $t$ -local distributions to describe the relaxations. For a  $k$ -CSP instance  $\mathfrak{J} = (H, \mathcal{C}, w)$  on  $n$  variables, we consider the following semidefinite relaxation given by  $t$ -levels of the SoS hierarchy, with vectors  $v_{(S, \alpha)}$  for all  $S \subseteq [n]$  with  $|S| \leq t$ , and all  $\alpha \in [q]^S$ . Here, for  $\alpha_1 \in [q]^{S_1}$  and  $\alpha_2 \in [q]^{S_2}$ ,  $\alpha_1 \circ \alpha_2 \in [q]^{S_1 \cup S_2}$  denotes the partial assignment obtained by concatenating  $\alpha_1$  and  $\alpha_2$ .

maximize $\mathbb{E}_{\mathfrak{a} \sim w} \left[ \sum_{\mathfrak{a} \in \mathcal{C}_{\mathfrak{a}}} \ v_{(\mathfrak{a}, \alpha)}\ ^2 \right] =: \text{SDP}(\mathfrak{J})$
subject to $\langle v_{(S_1, \alpha_1)}, v_{(S_2, \alpha_2)} \rangle = 0 \quad \forall \alpha_1 _{S_1 \cap S_2} \neq \alpha_2 _{S_1 \cap S_2}$
$\langle v_{(S_1, \alpha_1)}, v_{(S_2, \alpha_2)} \rangle = \langle v_{(S_3, \alpha_3)}, v_{(S_4, \alpha_4)} \rangle \quad \forall S_1 \cup S_2 = S_3 \cup S_4,$
$\alpha_1 \circ \alpha_2 = \alpha_3 \circ \alpha_4$
$\sum_{j \in [q]} \ v_{(\{i, j\})}\ ^2 = 1 \quad \forall i \in [n]$
$\ v_{(\emptyset, \emptyset)}\  = 1$

For any set  $S$  with  $|S| \leq t$ , the vectors  $v_{(S, \alpha)}$  induce a probability distribution  $\mu_S$  over  $[q]^S$  such that the assignment  $\alpha \in [q]^S$  appears with probability  $\|v_{(S, \alpha)}\|^2$ . Moreover, these distributions are consistent on intersections i.e., for  $T \subseteq S \subseteq [n]$ , we have  $\mu_S|_T = \mu_T$ , where  $\mu_S|_T$  denotes the restriction of the distribution  $\mu_S$  to the set  $T$ . We use these distributions to define a collection of random variables  $\mathbf{Y}_1, \dots, \mathbf{Y}_n$  taking values in  $[q]$ , such that for any set  $S$  with  $|S| \leq t$ , the collection of variables  $\{\mathbf{Y}_i\}_{i \in S}$  have a joint distribution  $\mu_S$ . Note that the entire collection  $(\mathbf{Y}_1, \dots, \mathbf{Y}_n)$  may not have a joint distribution: this property is only true for sub-collections of size  $t$ . We will refer to the collection  $(\mathbf{Y}_1, \dots, \mathbf{Y}_n)$  as a  $t$ -local ensemble of random variables.

We also have that that for any  $T \subseteq [n]$  with  $|T| \leq t-2$ , and any  $\beta \in [q]^T$ , we can define a  $(t-|T|)$ -local ensemble  $(\mathbf{Y}'_1, \dots, \mathbf{Y}'_n)$  by “conditioning” the local distributions on the event  $\mathbf{Y}_T = \beta$ , where  $\mathbf{Y}_T$  is shorthand for the collection  $\{\mathbf{Y}_i\}_{i \in T}$ . For any  $S$  with  $|S| \leq t-|T|$ , we define the distribution of  $\mathbf{Y}'_S$  as  $\mu'_S := \mu_{S \cup T} | \{\mathbf{Y}_T = \beta\}$ . Finally, the semidefinite program also ensures that for any such conditioning,

the conditional covariance matrix

$$M_{(S_1, \alpha_1)(S_2, \alpha_2)} = \text{Cov}\left(\mathbf{1}[\mathbf{Y}'_{S_1} = \alpha_1], \mathbf{1}[\mathbf{Y}'_{S_2} = \alpha_2]\right)$$

is positive semidefinite, where  $|S_1|, |S_2| \leq (t - |T|)/2$ . Here, for each pair  $S_1, S_2$  the covariance is computed using the joint distribution  $\mu'_{S_1 \cup S_2}$ . The PSD-ness can be easily verified by noticing that the above matrix can be written as the Gram matrix of the vectors

$$w_{(S, \alpha)} := \frac{1}{\|\mathbf{v}_{(T, \beta)}\|} \cdot \mathbf{v}_{(T \cup S, \beta \circ \alpha)} - \frac{\|\mathbf{v}_{(T \cup S, \beta \circ \alpha)}\|^2}{\|\mathbf{v}_{(T, \beta)}\|^3} \cdot \mathbf{v}_{(T, \beta)}$$

In this paper, we will only consider  $t$ -local ensembles such that for every conditioning on a set of size at most  $t - 2$ , the conditional covariance matrix is PSD. We will refer to these as  $t$ -local PSD ensembles. We will also need a simple corollary of the above definitions.

**Fact II.7.** *Let  $(\mathbf{Y}_1, \dots, \mathbf{Y}_n)$  be a  $t$ -local PSD ensemble, and let  $X$  be any simplicial complex with  $X(1) = [n]$ . Then, for all  $s \leq t/2$ , the collection  $\{\mathbf{Y}_\alpha\}_{\alpha \in X(\leq s)}$  is a  $(t/s)$ -local PSD ensemble, where  $X(\leq s) = \bigcup_{i=1}^s X(i)$ .*

For random variables  $\mathbf{Y}_S$  in a  $t$ -local PSD ensemble, we use the notation  $\{\mathbf{Y}_S\}$  to denote the distribution of  $\mathbf{Y}_S$  (which exists when  $|S| \leq t$ ). We also define  $\text{Var}[\mathbf{Y}_S]$  as  $\sum_{\alpha \in [q]^S} \text{Var}[\mathbf{1}[\mathbf{Y}_S = \alpha]]$ .

### III. PROOF OVERVIEW: APPROXIMATING MAX 4-XOR

We consider a simple example of a specific  $k$ -CSP, which captures most of the key ideas in our proof. Let  $\mathcal{J}$  be an unweighted instance of 4-XOR on  $n$  Boolean variables. Let  $H$  be a 4-uniform hypergraph on vertex set  $[n]$ , with a hyperedge corresponding to each constraint i.e., each  $\alpha = \{i_1, i_2, i_3, i_4\} \in H$  corresponds to a constraint in  $\mathcal{J}$  of the form

$$x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = b_\alpha \pmod{2},$$

for some  $b_\alpha \in \{0, 1\}$ . Let  $X$  denote the constraint complex for the instance  $\mathcal{J}$  such that  $X(1) = [n]$ ,  $X(4) = H$  and let  $\Pi_1, \dots, \Pi_4$  be the associated distributions (with  $\Pi_4$  being uniform on  $H$ ).

**Local vs global correlation: the BRS strategy.** We first recall the strategy used by [BRS11], which also suggests a natural first step for our proof. Given a 2-CSP instance with an associated graph  $G$ , and a  $t$ -local PSD ensemble  $\mathbf{Y}_1, \dots, \mathbf{Y}_n$  obtained from the SoS relaxation, they consider if the ‘‘local correlation’’ of the ensemble is small across the edges of  $G$  (which correspond to constraints) i.e.,

$$\mathbb{E}_{\{i,j\} \sim G} \left\| \{\mathbf{Y}_i \mathbf{Y}_j\} - \{\mathbf{Y}_i\} \{\mathbf{Y}_j\} \right\|_1 \leq \varepsilon.$$

If the local correlation is indeed small, we easily produce an assignment achieving a value  $\text{SDP} - \varepsilon$  in expectation, simply by rounding each variable  $x_i$  independently according to the distribution  $\{\mathbf{Y}_i\}$ . On the other hand, if this is not satisfied, they show (as a special case of their proof) that if  $G$  is an expander with second eigenvalue  $\lambda \leq c \cdot (\varepsilon^2/q^2)$ , then variables also have a high ‘‘global correlation’’, between a typical pair  $(i, j) \in [n]^2$ . Here,  $q$  is the alphabet size and  $c$  is a fixed constant. They use this to show that for  $(\mathbf{Y}'_1, \dots, \mathbf{Y}'_n)$  obtained by conditioning on the value of a randomly chosen  $\mathbf{Y}_{i_0}$ , we have

$$\mathbb{E}_i [\text{Var}[\mathbf{Y}_i]] - \mathbb{E}_{i_0, \mathbf{Y}_{i_0}} \mathbb{E}_i [\text{Var}[\mathbf{Y}'_i]] \geq \Omega(\varepsilon^2/q^2),$$

where the expectations over  $i$  and  $i_0$  are both according to the stationary distribution on the vertices of  $G$ . Since the variance is bounded between 0 and 1, this essentially shows that the local correlation must be at most  $\varepsilon$  after conditioning on a set of size  $O(q^2/\varepsilon^2)$  (although the actual argument requires a bit more care and needs to condition on a somewhat larger set).

**Extension to 4-XOR.** As in [BRS11], we check if the  $t$ -local PSD ensemble  $(\mathbf{Y}_1, \dots, \mathbf{Y}_n)$  obtained from the SDP solution satisfies

$$\mathbb{E}_{\{i_1, i_2, i_3, i_4\} \in H} \left\| \{\mathbf{Y}_{i_1} \mathbf{Y}_{i_2} \mathbf{Y}_{i_3} \mathbf{Y}_{i_4}\} - \{\mathbf{Y}_{i_1}\} \{\mathbf{Y}_{i_2}\} \{\mathbf{Y}_{i_3}\} \{\mathbf{Y}_{i_4}\} \right\|_1 \leq \varepsilon.$$

As before, independently sampling each  $x_i$  from  $\{\mathbf{Y}_i\}$  gives an expected value at least  $\text{SDP} - \varepsilon$  in this case. If the above inequality is not satisfied, an application of triangle inequality gives

$$\mathbb{E}_{\{i_1, i_2, i_3, i_4\} \in H} \left[ \begin{aligned} & \left\| \{\mathbf{Y}_{i_1} \mathbf{Y}_{i_2} \mathbf{Y}_{i_3} \mathbf{Y}_{i_4}\} - \{\mathbf{Y}_{i_1} \mathbf{Y}_{i_2}\} \{\mathbf{Y}_{i_3} \mathbf{Y}_{i_4}\} \right\|_1 + \\ & \left\| \{\mathbf{Y}_{i_1} \mathbf{Y}_{i_2}\} - \{\mathbf{Y}_{i_1}\} \{\mathbf{Y}_{i_2}\} \right\|_1 + \\ & \left\| \{\mathbf{Y}_{i_3} \mathbf{Y}_{i_4}\} - \{\mathbf{Y}_{i_3}\} \{\mathbf{Y}_{i_4}\} \right\|_1 \end{aligned} \right] > \varepsilon.$$

Symmetrizing over all orderings of  $\{i_1, i_2, i_3, i_4\}$ , we can write the above as

$$\varepsilon_2 + 2 \cdot \varepsilon_1 > \varepsilon,$$

which gives  $\max\{\varepsilon_1, \varepsilon_2\} \geq \varepsilon/3$ . Here,

$$\varepsilon_1 := \mathbb{E}_{\{i_1, i_2\} \sim \Pi_2} \left\| \{\mathbf{Y}_{i_1} \mathbf{Y}_{i_2}\} - \{\mathbf{Y}_{i_1}\} \{\mathbf{Y}_{i_2}\} \right\|_1, \quad \text{and}$$

$$\begin{aligned} \varepsilon_2 &:= \mathbb{E}_{\{i_1, i_2, i_3, i_4\} \sim \Pi_4} \left\| \{\mathbf{Y}_{i_1} \mathbf{Y}_{i_2} \mathbf{Y}_{i_3} \mathbf{Y}_{i_4}\} - \{\mathbf{Y}_{i_1} \mathbf{Y}_{i_2}\} \{\mathbf{Y}_{i_3} \mathbf{Y}_{i_4}\} \right\|_1 \\ &= \mathbb{E}_{\{i_1, i_2, i_3, i_4\} \sim \Pi_4} \left\| \{\mathbf{Y}_{\{i_1, i_2\}} \mathbf{Y}_{\{i_3, i_4\}}\} - \{\mathbf{Y}_{\{i_1, i_2\}}\} \{\mathbf{Y}_{\{i_3, i_4\}}\} \right\|_1. \end{aligned}$$

As before,  $\varepsilon_1$  measures the local correlation across edges of a weighted graph  $G_1$  with vertex set  $X(1) = [n]$  and edge-weights given by  $\Pi_2$ . Also,  $\varepsilon_2$  measures the analogous quantity for a graph  $G_2$  with vertex set  $X(2)$  (pairs of variables occurring in constraints) and edge-weights given by  $\Pi_4$ .

Recall that the result from [BRS11] can be applied to *any* graph  $G$  over variables in a 2-local PSD ensemble, as long as the  $\sigma_2(G)$  is small. Since  $\{\mathbf{Y}_i\}_{i \in [n]}$  and  $\{\mathbf{Y}_s\}_{s \in X(2)}$  are both  $(t/2)$ -local PSD ensembles (by Fact II.7), we will apply the result to the graph  $G_1$  on the first ensemble and  $G_2$  on the second ensemble. We consider the potential

$$\Phi(\mathbf{Y}_1, \dots, \mathbf{Y}_n) := \mathbb{E}_{i \sim \Pi_1} \text{Var}[\mathbf{Y}_i] + \mathbb{E}_{s \sim \Pi_2} \text{Var}[\mathbf{Y}_s].$$

Since local correlation is large along at least one of the graphs  $G_1$  and  $G_2$ , using the above arguments (and the non-decreasing nature of variance under conditioning) it is easy to show that in expectation over the choice of  $\{i_0, j_0\} \sim \Pi_2$  and  $\beta \in [q]^2$  chosen from  $\{\mathbf{Y}_{\{i_0, j_0\}}\}$ , the conditional ensemble  $(\mathbf{Y}'_1, \dots, \mathbf{Y}'_n)$  satisfies

$$\Phi(\mathbf{Y}_1, \dots, \mathbf{Y}_n) - \mathbb{E}_{i_0, j_0, \beta} \Phi(\mathbf{Y}'_1, \dots, \mathbf{Y}'_n) = \Omega(\varepsilon^2),$$

provided  $G_1$  and  $G_2$  satisfy  $\sigma_2(G_1), \sigma_2(G_2) \leq c \cdot \varepsilon^2$  for an appropriate constant  $c$ .

The bound on the eigenvalue of  $G_1$  follows simply from the fact that it is the skeleton of  $X$ , which is a  $\gamma$ -HDX. Obtaining bounds on the eigenvalues of  $G_2$  and similar higher-order graphs, constitutes much of the technical part of this paper. Note that for a random sparse instance of MAX 4-XOR, the graph  $G_2$  will be a matching with high probability (since  $\{i_1, i_2\}$  in a constraint will only be connected to  $\{i_3, i_4\}$  in the same constraint). However, we show that in case of a  $\gamma$ -HDX, this graph has second eigenvalue  $O(\gamma)$ . We analyze these graphs in terms of modified high-dimensional random walks, which we call “swap walks”.

We remark that our potential and choice of a “seed set” of variables to condition on, is slightly different from [BRS11]. To decrease the potential function above, we need that for each level  $X(i)$  ( $i = 1, 2$  in the example above) the seed set must contain sufficiently many independent samples from  $X(i)$  sampled according to  $\Pi_i$ . This can be ensured by drawing independent samples from the top level  $X(k)$  (though  $X(2)$  suffices in the above example). In contrast, the seed set in [BRS11] consists of random samples from  $\Pi_1$ . The graph  $G_2$  defined above can be thought of as a random walk on  $X(2)$ , which starts at a face  $s \in X(2)$ , moves up to a face (constraint)  $s' \in X(4)$  containing it, and then descends to a face  $t \in X(2)$  such that  $t \subset s'$  and  $s \cap t = \emptyset$  i.e., the walk “swaps out” the elements in  $s$  for other elements in  $s'$ . Several walks considered on simplicial complexes allow for the possibility of a non-trivial intersection, and hence have second eigenvalue lower bounded by a constant. On the other hand, swap

walks completely avoid any laziness and thus turn out to have eigenvalues which can be made arbitrarily small. To understand the eigenvalues for this walk, we will express it in terms of other canonical walks defined on simplicial complexes.

Recall that the up and down operators can be used to define random walks on simplicial complexes. The up operator  $U_i : C^i \rightarrow C^{i+1}$  defines a walk that moves *down* from a face  $s \in X(i+1)$  to a random face  $t \in X(i), t \subset s$  (the operator thus “lifts” a function in  $C^i$  to a function in  $C^{i+1}$ ). Similarly, the down operator  $D_i : C^i \rightarrow C^{i-1}$  moves *up* from a face  $s \in X(i-1)$  to  $t \in X(i), t \supset s$ , with probability  $\Pi_i(t)/(i \cdot \Pi_{i-1}(s))$ . These can be used to define a canonical random walk

$$N_{2,2}^{(u)} := D_3 \cdots D_{u+2} U_{u+1} \cdots U_2, \quad N_{2,2}^{(u)} : C^2 \rightarrow C^2,$$

which moves from up for  $u$  steps  $s \in X(2)$  to  $s' \in X(u+2)$ , and then descends back to  $t \in X(2)$ . Such walks were analyzed optimally by Dinur and Kaufman [DK17], who proved that  $\lambda_2(N_{2,2}^{(u)}) = 2/(u+2) \pm O_u(\gamma)$  when  $X$  is a  $\gamma$ -HDX. Thus, while this walk gives an expanding graph with vertex set  $X(2)$ , the second eigenvalue cannot be made arbitrarily small for a fixed  $u$  (recall that we are interested in showing that  $\sigma_2(G_2) \leq c \cdot \varepsilon^2$ ). However, note that we are only interested in  $N_{2,2}^{(2)}$  conditioned on the event that the two elements from  $s$  are “swapped out” with new elements in the final set  $t$  i.e.,  $s \cap t = \emptyset$ . We define

$$S_{2,2}^{(u,j)}(s, t) := \begin{cases} \frac{\binom{u+2}{2}}{\binom{u}{j} \cdot \binom{2}{2-j}} \cdot N_{2,2}^{(u)} & \text{if } |t \setminus s| = j \\ 0 & \text{otherwise} \end{cases},$$

where the normalization is to ensure stochasticity of the matrix. In this notation, the graph  $G_2$  corresponds to the random-walk matrix  $S_{2,2}^{(2,2)}$ . We show that while  $\sigma_2(N_{2,2}^{(2)}) \approx 1/2$ , we have that  $\sigma_2(S_{2,2}^{(2,2)}) = O(\gamma)$ . We first write the canonical walks in terms of the swap walks. Note that

$$N_{2,2}^{(2)} = \frac{1}{6} \cdot I + \frac{2}{3} \cdot S_{2,2}^{(2,1)} + \frac{1}{6} \cdot S_{2,2}^{(2,2)},$$

since the “descent” step from  $s' \in X(4)$  containing  $s \in X(2)$ , produces a  $t \in X(2)$  which “swaps out” 0, 1 and 2 elements with probabilities  $1/6, 2/3$  and  $1/6$  respectively. Similarly,

$$N_{2,2}^{(1)} = \frac{1}{3} \cdot I + \frac{2}{3} \cdot S_{2,2}^{(1,1)}.$$

Finally, we use the fact (proved in Section IV) that while the canonical walks do depend on the “height”  $u$  (i.e.,  $N_{2,2}^{(u)} \neq N_{2,2}^{(u')}$ ) the swap walks (for a fixed number

of swaps  $j$ ) are independent of the height to which they ascend! In particular, we have

$$S_{2,2}^{(2,1)} = S_{2,2}^{(1,1)}.$$

Using these, we can derive an expression for the swap walk  $S_{2,2}^{(2,2)}$  as

$$S_{2,2}^{(2,2)} = 1 + 6 \cdot N_{2,2}^{(2)} - 6 \cdot N_{2,2}^{(1)} = 1 + 6 \cdot (D_3 D_4 U_3 U_2 - D_3 U_2)$$

To understand the spectrum of operators such as the ones given by the above expression, we use the beautiful machinery for harmonic analysis over HDXs (and more generally over expanding posets) developed by Dikstein et al. [DDFH18]. They show how to decompose the spaces  $C^k$  into approximate eigenfunctions for operators of the form DU. Using these decompositions and the properties of expanding posets, we can show that distinct eigenvalues of the above operator are approximately the same (up to  $O(\gamma)$  errors) when analyzing the walks on the complete complex. Finally, we use the fact that swap walks in a complete complex correspond to Kneser graphs (for which the eigenvectors and eigenvalues are well-known) to show that  $\lambda_2(S_{2,2}^{(2,2)}) = O(\gamma)$ .

**Splittable CSPs and high-dimensional threshold rank.** We note that the ideas used above can be generalized (at least) in two ways. In the analysis of distance from product distribution for a 4-tuple of random variables forming a constraint, we split it in 2-tuples. In general, we can choose to split tuples in a  $k$ -CSP instance along *any* binary tree  $\mathcal{T}$  with  $k$  leaves, with each parent node corresponding to a swap walk between tuples forming its children. Finally, the analysis from [BRS11] also works if the each of the swap walks in some  $\mathcal{T}$  have a bounded number (say  $r$ ) of eigenvalues above some threshold  $\tau$ , which provide a notion of high-dimensional threshold rank for hypergraphs. We refer to such an instance as a  $(\mathcal{T}, \tau, r)$ -splittable.

The arguments sketched above show that high-dimensional expanders are  $(\mathcal{T}, O(\gamma), 1)$ -splittable for all  $\mathcal{T}$ . Since the knowledge of  $\mathcal{T}$  is only required in our analysis and not in the algorithm, we say that  $\text{rank}_\tau(\mathcal{J}) \leq r$  (or that  $\mathcal{J}$  is  $(\tau, r)$ -splittable) if  $\mathcal{J}$  is  $(\mathcal{T}, \tau, r)$ -splittable for any  $\mathcal{T}$ . We defer the precise statement of results for  $(\tau, r)$ -splittable instances to Section VII.

#### IV. WALKS

It is important to note that both  $U_i$  and  $D_{i+1}$  are row-stochastic matrices, i.e. we can think of them as the probability matrices describing the movement of a

walk from  $X(i+1)$  to  $X(i)$ ; and from  $X(i)$  to  $X(i+1)$  respectively.

Concretely, we will think  $[D_{i+1}^\top e_s](t)$  as the probability of the walk moving up from  $s \in X(i)$  to  $t \in X(i+1)$ . Similarly, we will think of  $[U_i^\top e_t](s)$  as the walk moving down from  $t \in X(i+1)$  to  $s \in X(i)$ .

By referring to the definition of the up and down operators in Section II, we observe

$$[D_{i+1}^\top e_s](t) = \frac{\mathbf{1}[t \supseteq s]}{i+1} \frac{\Pi_{i+1}(t)}{\Pi_i(s)} \quad \text{and} \quad [U_i^\top e_t](s) = \frac{\mathbf{1}[s \subseteq t]}{i+1}.$$

With this observation it is easy to see that our notion of random walk respects the probability distributions  $\Pi_j$ , i.e. we have

$$U_i^\top \Pi_{i+1} = \Pi_i \quad \text{and} \quad D_{i+1}^\top \Pi_i = \Pi_{i+1},$$

i.e., randomly moving up from a sample of  $\Pi_j$  gives a sample of  $\Pi_{j+1}$  and similarly, moving down from a sample of  $\Pi_{j+1}$  results in a sample of  $\Pi_j$ .

Instead of going up and down by one dimension, one can try going up or down by multiple dimensions since  $(D_{i+1} \cdots D_{i+\ell})$  and  $(U_{i+\ell} \cdots U_i)$  are still row-stochastic matrices. Further, the corresponding probability vectors still have intuitive explanations in terms of the distributions  $\Pi_j$ . For a face  $s \in X(k)$ , we introduce the notation

$$p_s^{(u)} = (D_{k+1} \cdots D_{k+u})^\top e_s$$

where we take  $p_s^{(0)} = e_s$ . This notation will be used to denote the probability distribution of the up-walk starting from  $s \in X(k)$  and ending in a random face  $t \in X(k+u)$  satisfying  $t \supseteq s$ .

Note that the following Lemma together with Proposition II.4 implies that  $p_s^{(u)}$  is indeed a probability distribution.

**Proposition IV.1.** *For  $s \in X(k)$  and  $a \in X(k+u)$  one has,*

$$p_s^{(u)}(a) = \mathbf{1}[a \supseteq s] \cdot \frac{1}{\binom{k+u}{u}} \cdot \frac{\Pi_{k+u}(a)}{\Pi_k(s)}.$$

Similarly, we introduce the notation  $q_a^{(u)}$ , as

$$q_a^{(u)}(s) = (U_{k+u-1} \cdots U_k)^\top e_s,$$

i.e. for the probability distribution of the down-walk starting from  $a \in X(k+u)$  and ending in a random face of  $X(k)$  contained in  $a$ . The following can be verified using Proposition IV.1, and the fact that  $(U_{k+u-1} \cdots U_k)^\dagger = D_{k+u} \cdots D_{k+1}$ .

**Corollary IV.2.** Let  $X(\leq d)$  be a simplicial complex, and  $k, u \geq 0$  be parameters satisfying  $k + u \leq d$ . For  $\mathfrak{a} \in X(k + u)$  and  $\mathfrak{s} \in X(k)$ , one has

$$q_{\mathfrak{a}}^{(u)}(\mathfrak{s}) = \frac{1}{\binom{k+u}{u}} \cdot \mathbf{1}[\mathfrak{s} \subseteq \mathfrak{a}].$$

In the remainder of this section, we will try to construct more intricate walks on  $X$  from  $X(k)$  to  $X(l)$ .

#### A. The Canonical and the Swap Walks on a Simplicial Complex

**Definition IV.3** (Canonical and Swap  $u$ -Walks). Let  $d \geq 0$ ,  $X(\leq d)$  be a simplicial complex, and  $k, l, u \geq 0$  be parameters satisfying  $l \leq k$ ,  $u \leq l$  and  $d \geq k + u$ ; where the constraints on these parameters are to ensure well-definedness. We will define the following random walks,

- **canonical  $u$ -walk from  $X(k)$  to  $X(l)$ .** Let  $N_{k,l}^{(u)}$  be the (row-stochastic) Markov operator that represents the following random walk: Starting from a face  $\mathfrak{s} \in X(k)$ ,
- (random ascent/up-walk) randomly move up a face  $\mathfrak{s}'' \in X(k + u)$  that contains  $\mathfrak{s}$ , where  $\mathfrak{s}''$  is picked with probability

$$p_{\mathfrak{s}}^{(u)}(\mathfrak{s}'') = [(D_{k+1} \cdots D_{k+u})^{\top} e_{\mathfrak{s}}](\mathfrak{s}'').$$

- (random descent/down-walk) go to a face  $\mathfrak{s}' \in X(l)$  picked uniformly among all the  $l$ -dimensional faces that are contained in  $\mathfrak{s}''$ , i.e., the set  $\mathfrak{s}'$  is picked with probability

$$q_{\mathfrak{s}''}(\mathfrak{s}') = \frac{\mathbf{1}[\mathfrak{s}' \subseteq \mathfrak{s}'']}{\binom{k+u}{l}} = [(U_{k+u-1} \cdots U_l)^{\top} e_{\mathfrak{s}''}](\mathfrak{s}').$$

The operator  $N_{k,l}^{(u)}: C^l \rightarrow C^k$  satisfies the following equation,

$$N_{k,l}^{(u)} = D_{k+1} \cdots D_{k+u} U_{k+u-1} \cdots U_k \cdots U_l.$$

Notice that we have  $N_{k,k}^{(0)} = I$ , and  $N_{k,l}^{(0)} = (U_{k-1} \cdots U_l)$  for  $l < k$ .

- **swapping walk from  $X(k)$  to  $X(l)$ .** Let  $S_{k,l}$  be the Markov operator that represents the following random walk: Starting from a face  $\mathfrak{s} \in X(k)$ ,
- (random ascent/up-walk) randomly move up to a face  $\mathfrak{s}'' \in X(k + l)$  that contains  $\mathfrak{s}$ , where  $\mathfrak{s}''$  is picked with probability

$$p_{\mathfrak{s}}^{(l)}(\mathfrak{s}'') = [(D_{k+1} \cdots D_{k+l+1})^{\top} e_{\mathfrak{s}}](\mathfrak{s}'').$$

- (deterministic descent) deterministically go to  $\mathfrak{s}' = \mathfrak{s}'' \setminus \mathfrak{s} \in X(l)$ .

For our applications, we will need to show that the walk  $S_{k,l}$  has good spectral expansion whenever  $X$  is a

$d$ -dimensional  $\gamma$ -expander, for  $\gamma$  sufficiently small. To show this, we will relate the swapping walk operator  $S_{k,l}$  on  $X$  to the canonical random walk operators  $N_{k,l}^{(u)}$  (q.v. Lemma IV.4).

By the machinery of expanding posets (q.v. Section V) it is possible to argue that the spectral expansion of the random walk operator  $N_{k,l}^{(u)}$  on a high dimensional expander will be close to that of the complete complex. This will allow us to conclude using the relation between the swapping walks and the canonical walks (q.v. Lemma IV.4) that the spectral expansion of the swapping walk on  $X$ , will be comparable with the spectral expansion of the swap walk on the complete complex. More precisely, we will show

**Lemma IV.4** (Lemma V.34). For any  $d, k, l \geq 0$ , and the complete simplicial complex  $X(\leq d)$ , one has the following: If  $k \geq l \geq 0$  and  $d \geq k + l$ , we have

$$\sigma_2(S_{k,l}) = O_{k,l}\left(\frac{1}{n}\right).$$

Using these two, and the expanding poset machinery, we will conclude

**Theorem IV.5** (Theorem V.2 simplified). Let  $X$  be a  $d$ -dimensional  $\gamma$  expander. If  $k \geq l \geq 0$  satisfy  $d \geq l + k$  we have,

$$\sigma_2(S_{k,l}) = O_{k,l}(\gamma)$$

where  $S_{k,l}$  is the swapping walk on  $X$  from  $X(k)$  to  $X(l)$ .

To prove Theorem IV.5 we will need to define an intermediate random walk that we will call the  $j$ -swapping  $u$ -walk from  $X(k)$  to  $X(l)$ :

**Definition IV.6** ( $j$ -swapping  $u$ -walk from  $X(k)$  to  $X(l)$ ). Given  $d, u, j, k, l \geq 0$  satisfying  $l \leq k$ ,  $j \leq u$ ,  $u \leq l$ , and  $d \geq k + u$ . Let  $S_{k,l}^{(u,j)}$  be the Markov operator that represents the following random walk from  $X(k)$  to  $X(l)$  on a  $d$ -dimensional simplicial complex  $X$ : Starting from  $\mathfrak{s} \in X(k)$

- (random ascent/up-walk) randomly move up to a face  $\mathfrak{s}'' \in X(k + u)$  that contains  $\mathfrak{s}$ , where  $\mathfrak{s}''$  is picked with probability

$$p_{\mathfrak{s}}^{(u)}(\mathfrak{s}'') = [(D_{k+1} \cdots D_{k+u})^{\top} e_{\mathfrak{s}}](\mathfrak{s}'').$$

- (conditioned descent) go to a face  $\mathfrak{s}' \in X(l)$  sampled uniformly among all the subsets of  $\mathfrak{s}'' \in X(k + u)$  that have intersection  $j$  with  $\mathfrak{s}'' \setminus \mathfrak{s}$ , i.e.  $|\mathfrak{s}' \cap (\mathfrak{s}'' \setminus \mathfrak{s})| = j$ .

Notice that  $S_{k,l} = S_{k,l}^{(l,l)}$  for any  $k$  and  $l = S_{k,k}^{(u,0)}$  for any  $u$ .

**Remark IV.7.** We will prove that the parameter  $u$  does not effect the swapping walk  $S_{k,l}^{(u,j)}$  so long as  $u \geq j$ , i.e. for all  $u, u' \geq j$  we have  $S_{k,l}^{(u',j)} = S_{k,l}^{(u,j)}$ . Thus, we will often write  $S_{k,l}^{(j)}$  for  $S_{k,l}^{(j,j)}$ .

### B. Swap Walks are Height Independent

Recall that the swap walk  $S_{k,l}^{(u,j)}$  is the conditional walk defined in terms of  $N_{k,l}^{(u)}$  where  $\mathfrak{s} \in X(k)$  is connected to  $\mathfrak{t} \in X(l)$  only if  $|\mathfrak{t} \setminus \mathfrak{s}| = j$ . The parameter  $u$  is called the height of the walk, namely the number of times it moves up. Since up and down operators have second singular value bounded away from 1, the second singular value of  $N_{k,l}^{(u)}$  shrinks as  $u$  increases. In other words, the operator  $N_{k,l}^{(u)}$  depends on the height  $u$ . Surprisingly, the walk  $S_{k,l}^{(u,j)}$  which is defined in terms of  $N_{k,l}^{(u)}$  does not depend on the particular choice of  $u$  as long as it is well defined. More precisely, we have the following result.

**Lemma IV.8.** *If  $X$  is a  $d$ -dimensional simplicial complex,  $0 \leq l \leq k$ , and  $u, u' \in [j, d - k]$ , then*

$$S_{k,l}^{(u,j)} = S_{k,l}^{(u',j)}.$$

In order to obtain [Lemma IV.8](#), we will need a simple proposition:

**Proposition IV.9.** *Let  $\mathfrak{s} \in X(k)$ ,  $\mathfrak{s}' \subseteq \mathfrak{s}$  and  $|\mathfrak{t}'| = j$ . Suppose  $\mathfrak{s}' \sqcup \mathfrak{t}' \in X(l)$ . Then, we have*

$$S_{k,l}^{(u,j)}(\mathfrak{s}, \mathfrak{s}' \sqcup \mathfrak{t}') = \frac{1}{\binom{k}{l-j} \cdot \binom{u}{j}} \cdot \sum_{\substack{\mathfrak{a} \in X(k+u) \\ \mathfrak{a} \supseteq (\mathfrak{s} \sqcup \mathfrak{t}')}} p_{\mathfrak{s}}^{(u)}(\mathfrak{a}).$$

**Lemma IV.10** (Height Independence). *Let  $u \in [j, d - k]$ . For any  $\mathfrak{s} \in X(k)$ ,  $\mathfrak{s}' \subseteq \mathfrak{s}$  and  $\mathfrak{t}' \in X(j)$  satisfying  $\mathfrak{s}' \sqcup \mathfrak{t}' \in X(l)$  we have the following,*

$$S_{k,l}^{(u,j)}(\mathfrak{s}, \mathfrak{s}' \sqcup \mathfrak{t}') = \frac{1}{\binom{k}{l-j} \binom{k+j}{j}} \cdot \frac{\Pi_{k+j}(\mathfrak{s} \sqcup \mathfrak{t}')}{\Pi_k(\mathfrak{s})}.$$

*In particular, the choice of  $u \in [j, d - k]$  does not affect the swap walk.*

Since the choice of  $u$  does not affect the formula, we obtain [Lemma IV.8](#).

### C. Canonical Walks in Terms of the Swap Walks

We show that the canonical walks are given by an average of swap walks with respect to the hypergeometric distribution.

**Lemma IV.11.** *Let  $u, l, k, d \geq 0$  be given satisfying  $l \leq k$  and  $u \leq l$ . Then, we have the following formula for the canonical  $u$ -walk on any  $X(\leq d)$  satisfying  $d \geq k + u$*

$$N_{k,l}^{(u)} = \sum_{j=0}^u \frac{\binom{u}{j} \binom{k}{l-j}}{\binom{k+u}{l}} \cdot S_{k,l}^{(j)}.$$

### D. Inversion: Swap Walks in Terms of Canonical Walks

We show how the swap walks can be obtained as a signed sum of canonical walks. This result follows from binomial inversion which we recall next.

**Fact IV.12** (Binomial Inversion, [[BS02](#)]). *Let  $(a_n)_{n \geq 0}, (b_n)_{n \geq 0}$  be arbitrary sequences. Suppose for all  $n \geq 0$  we have,*

$$b_n = \sum_{j=0}^n \binom{n}{j} \cdot (-1)^j \cdot a_j.$$

*Then, we also have*

$$a_n = \sum_{j=0}^n \binom{n}{j} \cdot (-1)^j \cdot b_j.$$

**Corollary IV.13.** *Let  $k, l, d \geq 0$  be given parameters such that  $k + l \leq d$  and  $k \geq 1$ . For any simplicial complex  $X(\leq d)$ , one has the following formula for the  $u$ -swapping walk from  $X(k)$  to  $X(l)$  in terms of the canonical  $j$ -walks:*

$$\binom{k}{l-u} S_{k,l}^{(u)} = \sum_{j=0}^u (-1)^{u-j} \cdot \binom{k+j}{l} \cdot \binom{u}{j} \cdot N_{k,l}^{(j)}.$$

## V. SPECTRAL ANALYSIS OF SWAP WALKS

Swap walks arise naturally in our  $k$ -CSPs approximation scheme on HDXs where the running time and the quality of approximation depend on the expansion of these walks. For this reason, we analyze the spectra of swap walks. We show that swap walks  $S_{k,k}$  of  $\gamma$ -HDXs are indeed expanding for  $\gamma$  sufficiently small. More precisely, the first main result of this section is the following.

**Theorem V.1** (Swap Walk Spectral Bound). *Let  $X(\leq d)$  be a  $\gamma$ -HDX with  $d \geq 2k$ . Then the second largest singular value  $\sigma_2(S_{k,k})$  of the swap operator  $S_{k,k}$  is*

$$\sigma_2(S_{k,k}) \leq \gamma \cdot \left( 2^7 \cdot k^4 \cdot 2^{3k} \cdot k^k \right).$$

[Theorem V.1](#) is enough for the analysis of our  $k$ -CSP approximation scheme when  $k$  is a power of two. However, to analyze general  $k$ -CSPs on HDXs we need to understand the spectra of general swap walks  $S_{k,l}$  where  $k$  may differ from  $l$ . Therefore, we generalize the spectral analysis of  $S_{k,k}$  above to  $S_{k,l}$  obtaining [Theorem V.2](#), our second main result of this section.

**Theorem V.2** (Rectangular Swap Walk Spectral Bound). *Suppose  $X(\leq d)$  is a  $\gamma$ -HDX with  $d \geq k + l$  and  $k \leq l$ . Then the largest non-trivial singular value  $\sigma_2(S_{k,l})$  of the swap operator  $S_{k,l}$  is*

$$\sigma_2(S_{k,l}) \leq \sqrt{\gamma \cdot (2^8 \cdot k^2 \ell^2 \cdot 2^{2k+4l} \cdot k^k)}.$$

### A. Square Swap Walks $S_{k,k}$

We prove [Theorem V.1](#) by connecting the spectral structure of  $S_{k,k}$  of general  $\gamma$ -HDXs to the well behaved case of complete simplicial complexes. To distinguish these two cases we denote by  $S_{k,k}^\Delta$  the swap  $S_{k,k}$  of complete complexes <sup>2</sup>. In fact,  $S_{k,k}^\Delta$  is the random walk operator of the well known Kneser graph  $K(n,k)$  (see [Definition V.3](#)).

**Definition V.3** (Kneser Graph  $K(n,k)$  [[GM15](#)]). *The Kneser graph  $K(n,k)$  is the graph  $G = (V, E)$  where  $V = \binom{[n]}{k}$  and  $E = \{\{s, t\} \mid s \cap t = \emptyset\}$ .*

Then at least for complete complexes we know that  $S_{k,k}^\Delta$  is expanding. This is a direct consequence of [Fact V.4](#).

**Fact V.4** (Kneser Graph [[GM15](#)]). *The singular values <sup>3</sup> of the Kneser graph  $K(n,k)$  are*

$$\binom{n-k-i}{k-i},$$

for  $i = 0, \dots, k$ .

This means that  $\sigma_2(S_{k,k}^\Delta) = O_k(1/n)$  as shown in [Claim V.5](#).

**Claim V.5.** *Let  $d \geq 2k$  and  $\Delta_d(n)$  be the complete complex. The second largest singular value  $\sigma_2(S_{k,k}^\Delta)$  of the swap operator  $S_{k,k}^\Delta$  on  $\Delta_d(n)$  is*

$$\sigma_2(S_{k,k}^\Delta) = \frac{k}{n-k},$$

provided  $n \geq M_k$  where  $M_k \in \mathbb{N}$  only depends on  $k$ .

Therefore, if we could claim that  $\sigma_2(S_{k,k})$  of an arbitrary  $\gamma$ -HDX is close to  $\sigma_2(S_{k,k}^\Delta)$  (provided  $\gamma$  is sufficiently small), we would conclude that general  $S_{k,k}$  walks are also expanding. A priori there is no reason why this claim should hold since a general  $d$ -sized  $\gamma$ -HDX may have much fewer hyperedges ( $O_d(n)$  versus  $\binom{n}{d}$ ) in the complete  $\Delta_d(n)$ ). Fortunately, it turns out that this claim is indeed true (up to  $O_k(\gamma)$  errors).

To prove [Theorem V.1](#)<sup>4</sup> we employ the beautiful expanding poset (EPoset) machinery of Dikstein et al. [[DDFH18](#)]. Before we delve into the full technical analysis, it might be instructive to see how [Theorem V.1](#) is obtained from understanding the quadratic form  $\langle S_{k,k}f, f \rangle$  where  $f \in C^k$ .

<sup>2</sup>The precise parameters of the complete complex  $\Delta_d(n)$  where  $S_{k,k}^\Delta$  lives will not be important. We only require that  $S_{k,k}^\Delta$  is well defined in the sense that  $d \geq 2k$  and  $n > d$ .

<sup>3</sup>The precise eigenvalues are also well known, but singular values are enough in our analysis.

<sup>4</sup>Due to space constraints, we will omit some details. The full proof can be found in the full version of our paper [[AJT19](#)]

First we informally recall the decomposition  $C^k = \sum_{i=0}^k C_i^k$  from the EPoset machinery where  $C_i^k$  can be thought of as the space of approximate eigenfunctions of degree  $i$  of  $C^k$  (the precise definitions are deferred to [V-B](#)). In this decomposition,  $C_0^k$  is defined as the space of constant functions of  $C^k$ .

We prove the stronger result that the  $S_{k,k}$  operators of any  $\gamma$ -HDX has an approximate spectrum that only depends on  $k$  provided  $\gamma$  is small enough. More precisely, we prove [Lemma V.6](#).

**Lemma V.6** (Swap Quadratic Form). *Let  $f = \sum_{i=0}^k f_i$  with  $f_i \in C_i^k$ . Suppose  $X(\leq d)$  is a  $\gamma$ -HDX with  $d \geq 2k$ . If  $\gamma \leq \varepsilon \left(64k^{k+4}2^{3k+1}\right)^{-1}$ , then*

$$\langle S_{k,k}f, f \rangle = \sum_{i=0}^k \lambda_k(i) \cdot \langle f_i, f_i \rangle \pm \varepsilon,$$

where  $\lambda_k(i)$  depends only on  $k$  and  $i$ , i.e.,  $\lambda_k(i)$  is an approximate eigenvalue of  $S_{k,k}$  associated to space  $C_i^k$ .

**Remark V.7.** From [Lemma V.6](#), it might seem that we are done since there exist approximate eigenvalues  $\lambda_k(i)$  that only depend on  $k$  and  $i$ . However, giving an explicit expression for these approximate eigenvalues is tricky. For this reason, we rely on the expansion of Kneser graphs as will be clear later.

Towards showing [Lemma V.6](#), we introduce the notion of *balanced* operators which in particular captures canonical and swap walks and we show that the quadratic form expression of [Lemma V.6](#) is a particular case of a general result for  $\langle Bf, f \rangle$  where  $B$  is a general *balanced* operator. A *balanced* operator in  $C^k$  is any operator that can be obtained as linear combination of *pure balanced* operators, the later being operators that are a formal product of an equal number of up and down operators.

**Lemma V.8** (General Quadratic Form). *Let  $\varepsilon \in (0, 1)$  and let  $\mathcal{Y} \subseteq \{Y \mid Y: C^k \rightarrow C^k\}$  be a collection of formal operators that are product of an equal number of up and down walks (i.e., pure balanced operators) not exceeding  $\ell$  walks. Let  $B = \sum_{Y \in \mathcal{Y}} \alpha^Y Y$  where  $\alpha^Y \in \mathbb{R}$  and let  $f = \sum_{i=0}^k f_i$  with  $f_i \in C_i^k$ . If  $\gamma \leq \varepsilon \left(16k^{k+2}\ell^2 \sum_{Y \in \mathcal{Y}} |\alpha^Y|\right)^{-1}$ , then*

$$\langle Bf, f \rangle = \sum_{i=0}^k \left( \sum_{Y \in \mathcal{Y}} \alpha^Y \lambda_k^Y(i) \right) \cdot \langle f_i, f_i \rangle \pm \varepsilon,$$

where  $\lambda_k^Y(i)$  depends only on the operators appearing in the formal expression of  $Y$ ,  $i$  and  $k$ , i.e.,  $\lambda_k^Y(i)$  is the approximate eigenvalue of  $Y$  associated to  $C_i^k$ .

**Remark V.9.** Note that our result generalizes the analysis of [[DDFH18](#)] for expanding posets of HDXs which

considered the particular case  $B = D_{k+1}U_k$ . Moreover, their error term analysis treated all the parameters not depending on the number of vertices  $n$  as constants. In this work we make the dependence on the parameters explicit since this dependence is important in understanding the limits of our  $k$ -CSPs approximation scheme on HDXs. The beautiful EPoset machinery [DDFH18] is instrumental in our analysis.

Now, we are ready to prove [Theorem V.1](#). For convenience we restate it below.

**Theorem V.10** (Swap Walk Spectral Bound (restatement of [Theorem V.1](#))). *Let  $X(\leq d)$  be a  $\gamma$ -HDX with  $d \geq 2k$ . For every  $\sigma \in (0, 1)$ , if  $\gamma \leq \sigma \cdot (64k^{k+4}2^{3k+1})^{-1}$ , then the second largest singular value  $\sigma_2(S_{k,k})$  of the swap operator  $S_{k,k}$  is*

$$\sigma_2(S_{k,k}) \leq \sigma.$$

### B. Expanding Posets and Balanced Operators

We state the definitions used in our technical proofs starting with  $\gamma$ -EPoset from [DDFH18].

**Definition V.11** ( $\gamma$ -EPoset adapted from [DDFH18]). *A complex  $X(\leq d)$  with operators  $U_0, \dots, U_{d-1}, D_1, \dots, D_d$  is said to be a  $\gamma$ -EPoset<sup>5</sup> provided*

$$\|M_i^+ - U_{i-1}D_i\|_{\text{op}} \leq \gamma, \quad (1)$$

for every  $i = 1, \dots, d-1$ , where

$$M_i^+ := \frac{i+1}{i} \left( D_{i+1}U_i - \frac{1}{i+1}I \right),$$

i.e.,  $M_i^+$  is the non-lazy version of the random walk  $N_{i,i}^{(1)} = D_{i+1}U_i$ .

[Definition V.11](#) can be directly used as an operational definition of high-dimension expansion as done in [DDFH18]. To us it is important that  $\gamma$ -HDXs are also  $\gamma$ -EPosets as established in [Lemma V.12](#). In fact, these two notions are known to be closely related.

**Lemma V.12** (From [DDFH18]). *Let  $X$  be a  $d$ -sized simplicial complex.*

- If  $X$  is a  $\gamma$ -HDX, then  $X$  is a  $\gamma$ -EPoset.
- If  $X$  is a  $\gamma$ -EPoset, then  $X$  is a  $3d\gamma$ -HDX.

Naturally the complete complex  $\Delta_d(n)$  is a  $\gamma$ -EPoset since it is a  $\gamma$ -HDX. Moreover, in this particular case  $\gamma$  vanishes as  $n$  grows.

**Lemma V.13** (From [DDFH18]). *The complete complex  $\Delta_d(n)$  is a  $\gamma$ -EPoset with  $\gamma = O_d(1/n)$ .*

<sup>5</sup>We tailor their general EPoset definition to HDXs. In fact, what they call  $\gamma$ -HDX we call  $\gamma$ -EPoset. Moreover, what they call  $\gamma$ -HD expander we call  $\gamma$ -HDX.

*Harmonic Analysis on Simplicial Complexes:* The space  $C^k$  defined in [Section II-B2](#) can be decomposed into subspaces  $C_i^k$  of functions of degree  $i$  for  $0 \leq i \leq k$  where

$$C_i^k := \{U^{k-i}h_i \mid h_i \in H_i\},$$

with  $H_i := \ker(D_i)$ , and

$$C_0^k := \{f: X(k) \rightarrow \mathbb{R} \mid f \text{ is constant}\}.$$

More precisely, we have the following.

**Lemma V.14** (From [DDFH18]).

$$C^k = \sum_{i=0}^k C_i^k.$$

For convenience set  $\vec{\delta} \in \mathbb{R}^{d-1}$  such that  $\delta_i = 1/(i+1)$  for  $i \in [d-1]$ . It will also be convenient to work with the following equivalent version of [Eq. \(1\)](#)

$$\|D_{i+1}U_i - (1 - \delta_i)U_{i-1}D_i - \delta_i I\|_{\text{op}} \leq \frac{i}{i+1}\gamma. \quad (2)$$

Towards our goal of understanding quadratic forms of swap operators we study the approximate spectrum of operators of the form  $Y = Y_\ell \dots Y_1$  where each  $Y_i$  is either an up or down operator, namely,  $Y$  is a generalized random walk of  $\ell$  steps. We regard the expression  $Y_\ell \dots Y_1$  defining  $Y$  as a formal product.

**Definition V.15** (Pure Balanced Operator). *We call  $Y: C^k \rightarrow C^k$  a pure balanced operator if  $Y$  can be defined as product  $Y_\ell \dots Y_1$ <sup>6</sup> where each  $Y_i$  is either an up or down operator. When we say that the spectrum of  $Y$  depends on  $Y$  we mean that it depends on  $k$  and on the formal expression  $Y_\ell \dots Y_1$  (i.e., pattern of up and down operators).*

**Remark V.16.** *By definition canonical walks  $N_{k,k}^{(u)}$  are pure balanced operators.*

Taking linear combinations of pure balanced operators leads to the notion of balanced operators.

**Definition V.17** (Balanced Operator). *We call  $B: C^k \rightarrow C^k$  a balanced operator provided there exists a set of pure balanced operators  $\mathcal{Y}$  such that*

$$B = \sum_{Y \in \mathcal{Y}} \alpha^Y \cdot Y,$$

where  $\alpha^Y \in \mathbb{R}$ .

**Remark V.18.** *Corollary IV.13 establishes that  $S_{k,k}^{(u)}$  are balanced operators. In particular,  $S_{k,k}$  is a balanced operator.*

It turns out that at a more crude level the behavior of  $Y$  is governed by how the number

<sup>6</sup>For the analysis it is convenient to order the indices appearing in  $Y_\ell \dots Y_1$  in decreasing order from left to right.

of up operators compares to the number of down operators. For this reason, it is convenient to define  $\mathcal{U}(Y) = \{Y_i \mid Y_i \text{ is an up operator}\}$  and  $\mathcal{D}(Y) = \{Y_i \mid Y_i \text{ is a down operator}\}$  where  $Y$  is a *pure balanced* operator. When  $Y$  is clear in the context we use  $\mathcal{U} = \mathcal{U}(Y)$  and  $\mathcal{D} = \mathcal{D}(Y)$ .

Henceforth we assume  $h_i \in H_i = \ker(D_i)$ ,  $f_i \in C_i^k$  and  $g \in C^k$ . This convention will make the statements of the technical results of [Section V-C](#) cleaner.

### C. Quadratic Forms over Balanced Operators

Now we establish all the technical results leading to and including the analysis of quadratic forms over *balanced operators*. By considering this general class of operators our analysis generalizes the analysis given in [\[DDFH18\]](#). At the same time we refine their error terms analysis by making the dependence on the EPoset parameters explicit. Recall that an explicit dependence on these parameters is important in understanding the limits of our  $k$ -CSP approximation scheme.

**Lemma V.19** (General Quadratic Form (restatement of [Lemma V.8](#))). *Let  $\varepsilon \in (0, 1)$  and let  $\mathcal{Y} \subseteq \{Y \mid Y: C^k \rightarrow C^k\}$  be a collection of formal operators that are product of an equal number of up and down walks (i.e., pure balanced operators) not exceeding  $\ell$  walks. Let  $B = \sum_{Y \in \mathcal{Y}} \alpha^Y$  where  $\alpha^Y \in \mathbb{R}$  and let  $f = \sum_{i=0}^k f_i$  with  $f_i \in C_i^k$ . If  $\gamma \leq \varepsilon \left(16k^{k+2}\ell^2 \sum_{Y \in \mathcal{Y}} |\alpha^Y|\right)^{-1}$ , then*

$$\langle Bf, f \rangle = \sum_{i=0}^k \left( \sum_{Y \in \mathcal{Y}} \alpha^Y \lambda_k^Y(i) \right) \cdot \langle f_i, f_i \rangle \pm \varepsilon,$$

where  $\lambda_k^Y(i)$  depends only on the operators appearing in the formal expression of  $Y$ ,  $i$  and  $k$ , i.e.,  $\lambda_k^Y(i)$  is the approximate eigenvalue of  $Y$  associated to  $C_i^k$ .

Since swap walks are *balanced operators*, we will deduce the following.

**Lemma V.20** (Swap Quadratic Form (restatement of [Lemma V.6](#))). *Let  $f = \sum_{i=0}^k f_i$  with  $f_i \in C_i^k$ . Suppose  $X(\leq d)$  is a  $\gamma$ -HDX with  $d \geq 2k$ . If  $\gamma \leq \varepsilon \left(64k^{k+4}2^{3k+1}\right)^{-1}$ , then*

$$\langle S_{k,k}f, f \rangle = \sum_{i=0}^k \lambda_k(i) \cdot \langle f_i, f_i \rangle \pm \varepsilon,$$

where  $\lambda_k(i)$  depends on only on  $k$  and  $i$ , i.e.,  $\lambda_k(i)$  is an approximate eigenvalue of  $S_{k,k}$  associated to space  $C_i^k$ .

The next result, [Lemma V.21](#), (implicit in [\[DDFH18\]](#)) will be key in establishing that the spectral structure of  $\gamma$ -EPosets is fully determined by

the parameters in  $\vec{\delta}$  provided  $\gamma$  is small enough. Note that the Eposet [Definition V.11](#) provides a ‘‘calculus’’ for rearranging a single pair of up and down DU. The next result treats the more general case of  $DU \dots U$ .

**Lemma V.21** (Structure Lemma). *Suppose  $|\mathcal{D}| = 1$ . Let  $Y_c \in \mathcal{D}$  be the unique down operator in  $Y_\ell \dots Y_1$ . If  $\|A\|_{\text{op}} \leq 1$ , then*

$$\langle AY_\ell \dots Y_1 h_i, g \rangle = \begin{cases} 0 & \text{if } \ell = 1 \text{ or } \\ Q_{c,i}(\vec{\delta}) \cdot \langle AU^{\ell-2} h_i, g \rangle \pm (c-1) \cdot \gamma \|h_i\| \|g\| & c = 1 \\ & \text{otherwise,} \end{cases}$$

where  $Q_{c,i}$  is a polynomial in the variables  $\vec{\delta}$  depending on  $c, i$  such that  $Q_{c,i}(\vec{\delta}) \leq 1$ .

With [Lemma V.21](#) we are close to recover the approximate spectrum of  $D_{k+1}U_k$  from [\[DDFH18\]](#). However, in our application we will need to analyze more general operators, namely, *pure balanced* and *balanced operators*.

**Lemma V.22** (Refinement of [\[DDFH18\]](#)). *If  $\|A\|_{\text{op}} \leq 1$ , then*

$$\langle AD_{k+1}U_k f_i, g \rangle = \lambda_i \cdot \langle A f_i, g \rangle \pm (k-i+1) \cdot \gamma \|h_i\| \|g\|,$$

where  $\lambda_i = Q_{k-i+2,i}(\vec{\delta})$ .

Then powers of the operator  $D_{k+1}U_k$  behave as expected.

**Lemma V.23** (Exponentiation Lemma).

$$\langle (D_{k+1}U_k)^s f_i, f_i \rangle = \lambda_i^s \cdot \|f_i\|^2 \pm s \cdot (k-i+1) \cdot \gamma \|h_i\| \|f_i\|,$$

where  $\lambda_i$  is given in [Lemma V.22](#).

In case  $|\mathcal{D}| > |\mathcal{U}|$ ,  $Y: C^i \rightarrow C^j$  is an operator whose kernel approximately contains  $\ker(D_i)$  as the following lemma makes precise.

**Lemma V.24** (Refinement of [\[DDFH18\]](#)). *If  $|\mathcal{D}| > |\mathcal{U}|$  and  $h_i \in \ker(D_i)$ , then*

$$\langle AY_\ell \dots Y_1 h_i, g \rangle = \pm \ell^2 \cdot \gamma \|h_i\| \|g\|,$$

provided  $\|A\|_{\text{op}} \leq 1$ .

We turn to an important particular case of  $|\mathcal{D}| = |\mathcal{U}|$ , namely, the canonical walks. We show that  $N_{k,k}^{(u)}$  is approximately a polynomial in the operator  $D_{k+1}U_k$ . As a warm up consider the case  $N_{k,k}^{(2)} = D_{k+1}D_{k+2}U_{k+1}U_k$ . Using the [Eq. \(2\)](#), we get

$$\begin{aligned} N_{k,k}^{(2)} &\approx (1 - \delta_{k+1}) \cdot D_{k+1}U_k D_{k+1}U_k + \delta_{k+1} \cdot D_{k+1}U_k \\ &= (1 - \delta_{k+1}) \cdot (D_{k+1}U_k)^2 + \delta_{k+1} \cdot D_{k+1}U_k. \end{aligned}$$

Inspecting this polynomial more carefully we see that that its coefficients form a probability distribution. This

property holds in general as the following [Lemma V.25](#) shows. This gives an alternative (approximate) random walk interpretation of  $N_{k,k}^{(u)}$  as the walk that first selects the power  $s$  according to the distribution encoded in the polynomial and then moves according to  $(D_{k+1}U_k)^s$ .

**Lemma V.25** (Canonical Polynomials). *For  $k, u \geq 0$  there exists a degree  $u$  univariate polynomial  $F_{u,k,\vec{\delta}}^N$  depending only on  $u, k, \vec{\delta}$  such that*

$$\left\| N_{k,k}^{(u)} - F_{u,k,\vec{\delta}}^N(D_{k+1}U_k) \right\|_{\text{op}} \leq (u-1)^2 \cdot \gamma.$$

Moreover, the coefficients of this polynomial form a probability distribution, i.e.,  $F_{u,k,\vec{\delta}}^N(x) = \sum_{i=0}^u c_i x^i$  where  $\sum_{i=0}^u c_i = 1$  and  $c_i \geq 0$  for  $i = 0, \dots, u$ .

**Remark V.26.** *Having a polynomial expression  $F_{u,k,\vec{\delta}}^N(D_{k+1}U_k) \approx N_{k,k}^{(u)}$  and knowing that  $S_{k,k}$  can be written as linear combination of canonical walks, we could deduce that  $S_{k,k}$  is also approximately a polynomial in  $D_{k+1}U_k$ . Using an error refined version of the [Lemma V.23](#) (showing that exponentiation of  $D_{k+1}U_k$  behaves naturally), we could deduce the approximate spectrum of  $S_{k,k}$ . We avoid this approach since it analysis introduces unnecessary error terms and we can understand quadratic forms of pure balanced operators directly.*

**Remark V.27.** *The canonical polynomial  $F_{u,k,\vec{\delta}}^N(D_{k+1}U_k)$  is used later in the error analysis that relates the norms  $\|h_i\|$  and  $\|f_i\|$  ([Lemma V.30](#)).*

Now we consider  $Y$  where  $|\mathcal{D}| = |\mathcal{U}|$  in full generality. We show how the quadratic form of  $Y$  behaves in terms of the approximate eigenspace decomposition  $C^k = \sum_{i=0}^k C_i^k$ .

**Lemma V.28** (Pure Balanced Walks). *Suppose  $Y = Y_\ell \dots Y_1$  is a product of an equal number of up and down operators, i.e.,  $|\mathcal{D}| = |\mathcal{U}|$ . Then for  $f_i \in C_i^k$*

$$\langle Y f_i, f_i \rangle = \lambda_{k,i}^Y \cdot \langle f_i, f_i \rangle \pm \gamma \cdot (\ell^2 + \ell(k-i-1)) \|h_i\| \|f_i\|,$$

where  $\lambda_{k,i}^Y$  is an approximate eigenvalue depending only on  $Y, k$  and  $i$ .

To understand all errors in the analysis in [Lemma V.28](#) we need to use the approximate orthogonality of  $f_i$  and  $f_j$  for  $i \neq j$  from [\[DDFH18\]](#) in more detail. We start with the following bound in terms of  $h_i, h_j$ .

**Lemma V.29** (Refinement of [\[DDFH18\]](#)). *For  $i \neq j$ ,*

$$\langle f_i, f_j \rangle = \pm (2k - i - j)^2 \cdot \gamma \|h_i\| \|h_j\|.$$

To give a bound for [Lemma V.29](#) only in terms of the eigenfunction norms  $\|f_i\|$  and not in terms of  $\|h_i\|$ ,

we need to understand how the norms of  $h_i$  and  $f_i$  are related.

**Lemma V.30** (Refinement of [\[DDFH18\]](#)). *Let  $\eta_{k,i} = (k-i)^2 + 1$  and let  $\beta_i = \sqrt{|F_{k-i,i,\vec{\delta}}^N(\delta_i) \pm \gamma \cdot \eta_{k,i}|}$  where  $F_{k-i,i,\vec{\delta}}^N$  is a canonical polynomial of degree  $k-i$  from [Lemma V.25](#). Then*

$$\langle f_i, f_i \rangle = \beta_i^2 \cdot \langle h_i, h_i \rangle.$$

Let  $\theta_{k,i} = (i+1)^{k-i}$ . Furthermore, if  $\gamma \leq 1/(2 \cdot \eta_{k,i} \cdot \theta_{k,i})$ , then  $\beta_i \geq \frac{1}{2\theta_{k,i}}$ .

Now, we can state the approximate orthogonality [Lemma V.31](#) in terms of the eigenfunction norms.

**Lemma V.31** (Approximate Orthogonality (refinement of [\[DDFH18\]](#))). *Let  $\eta_{k,s}, \theta_{k,s}, \beta_s$  for  $s \in \{i, j\}$  be given as in [Lemma V.30](#). If  $i \neq j$  and  $\beta_i, \beta_j > 0$ , then*

$$\langle f_i, f_j \rangle = \pm \frac{\gamma \cdot (2k - i - j)^2}{\beta_i \beta_j} \|f_i\| \|f_j\|.$$

Furthermore, if  $\gamma \leq \min(1/(2 \cdot \eta_{k,i} \cdot \theta_{k,i}), 1/(2 \cdot \eta_{k,j} \cdot \theta_{k,j}))$ , then  $\beta_i, \beta_j > 0$  and

$$\langle f_i, f_j \rangle = \pm \gamma \cdot \theta_{k,i} \cdot \theta_{k,j} \cdot (2k - i - j)^2 \|f_i\| \|f_j\|.$$

We generalize the quadratic form of [Lemma V.28](#) to linear combinations of general pure balanced operators  $Y$ , namely, to balanced operators.

**Lemma V.32** (General Quadratic Form (restatement of [Lemma V.8](#))). *Let  $\varepsilon \in (0, 1)$  and let  $\mathcal{Y} \subseteq \{Y \mid Y: C^k \rightarrow C^k\}$  be a collection of formal operators that are product of an equal number of up and down walks (i.e., pure balanced operators) not exceeding  $\ell$  walks. Let  $B = \sum_{Y \in \mathcal{Y}} \alpha^Y Y$  where  $\alpha^Y \in \mathbb{R}$  and let  $f = \sum_{i=0}^k f_i$  with  $f_i \in C_i^k$ . If  $\gamma \leq \varepsilon \left(16k^{k+2} \ell^2 \sum_{Y \in \mathcal{Y}} |\alpha^Y|\right)^{-1}$ , then*

$$\langle Bf, f \rangle = \sum_{i=0}^k \left( \sum_{Y \in \mathcal{Y}} \alpha^Y \lambda_k^Y(i) \right) \cdot \langle f_i, f_i \rangle \pm \varepsilon,$$

where  $\lambda_k^Y(i)$  depends only on the operators appearing in the formal expression of  $Y, i$  and  $k$ , i.e.,  $\lambda_k^Y(i)$  is the approximate eigenvalue of  $Y$  associated to  $C_i^k$ .

We instantiate [Lemma V.31](#) for swap walks with their specific parameters. First, we introduce some notation. Using [Corollary IV.13](#), we have

$$S_{k,k} = \sum_{j=0}^k (-1)^{k-j} \cdot \binom{k+j}{k} \cdot \binom{k}{j} \cdot N_{k,k}^{(j)} = \sum_{j=0}^k \alpha_j \cdot N_{k,k}^{(j)}$$

where  $\alpha_j = (-1)^{k-j} \cdot \binom{k+j}{k} \cdot \binom{k}{j}$ .

Finally, we have all the pieces to prove [Lemma V.6](#) restated below.

**Lemma V.33** (Swap Quadratic Form (restatement of [Lemma V.6](#))). Let  $f = \sum_{i=0}^k f_i$  with  $f_i \in C_i^k$ . Suppose  $X(\leq d)$  is a  $\gamma$ -HDX with  $d \geq 2k$ . If  $\gamma \leq \varepsilon \left(64k^{k+4}2^{3k+1}\right)^{-1}$ , then

$$\langle S_{k,k}f, f \rangle = \sum_{i=0}^k \lambda_k(i) \cdot \langle f_i, f_i \rangle \pm \varepsilon,$$

where  $\lambda_k(i)$  depends on only on  $k$  and  $i$ , i.e.,  $\lambda_k(i)$  is an approximate eigenvalue of  $S_{k,k}$  associated to space  $C_i^k$ .

#### D. Rectangular Swap Walks $S_{k,l}$

We turn to the spectral analysis of rectangular swap walks, i.e.,  $S_{k,l}$  where  $k \neq l$ . Recall that to bound  $\sigma_2(S_{k,k})$  in [Section V-A](#) we proved that the spectrum of  $S_{k,k}$  for a  $\gamma$ -HDX is close to the spectrum of  $S_{k,k}^\Delta$  using the analysis of quadratic forms over *balanced* operators from [Section V-C](#). Then we appealed to the fact that  $S_{k,k}^\Delta$  is expanding since it is the walk operator of the well known Kneser graph. In this rectangular case, we do not have a classical result establishing that  $S_{k,l}^\Delta$  is expanding, but we were able to establish it [Lemma V.34](#).

**Lemma V.34.** Let  $d \geq k+l$  and  $\Delta_d(n)$  be the complete complex. The second largest singular value  $\sigma_2(S_{k,l}^\Delta)$  of the swap operator  $S_{k,l}^\Delta$  on  $\Delta_d(n)$  is

$$\sigma_2(S_{k,l}^\Delta) \leq \max\left(\frac{k}{n-k}, \frac{l}{n-l}\right),$$

provided  $n \geq M_{k,l}$  where  $M_{k,l} \in \mathbb{N}$  only depends on  $k$  and  $l$ .

Towards proving [Lemma V.34](#) we first introduce a generalization of Kneser graphs which we denote *bipartite Kneser graphs* defined as follows.

**Definition V.35** (General Bipartite Kneser Graph). Let  $X(\leq d)$  where  $d \geq k+l$ . We denote by  $K^X(n, k, l)$  the bipartite graph on (vertex) partition  $(X(k), X(l))$  where  $s \in X(k)$  is adjacent to  $t \in X(l)$  if and only if  $s \cap t$  is empty. We also refer to graphs of the form  $K^X(n, k, l)$  as *bipartite Kneser graphs*.

It will be convenient to distinguish *bipartite Kneser graphs* coming from general  $\gamma$ -HDX and the complete complex  $\Delta_d(n)$ .

**Definition V.36** (Complete Bipartite Kneser Graph). Let  $X(\leq d)$  where  $d \geq k+l$ . If  $X$  is the complete complex, i.e.,  $X = \Delta_d(n)$ , then we denote  $K^X(n, k, l)$  as simply as  $K(n, k, l)$  and we refer to it as *complete bipartite Kneser*.

We obtain the spectra of *bipartite Kneser graphs* generalizing <sup>7</sup> the classical result of [Fact V.4](#). More precisely, we prove [Lemma V.37](#).

**Lemma V.37** (Bipartite Kneser Spectrum). The non-zero eigenvalues of the (normalized) walk operator of  $K(n, k, l)$  are  $\pm \lambda_i$  where

$$\lambda_i = \frac{\binom{n-k-i}{l-i} \binom{n-l-i}{k-i}}{\binom{n-k}{l} \binom{n-l}{k}},$$

for  $i = 0, \dots, \min(k, l)$ .

Now the proof follows a similar strategy to the  $S_{k,k}$ , namely, we analyze quadratic forms over  $S_{k,k}$  using the results from [Section V-C](#)

Let  $X(\leq d)$  where  $d \geq k+l$ . Let  $A_{k,l}$  be the (normalized) walk operator of  $K^X(n, k, l)$ , i.e.,

$$A_{k,l} = \begin{pmatrix} 0 & S_{k,l}^{(l)} \\ \left(S_{k,l}^{(l)}\right)^\dagger & 0 \end{pmatrix}.$$

To determine the spectrum of  $A_{k,l}$  it is enough to consider the spectrum of  $B = S_{k,l}^{(l)} \left(S_{k,l}^{(l)}\right)^\dagger$ . Using [Corollary IV.13](#), we have

$$\begin{aligned} B &= \left( \sum_{j=0}^l (-1)^{l-j} \binom{k+j}{l} \cdot \binom{l}{j} \cdot \mathbf{N}_{k,l}^{(j)} \right) \\ &\quad \left( \sum_{j'=0}^l (-1)^{l-j'} \binom{k+j'}{l} \cdot \binom{l}{j'} \cdot \left(\mathbf{N}_{k,l}^{(j')}\right)^\dagger \right) \\ &= \sum_{j,j'=0}^l \alpha_{k,l,j,j'} \mathbf{N}_{k,l}^{(j)} \mathbf{N}_{l,k}^{(j'+k-l)}, \end{aligned}$$

for some coefficients  $\alpha_{k,l,j,j'}$  depending only on  $k, l, i, j$  and  $j'$ . Since we have not yet used any specific property of HDXs, these coefficients are the same for the complete complex and general HDXs.

**Lemma V.38.** Let  $X(\leq d)$  be a  $\gamma$ -HDX with  $d \geq k+l$ . Let  $f = \sum_{i=0}^k f_i$  with  $f_i \in C_i^k$ . For  $\varepsilon \in (0, 1)$ , if  $\gamma \leq \varepsilon \left(64k^{k+2} \ell^2 2^{2k+4l+2}\right)^{-1}$ , then

$$\langle Bf, f \rangle = \sum_{i=0}^k \left( \sum_{j,j'=0}^l \alpha_{k,l,j,j'} \lambda_{k,l,j,j'}(i) \right) \cdot \langle f_i, f_i \rangle + \varepsilon,$$

where  $\lambda_{k,l,j,j'}(i)$  is the approximate eigenvalues of  $\mathbf{N}_{k,l}^{(j)} \mathbf{N}_{l,k}^{(j'+k-l)}$  corresponding to space  $C_i^k$ . Furthermore,  $\lambda_{k,l,j,j'}(i)$  depends only on  $k, l, i, j$  and  $j'$ .

<sup>7</sup>Note that the singular values of  $K(n, k)$  can be deduced from the bipartite case.

Let  $B$  and  $B^\Delta$  stand for the  $B$  operator for general  $\gamma$ -HDX and the complete complex, respectively.

**Lemma V.39.** *Suppose  $X(\leq d)$  is a  $\gamma$ -HDX with  $d \geq k + l$ . For  $\varepsilon \in (0, 1)$ , if  $\gamma \leq \varepsilon^2 \left(64k^{k+2}\ell^2 2^{2k+4l+2}\right)^{-1}$ , then the second largest singular value  $\sigma_2(B)$  of  $B$  is*

$$\sigma_2(B) \leq \varepsilon^2.$$

Furthermore, the second largest non-trivial eigenvalue  $\lambda(A_{k,l})$  of the walk matrix of  $K(n, k, l)$  is

$$\lambda(A_{k,l}) \leq \varepsilon.$$

Now the proof of [Theorem V.2](#) follows. For convenience, we restate it.

**Theorem V.40** (Rectangular Swap Walk Spectral Bound (restatement of [Theorem V.2](#))). *Suppose  $X(\leq d)$  is a  $\gamma$ -HDX with  $d \geq k + l$  and  $k \leq l$ . For  $\sigma \in (0, 1)$ , if  $\gamma \leq \sigma^2 \cdot \left(64k^{k+2}\ell^2 2^{2k+4l+2}\right)^{-1}$ , then the largest non-trivial singular value  $\sigma_2(S_{k,l})$  of the swap operator  $S_{k,l}$  is*

$$\sigma_2(S_{k,l}) \leq \sigma.$$

#### E. Bipartite Kneser Graphs - Complete Complex

Now we determine the spectrum of the complete bipartite Kneser graph  $K(n, k, l)$ . More precisely, we prove the following.

**Lemma V.41** (Bipartite Kneser Spectrum (restatement of [Lemma V.37](#))). *The non-zero eigenvalues of the normalized walk operator of  $K(n, k, l)$  are  $\pm\lambda_i$  where*

$$\lambda_i = \frac{\binom{n-k-i}{l-i} \binom{n-l-i}{k-i}}{\binom{n-k}{l} \binom{n-l}{k}},$$

for  $i = 0, \dots, \min(k, l)$ .

Henceforth, set  $X = \Delta_d(n)$ . To prove [Lemma V.37](#) we work with the natural rectangular matrix associated with  $K(n, k, l)$ , namely, the matrix  $W \in \mathbb{R}^{X(k) \times X(l)}$  such that

$$W(\mathfrak{s}, \mathfrak{t}) = \mathbb{1}_{[\mathfrak{s} \cap \mathfrak{t} = \emptyset]}$$

for every  $\mathfrak{s} \in X(k)$  and  $\mathfrak{t} \in X(l)$ .

Observe that the entries of  $WW^\top$  and  $W^\top W$  only depend on the size of the intersection of the sets indexing the row and columns. Hence, these matrices belong to the Johnson scheme [\[GM15\]](#)  $J(n, k)$  and  $J(n, l)$ , respectively. Moreover, the left and right singular vectors of  $W$  are eigenvectors of these schemes.

We adopt the eigenvectors used in Filmus' work [\[Fil16\]](#), i.e., natural basis vectors coming from some irreducible representation of  $S_n$  (see [\[Sag13\]](#)). First we introduce some notation. Let  $\mu = (n - i, i)$  be a

partition of  $n$  and let  $\tau_\mu$  be a standard tableau of shape  $\mu$ . Suppose the first row  $\tau_\mu$  contains  $a_1 < \dots < a_{n-i}$  whereas the second contains  $b_1 < \dots < b_i$ . To  $\tau_\mu$  we associate the function  $\varphi_{\tau_\mu} \in \mathbb{R}^{\binom{[n]}{k}}$  as follows

$$\varphi_{\tau_\mu} = (\mathbb{1}_{a_1} - \mathbb{1}_{b_1}) \dots (\mathbb{1}_{a_i} - \mathbb{1}_{b_i}),$$

where  $\mathbb{1}_a \in \mathbb{R}^{\binom{[n]}{k}}$  is the containment indicator of element  $a$ , i.e.,  $\mathbb{1}_a(\mathfrak{s}) = 1$  if and only if  $a \in \mathfrak{s}$ . Filmus proved that

$$\left\{ \varphi_{\tau_\mu} \mid 0 \leq i \leq k, \mu \vdash (n - i, i), \tau_\mu \text{ standard} \right\}$$

is an eigenbasis of  $\mathcal{J}(n, k)$ . We abuse the notation by considering  $\varphi_{\tau_\mu}$  as both a function in  $\mathbb{R}^{\binom{[n]}{k}}$  and  $\mathbb{R}^{\binom{[n]}{l}}$  as long as these functions are well defined.

**Claim V.42.** *If  $\mu = (n - i, i)$  and  $k, l \geq i$ , then*

$$W\varphi_{\tau_\mu} = (-1)^i \cdot \binom{n-k-i}{l-i} \cdot \varphi_{\tau_\mu}.$$

Since we are working with singular vectors, we need to be careful with their normalization when deriving the singular values. We stress that the norm of  $\varphi_{\tau_\mu}$  depends on the space where  $\varphi_{\tau_\mu}$  lies.

**Claim V.43.** *If  $\mu = (n - i, i)$  and  $\varphi_{\tau_\mu} \in \mathbb{R}^{\binom{[n]}{k}}$ , then*

$$\|\varphi_{\tau_\mu}\|_2 = \sqrt{2^i \binom{n-2i}{k-i}}.$$

Now the singular values of  $W$  follow.

**Corollary V.44** (Singular Values). *The singular values of  $W$  are*

$$\sigma_i = \binom{n-k-i}{l-i} \cdot \frac{\|\varphi_{\tau_\mu}^k\|_2}{\|\varphi_{\tau_\mu}^l\|_2},$$

for  $i = 0, \dots, \min(k, l)$ .

Note that for  $k = l$  we recover the well know result of [Fact V.4](#).

Finally we compute the eigenvalues of the bipartite graph  $K(n, k, l)$ . Let  $A_{n,k,l}$  be its normalized adjacency matrix, i.e.,

$$A_{n,k,l} = \begin{pmatrix} 0 & \frac{1}{\binom{n-k}{l}} W \\ \frac{1}{\binom{n-l}{k}} W^\top & 0 \end{pmatrix}.$$

**Lemma V.45** (Bipartite Kneser Spectrum (restatement of [Lemma V.37](#))). *The non-zero eigenvalues of the normalized walk operator of  $K(n, k, l)$  are  $\pm\lambda_i$  where*

$$\lambda_i = \frac{\binom{n-k-i}{l-i} \binom{n-l-i}{k-i}}{\binom{n-k}{l} \binom{n-l}{k}},$$

for  $i = 0, \dots, \min(k, l)$ .

## VI. APPROXIMATING MAX- $k$ -CSP

In the following, we will show that  $k$ -CSP instances  $\mathcal{J}$  whose constraint complex  $X_{\mathcal{J}}(\leq k)$  is a suitable expander admit an efficient approximation algorithm. We will assume throughout that  $X_{\mathcal{J}}(1) = [n]$ , and drop the subscript  $\mathcal{J}$ .

This was shown for 2-CSPs in [BRS11]. In extending this result to  $k$ -CSPs we will rely on a central Lemma of their paper. Before, we explain our algorithm we give a basic outline of our idea:

We will work with the SDP relaxation for the  $k$ -CSP problem given by  $L$ -levels of SoS hierarchy, as defined in Section II-D (for  $L$  to be specified later). This will give us an  $L$ -local PSD ensemble  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$ , which attains some value  $\text{SDP}(\mathcal{J}) \geq \text{OPT}(\mathcal{J})$ . Since  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$ , is a local PSD ensemble, and not necessarily a probability distribution, we cannot sample from it directly. Nevertheless, since  $\{\mathbf{Y}_j\}$  will be actual probability distributions for all  $j \in [n]$ , one can independently sample  $\eta_j \sim \{\mathbf{Y}_j\}$  and use  $\eta = (\eta_1, \dots, \eta_n)$  as the assignment for the  $k$ -CSP instance  $\mathcal{J}$ .

Unfortunately, while we know that the local distributions  $\{\mathbf{Y}_a\}_{a \in X(k)}$  induced by  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$  will satisfy the constraints of  $\mathcal{J}$  with good probability, i.e.,

$$\begin{aligned} \mathbb{E}_{a \sim \Pi_k} \mathbb{E}_{\{\mathbf{Y}_a\}} \left[ \mathbf{1}_{\left[ \underbrace{\mathbf{Y}_a \text{ satisfies the constraint on } a}_{\iff \mathbf{Y}_a \in C_a} \right]} \right] \\ = \text{SDP}(\mathcal{J}) \geq \text{OPT}(\mathcal{J}), \end{aligned}$$

this might not be the case for the assignment  $\eta$  sampled as before. It might be that the random variables  $\mathbf{Y}_{a_1}, \dots, \mathbf{Y}_{a_k}$  are highly correlated for  $a \in X(k)$ , i.e.,  $\mathbb{E}_{a \sim \Pi_k} \|\{\mathbf{Y}_a\} - \{\mathbf{Y}_{a_1}\} \cdots \{\mathbf{Y}_{a_k}\}\|_1$  is large. One strategy employed by [BRS11] to ensure that the quantity above is small, is making the local PSD ensemble  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$  be consistent with a randomly sampled partial assignment for a small subset of variables (q.v. Section II-D). We will show that this strategy is successful if  $X(\leq k)$  is a  $\gamma$ -HDX (for  $\gamma$  sufficiently small). Our final algorithm is Algorithm VI.1.

In our setting, we will apply Algorithm VI.1 with the distribution  $\Pi_k$  and the  $L$ -local PSD ensemble  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$ . Notice that in expectation, the marginals of  $\mathbf{Y}'$  on faces  $a \in X(k)$  – which are actual distributions – will agree with the marginals of  $\mathbf{Y}$ , i.e.  $\mathbb{E}_{S, \eta_S} \mathbb{E} \mathbf{Y}'_a = \mathbb{E} \mathbf{Y}_a$ . In particular, the approximation quality of Algorithm VI.1 will depend on the average correlation of  $\mathbf{Y}'_{a_1}, \dots, \mathbf{Y}'_{a_k}$  on the constraints  $a \in X(k)$ , where  $\mathbf{Y}'$  is the local PSD ensemble obtained at the end of the first phase of Algorithm VI.1.

In the case where  $k = 2$ , the following is known

**Algorithm VI.1** (Propagation Rounding Algorithm).

**Input** An  $L$ -local PSD ensemble  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$ , and a distribution  $\Pi$  on  $X(\leq \ell)$ .

**Output** A random assignment  $\eta : [n] \rightarrow [q]$ .

- 1) Choose  $m \in \{1, \dots, L/\ell\}$  uniformly at random.
- 2) Independently sample  $m$   $\ell$ -faces,  $\mathfrak{s}_j \sim \Pi$  for  $j = 1, \dots, m$ .
- 3) Write  $S = \bigcup_{j=1}^m \mathfrak{s}_j$ , for the set of the seed vertices.
- 4) Sample assignment  $\eta_S : S \rightarrow [q]$  according to the local distribution,  $\{\mathbf{Y}_S\}$ .
- 5) Set  $\mathbf{Y}' = \{\mathbf{Y}_1, \dots, \mathbf{Y}_n | \mathbf{Y}_S = \eta_S\}$ , i.e. the local ensemble  $\mathbf{Y}$  conditioned on agreeing with  $\eta_S$ .
- 6) For all  $j \in [n]$ , sample independently  $\eta_j \sim \{\mathbf{Y}'_j\}$ .
- 7) Output  $\eta = (\eta_1, \dots, \eta_n)$ .

**Theorem VI.2** (Theorem 5.6 from [BRS11]). Suppose a weighted undirected graph  $G = ([n], E, \Pi_2)$  and an  $L$ -local PSD ensemble  $\mathbf{Y} = \{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$  are given. There exists absolute constants  $c \geq 0$  and  $C \geq 0$  satisfying the following: If  $L \geq c \cdot \frac{q}{\varepsilon^4}$ ,  $\text{Supp}(\mathbf{Y}_i) \leq q$  for all  $i \in V$ , and  $\lambda_2(G) \leq C \cdot \varepsilon^2/q^2$  then we have

$$\mathbb{E}_{\{i,j\} \sim \Pi_2} \left\| \{\mathbf{Y}'_i, \mathbf{Y}'_j\} - \{\mathbf{Y}'_i\} \{\mathbf{Y}'_j\} \right\|_1 \leq \varepsilon,$$

where  $\mathbf{Y}'$  is as defined in Algorithm VI.1 on the input of  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$  and  $\Pi_1$ .

To approximate  $k$ -CSPs well, we will show the following generalization of Theorem VI.2 for  $k$ -CSP instances  $\mathcal{J}$ , whose constraint complex  $X(\leq k)$  is  $\gamma$ -HDX, for  $\gamma$  sufficiently small.

**Theorem VI.3.** Suppose a simplicial complex  $X(\leq k)$  with  $X(1) = [n]$  and an  $L$ -local PSD ensemble  $\mathbf{Y} = \{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$  are given.

There exists some universal constants  $c' \geq 0$  and  $C' \geq 0$  satisfying the following: If  $L \geq c' \cdot (q^k \cdot k^5 / \varepsilon^4)$ ,  $\text{Supp}(\mathbf{Y}_j) \leq q$  for all  $j \in [n]$ , and  $X$  is a  $\gamma$ -HDX for  $\gamma \leq C' \cdot \varepsilon^4 / (k^{8+k} \cdot 2^{6k} \cdot q^{2k})$ . Then, we have

$$\mathbb{E}_{a \sim \Pi_k} \left\| \{\mathbf{Y}'_a\} - \{\mathbf{Y}'_{a_1}\} \cdots \{\mathbf{Y}'_{a_k}\} \right\|_1 \leq \varepsilon, \quad (3)$$

where  $\mathbf{Y}'$  is as defined in Algorithm VI.1 on the input of  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$  and  $\Pi_k$ .

Indeed, using Theorem VI.3, it is straightforward to prove the following,

**Corollary VI.4.** Suppose  $\mathcal{J}$  is a  $q$ -ary  $k$ -CSP instance whose constraint complex  $X(\leq k)$  is a  $\gamma$ -HDX.

There exists absolute constants  $C' \geq 0$  and  $c' \geq 0$ , satisfying the following: If  $\gamma \leq C' \cdot \varepsilon^4 / (k^{8+k} \cdot 2^{6k} \cdot q^{2k})$ , there is an algorithm that runs in time  $n^{O(k^5 \cdot q^{2k} \cdot \varepsilon^{-4})}$  based

on  $(\frac{c \cdot k^5 \cdot q^k}{\varepsilon^4})$ -levels of SoS-hierarchy and [Algorithm VI.1](#) that outputs a random assignment  $\eta : [n] \rightarrow [q]$  that in expectation ensures  $\text{SAT}_{\mathcal{J}}(\eta) = \text{OPT}(\mathcal{J}) - \varepsilon$ .

Our proof of [Theorem VI.3](#)<sup>8</sup> will hinge on the fact that we can upper-bound the expected correlation of a face of large cardinality  $\ell$ , in terms of expected correlation over faces of smaller cardinality and expected correlations along the edges of a swap graph. The swap graph here is defined as a weighted graph  $G_{\ell_1, \ell_2} = (X(\ell_1) \sqcup X(\ell_2), E(\ell_1, \ell_2), w_{\ell_1, \ell_2})$ , where

$$E(\ell_1, \ell_2) = \left\{ \{\mathbf{a}, \mathbf{b}\} : \begin{array}{l} \mathbf{a} \in X(\ell_1), \mathbf{b} \in X(\ell_2), \\ \text{and } \mathbf{a} \sqcup \mathbf{b} \in X(\ell_1 + \ell_2) \end{array} \right\}.$$

We will assume  $\ell_1 \geq \ell_2$ , and if  $\ell_1 = \ell_2$  we are going to identify the two copies of every vertex. We will endow  $E(\ell_1, \ell_2)$  with the weight function,

$$w_{\ell_1, \ell_2}(\mathbf{a}, \mathbf{b}) = \frac{\Pi_{\ell_1 + \ell_2}(\mathbf{a} \sqcup \mathbf{b})}{\binom{\ell_1 + \ell_2}{\ell_1}},$$

which can easily be verified to be a probability distribution on  $E(\ell_1, \ell_2)$ . Notice that in the case where  $\ell_1 \neq \ell_2$  the random walk matrix of  $G_{\ell_1, \ell_2}$  is given by

$$A_{\ell_1, \ell_2} = \begin{pmatrix} 0 & S_{\ell_1, \ell_2} \\ S_{\ell_1, \ell_2}^\dagger & 0 \end{pmatrix},$$

and if  $\ell_1 = \ell_2$  we have  $A_{\ell_1, \ell_1} = S_{\ell_1, \ell_1}$ . The stationary distribution of  $A_{\ell_1, \ell_2}$  is  $\Pi_{\ell_1, \ell_2}$  defined by,

$$\Pi_{\ell_1, \ell_2}(\mathbf{b}) = \frac{\mathbf{1}[\mathbf{b} \in X(\ell_1)]}{2} \cdot \Pi_{\ell_1}(\mathbf{b}) + \frac{\mathbf{1}[\mathbf{b} \in X(\ell_2)]}{2} \cdot \Pi_{\ell_2}(\mathbf{b}). \quad (4)$$

When we write an expectation of  $f(\bullet, \bullet)$  over the edges in  $E(\ell_1, \ell_2)$  with respect to  $w_{\ell_1, \ell_2}$ , it is important to note,

$$\begin{aligned} \mathbb{E}_{\{\mathbf{s}, \mathbf{t}\} \sim w_{\ell_1, \ell_2}} [f(\mathbf{s}, \mathbf{t})] &= \sum_{\{\mathbf{s}, \mathbf{t}\} \in E(\ell_1, \ell_2)} \frac{\Pi_{\ell_1 + \ell_2}(\mathbf{s} \sqcup \mathbf{t})}{\binom{\ell_1 + \ell_2}{\ell_1}} \cdot f(\mathbf{s}, \mathbf{t}), \\ &= \frac{1}{\binom{\ell}{\ell_1}} \mathbb{E}_{\mathbf{a} \sim \Pi_{\ell}} \left[ \sum_{\{\mathbf{s}, \mathbf{t}\} \sim \mathbf{a}} f(\mathbf{s}, \mathbf{t}) \right], \end{aligned} \quad (5)$$

where sum within the expectation in the RHS runs over the  $\binom{\ell_1 + \ell_2}{\ell_1}$  possible ways of splitting  $\mathbf{a}$  into  $\mathbf{s} \sqcup \mathbf{t}$  such that  $\mathbf{s} \in X(\ell_1)$  and  $\mathbf{t} \in X(\ell_2)$ . When we are speaking about the spectral expansion of  $G_{\ell_1, \ell_2}$ , we will be speaking with regards to  $\lambda_2(G_{\ell_1, \ell_2})$  and not with regards to  $\sigma_2(G_{\ell_1, \ell_2})$ .

**Remark VI.5.** By simple linear algebra, we have

$$\lambda_2(G_{\ell_1, \ell_2}) := \lambda_2(A_{\ell_1, \ell_2}) \leq \sigma_2(S_{\ell_1, \ell_2}),$$

where we employ the notation  $\lambda_2(\mathbf{M})$  to denote the second largest eigenvalue (signed) of the matrix  $\mathbf{M}$ .

<sup>8</sup>Due to space constraints, we will omit some details. The full proof can be found in the full version of our paper [\[AJT19\]](#).

With this, we will show

**Lemma VI.6** (Glorified Triangle Inequality). *For a simplicial complex  $X(\leq k)$ ,  $\ell_1 \geq \ell_2 \geq 0$ ,  $\ell = \ell_1 + \ell_2$ ,  $\ell \leq k$ , and an  $\ell$ -local ensemble  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$ , one has*

$$\begin{aligned} \mathbb{E}_{\mathbf{a} \in \Pi_{\ell}} \left\| \{\mathbf{Y}_{\mathbf{a}}\} - \prod_{i=1}^{\ell} \{\mathbf{Y}_{a_i}\} \right\|_1 &\leq \mathbb{E}_{\{\mathbf{s}, \mathbf{t}\} \sim w_{\ell_1, \ell_2}} \left\| \{\mathbf{Y}_{\mathbf{s}}, \mathbf{Y}_{\mathbf{t}}\} - \{\mathbf{Y}_{\mathbf{s}}\} \{\mathbf{Y}_{\mathbf{t}}\} \right\|_1 \\ &+ \mathbb{E}_{\mathbf{s} \sim \Pi_{\ell_1}} \left\| \{\mathbf{Y}_{\mathbf{s}}\} - \prod_{i=1}^{\ell_1} \{\mathbf{Y}_{s_i}\} \right\|_1 + \mathbb{E}_{\mathbf{t} \sim \Pi_{\ell_2}} \left\| \{\mathbf{Y}_{\mathbf{t}}\} - \prod_{i=1}^{\ell_2} \{\mathbf{Y}_{t_i}\} \right\|_1 \end{aligned} \quad (6)$$

One useful observation, is that by using [Lemma VI.6](#) repeatedly, we can reduce the problem of bounding  $\mathbb{E}_{\mathbf{a} \in \Pi_{\ell}} \left\| \{\mathbf{Y}_{\mathbf{a}}\} - \prod_{i=1}^{\ell} \{\mathbf{Y}_{a_i}\} \right\|_1$  to a problem of bounding

$$\mathbb{E}_{\{\mathbf{s}, \mathbf{t}\} \sim w_{\ell_1, \ell_2}} \left\| \{\mathbf{Y}_{\mathbf{s}}, \mathbf{Y}_{\mathbf{t}}\} - \{\mathbf{Y}_{\mathbf{s}}\} \{\mathbf{Y}_{\mathbf{t}}\} \right\|_1,$$

for  $\ell_1 + \ell_2 \leq k$ . Though it is not a direct implication, it is heavily suggested by [Fact II.7](#) and [Theorem VI.2](#), that if  $G_{\ell_1, \ell_2}$  is a good spectral expander, after an application of [Algorithm VI.1](#) with our chosen parameters, we should be able to bound these expressions. Using a key lemma used from [\[BRS11\]](#), we will prove that this is indeed the case. The only thing we need to make sure after this point, is that the second eigenvalue  $\lambda_2(G_{\ell_1, \ell_2})$  of the swap graphs  $G_{\ell_1, \ell_2}$  we will be using are small enough for our purposes. Indeed, our choice of  $\gamma$  in [Theorem VI.3](#) and [Corollary VI.4](#) is to make sure that the bound we get on  $\lambda_2(G_{\ell_1, \ell_2})$  from [Theorem V.2](#) (together with [Remark VI.5](#)) is good enough for our purposes.

#### A. Breaking Correlations for Expanding CSPs

Throughout this section, we will use the somewhat non-standard definition of variance introduced in [\[BRS11\]](#),

$$\text{Var}[\mathbf{Y}_{\mathbf{a}}] = \sum_{\eta \in [q]^{\mathbf{a}}} \text{Var}[\mathbf{1}[\mathbf{Y}_{\mathbf{a}} = \eta]].$$

We will use the following central lemma from [\[BRS11\]](#) in our proof of [Theorem VI.3](#):

**Lemma VI.7** (Lemma 5.4 from [\[BRS11\]](#)). *Let  $G = (V, E, \Pi_2)$  be a weighted graph,  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$  a local PSD ensemble, where we have  $\text{Supp}(\mathbf{Y}_i) \leq q$  for every  $i \in V$ , and  $q \geq 0$ . Suppose  $\varepsilon \geq 0$  is a lower bound on the expected statistical difference between independent and correlated sampling along the edges, i.e.,*

$$\varepsilon \leq \mathbb{E}_{\{i, j\} \sim \Pi_2} \left\| \{\mathbf{Y}_{ij}\} - \{\mathbf{Y}_i\} \{\mathbf{Y}_j\} \right\|_1.$$

*There exists absolute constants  $c_0 \geq 0$  and  $c_1 \geq 0$  that satisfy the following: If  $\lambda_2(G) \leq c_0 \cdot \frac{\varepsilon}{q^2}$ . Then, conditioning*

on a random vertex decreases the variances,

$$\mathbb{E}_{i \sim \Pi_1} \mathbb{E}_{j \sim \Pi_2} \mathbb{E}_{\{Y_j\}} \text{Var} [Y_i | Y_j] \leq \mathbb{E}_{i \sim \Pi_1} [\text{Var} [Y_i]] - c_1 \cdot \frac{\varepsilon^2}{q^2}.$$

For our applications, we will be instantiating [Lemma VI.7](#) with  $G_{\ell_1, \ell_2}$  as  $G$ ; and with the local PSD ensemble  $\{\mathbf{Y}_\alpha\}_{\alpha \in X}$  that is obtained from  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$  (q.v. [Fact II.7](#)). For convenience, we will write the concrete instance of the Lemma that we will use,

**Corollary VI.8.** *Let  $\ell_1 \geq \ell_2 \geq 0$  satisfying  $\ell_1 + \ell_2 \leq k$  be given parameters, and let  $G_{\ell_1, \ell_2}$  be the swap graph defined for a  $\gamma$ -HDX  $X(\leq k)$ . Let  $\{\mathbf{Y}_\alpha\}_{\alpha \in X}$  be a local PSD ensemble; satisfying  $\text{Supp}(Y_\alpha) \leq q^k$  for every  $\alpha \in X(\ell_1) \cup X(\ell_2)$  for some  $q \geq 0$ . Suppose  $\varepsilon \geq 0$  satisfies,*

$$\frac{\varepsilon}{4k} \leq \mathbb{E}_{\{s, t\} \in \omega_{\ell_1, \ell_2}} \|\{\mathbf{Y}_{s \sqcup t}\} - \{\mathbf{Y}_s\}\{\mathbf{Y}_t\}\|_1.$$

There exists absolute constants  $c_0 \geq 0$  and  $c_2 \geq 0$  that satisfy the following: If  $\lambda_2(G) \leq c_0 \cdot (\varepsilon / (4k \cdot q^k))^2$ . Then, conditioning on a random face  $\alpha \sim \Pi_{\ell_1, \ell_2}$  decreases the variances, i.e.

$$\begin{aligned} & \mathbb{E}_{\alpha, b \sim \Pi_{\ell_1, \ell_2}^2} \mathbb{E}_{\{Y_\alpha\}} [\text{Var} [Y_b | Y_\alpha]] \\ & \leq \mathbb{E}_{b \sim \Pi_{\ell_1, \ell_2}} [\text{Var} [Y_b]] - c_2 \cdot \frac{\varepsilon^2}{16 \cdot k^2 \cdot q^{2k}}. \end{aligned}$$

Here, it can be verified that the expansion criterion presupposed by [Lemma VI.7](#) is satisfied by [Corollary VI.8](#) by [Theorem V.2](#).

## VII. HIGH-DIMENSIONAL THRESHOLD RANK

In [\[BRS11\]](#), [Theorem VI.2](#) was proven for a more general class of graphs than expander graphs – namely, the class of low threshold rank graphs.

**Definition VII.1** (Threshold Rank of Graphs (from [\[BRS11\]](#))). *Let  $G = (V, E, w)$  be a weighted graph on  $n$  vertices and  $A$  be its normalized random walk matrix. Suppose the eigenvalues of  $A$  are  $1 = \lambda_1 \geq \dots \geq \lambda_n$ . Given a parameter  $\tau \in (0, 1)$ , we denote the threshold rank of  $G$  by  $\text{rank}_{\geq \tau}(A)$  (or  $\text{rank}_{\geq \tau}(G)$ ) and define it as*

$$\text{rank}_{\geq \tau}(A) := |\{i | \lambda_i \geq \tau\}|.$$

There [\[BRS11\]](#), the authors asked for the correct notion of threshold rank for  $k$ -CSPs. In this section, we give a candidate definition of low threshold rank motivated by our techniques.

To break  $k$ -wise correlations it is sufficient to assume that the involved swap graphs in the foregoing discussion are low threshold rank since this is enough to apply a version of [Lemma VI.7](#), already described in the work of [\[BRS11\]](#).

Moreover, we have some flexibility as to which swap graphs to consider as long as they satisfy some splitting conditions. To define a swap graph it is enough to have a distributions on the hyperedges of a (constraint) hypergraph. Hence, the notion of swap graph is independent of high-dimensional expansion. HDXs are just an interesting family of objects for which the swap graphs are good expanders.

To capture the many ways of splitting the statistical distance over hyperedges into the statistical distance over the edges of swap graphs, we first define the following notion. We say that a binary tree  $\mathcal{T}$  is a  $k$ -splitting tree if it has exactly  $k$  leaves. Thus, labeling every vertex with the number of leaves on the subtree rooted at that vertex ensures,

- the root of  $\mathcal{T}$  is labeled with  $k$  and all other vertices are labeled with positive integers,
- the leaves are labeled with 1, and
- each non-leaf vertex satisfy the property that its label is the sum of the labels of its two children.

Note that, we will think of each non-leaf node with left and right children labeled as  $a$  and  $b$  as representing the swap graph from  $X(a)$  to  $X(b)$  for some simplicial complex  $X(\leq k)$ . Let  $\text{Swap}(\mathcal{T}, X)$  be the set of all such swap graphs over  $X$  finding representation in the splitting tree  $\mathcal{T}$ . Indeed the tree  $\mathcal{T}$  used in the proof of [Theorem VI.3](#) is just one special instance of a  $k$ -splitting tree.

Given a threshold parameter  $\tau \leq 1$  and a set of normalized adjacency matrices  $\mathcal{A} = \{A_1, \dots, A_s\}$ , we define the threshold rank of  $\mathcal{A}$  as

$$\text{rank}_{\geq \tau}(\mathcal{A}) := \max_{A \in \mathcal{A}} \text{rank}_{\geq \tau}(A),$$

where  $\text{rank}_{\geq \tau}(A)$  is denotes usual threshold rank of  $A$  as in [Definition VII.1](#).

Now, we are ready to define the notion of a  $k$ -CSP instance being  $(\mathcal{T}, \tau, r)$ -splittable as follows.

**Definition VII.2** ( $(\mathcal{T}, \tau, r)$ -splittability). *A  $k$ -CSP instance  $\mathfrak{J}$  with the constraint complex  $X(\leq k)$  is said to be  $(\mathcal{T}, \tau, r)$ -splittable if  $\mathcal{T}$  is a  $k$ -splitting tree and*

$$\text{rank}_{\geq \tau}(\text{Swap}(\mathcal{T}, X)) \leq r.$$

*If there exists some  $k$ -splitting tree  $\mathcal{T}$  such that  $\mathfrak{J}$  is  $(\mathcal{T}, \tau, r)$ -splittable, the instance  $\mathfrak{J}$  will be called a  $(\tau, r)$ -splittable instance.*

Now, using this definition we can show that whenever  $\text{rank}_{\geq \tau}(\mathfrak{J})$  is bounded for the appropriate choice of  $\tau$ , after conditioning on a random partial assignment as in [Algorithm VI.1](#) we will have small correlation over the faces  $\alpha \in X(k)$ , i.e.,

**Theorem VII.3.** Suppose a simplicial complex  $X(\leq k)$  with  $X(1) = [n]$  and an  $L$ -local PSD ensemble  $\mathbf{Y} = \{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$  are given. There exists some universal constants  $c_4 \geq 0$  and  $C'' \geq 0$  satisfying the following: If  $L \geq C'' \cdot (q^{4k} \cdot k^7 \cdot r / \varepsilon^5)$ ,  $\text{Supp}(\mathbf{Y}_j) \leq q$  for all  $j \in [n]$ , and  $\mathcal{J}$  is  $(c_4 \cdot (\varepsilon / (4k \cdot q^k))^2, r)$ -splittable. Then, we have

$$\mathbb{E}_{\mathbf{a} \in X(k)} \left\| \{\mathbf{Y}'_{\mathbf{a}}\} - \{\mathbf{Y}'_{a_1}\} \cdots \{\mathbf{Y}'_{a_k}\} \right\|_1 \leq \varepsilon, \quad (7)$$

where  $\mathbf{Y}'$  is as defined in Algorithm VI.1 on the input of  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$  and  $\Pi_k$ .

It is important to note that the specific knowledge of the  $k$ -splitting tree  $\mathcal{T}$  that makes  $\mathcal{J}(\mathcal{T}, \tau, r)$ -splittable is only needed for the proof of Theorem VII.3. The conclusion of Theorem VII.3 can be used without the knowledge of the specific  $k$ -splitting tree  $\mathcal{T}$ . Theorem VII.3 can be thought of an extension of Theorem VI.3 to the case where not necessarily every tree is good, and where we can bound the threshold rank instead of the spectral expansion.

This, will readily imply an algorithm

**Corollary VII.4.** Suppose  $\mathcal{J}$  is a  $q$ -ary  $k$ -CSP instance whose constraint complex is  $X(\leq k)$ . There exists an absolute constant  $C'' \geq 0$  and  $c_4 \geq 0$  that satisfies the following: If  $\mathcal{J}$  is  $(c_4 \cdot (\varepsilon / (4k \cdot q^k))^2, r)$ -splittable, then there is an algorithm that runs in time  $n^{O\left(\frac{q^{4k} \cdot k^7 \cdot r}{\varepsilon^5}\right)}$  and that is based on  $\left(\frac{C'' \cdot k^5 \cdot q^k \cdot r}{\varepsilon^4}\right)$ -levels of SoS-hierarchy and Algorithm VI.1 that outputs a random assignment  $\eta : [n] \rightarrow [q]$  that in expectation ensures  $\text{SAT}_{\mathcal{J}}(\eta) = \text{OPT}(\mathcal{J}) - \varepsilon$ .

We will need the more general version of Lemma VI.7, proved in [BRS11].

**Lemma VII.5** (Lemma 5.4 from [BRS11]). <sup>9</sup>Let  $G = (V, E, \Pi_2)$  be a weighted graph,  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$  a local PSD ensemble, where we have  $\text{Supp}(\mathbf{Y}_i) \leq q$  for every  $i \in V$ , and  $q \geq 0$ . If  $\varepsilon \geq 0$  is a lower bound on the expected statistical difference between independent and correlated sampling along the edges, i.e.,

$$\varepsilon \leq \mathbb{E}_{\{i,j\} \sim \Pi_2} \left\| \{\mathbf{Y}_{ij}\} - \{\mathbf{Y}_i\} \{\mathbf{Y}_j\} \right\|_1.$$

There exists absolute constants  $c_3 \geq 0$  and  $c_4 \geq 0$  that satisfy the following: Then, conditioning on a random vertex

<sup>9</sup>The parameters we get differ from those of [BRS11]. The derivation of our parameters can be found in the appendix of the full version of this paper [AJT19].

decreases the variances,

$$\begin{aligned} & \mathbb{E}_{i \sim \Pi_1} \mathbb{E}_{j \sim \Pi_1} \mathbb{E}_{\{\mathbf{Y}_j\}} \text{Var}[\mathbf{Y}_i | \mathbf{Y}_j] \\ & \leq \mathbb{E}_{i \sim \Pi_1} [\text{Var}[\mathbf{Y}_i]] - c_3 \cdot \frac{\varepsilon^4}{q^4 \cdot \text{rank}_{\geq c_4 \varepsilon^2 / q^2}(G)}. \end{aligned}$$

Since we will use this lemma, only with the swap graphs  $G_{\ell_1, \ell_2}$  and  $(L/k)$ -local PSD ensemble  $\{\mathbf{Y}_{\mathbf{a}}\}_{\mathbf{a} \in X}$  obtained from the  $L$ -local PSD ensemble  $\{\mathbf{Y}_1, \dots, \mathbf{Y}_n\}$ , for convenience we will write the corollary we will use more explicitly

**Corollary VII.6.** Let  $\ell_1 \geq \ell_2 \geq 0$  satisfying  $\ell_1 + \ell_2 \leq k$  be given parameters, and let  $G_{\ell_1, \ell_2}$  be the swap graph defined for a  $\gamma$ -HDX  $X(\leq k)$ . Let  $\{\mathbf{Y}_{\mathbf{a}}\}_{\mathbf{a} \in X}$  be a local PSD ensemble; and suppose we have  $\text{Supp}(\mathbf{Y}_{\mathbf{a}}) \leq q^k$  for every  $\mathbf{a} \in X(\ell_1) \cup X(\ell_2)$  for some  $q \geq 0$ . Suppose  $\varepsilon > 0$  satisfies,

$$\frac{\varepsilon}{4k} \leq \mathbb{E}_{\{s,t\} \in E(\ell_1, \ell_2)} \left\| \{\mathbf{Y}_{s \cup t}\} - \{\mathbf{Y}_s\} \{\mathbf{Y}_t\} \right\|_1.$$

There exists absolute constants  $c_3 \geq 0$  and  $c_5 \geq 0$  that satisfy the following:

If  $\text{rank}_{\geq c_4 \cdot (\varepsilon / (4k \cdot q^k))^2}(G_{\ell_1, \ell_2}) \leq r$ , then conditioning on a random face  $\mathbf{a} \sim \Pi_{\ell_1, \ell_2}$  decreases the variances, i.e.

$$\begin{aligned} & \mathbb{E}_{\mathbf{a}, \mathbf{b} \sim \Pi_{\ell_1, \ell_2}^2} \mathbb{E}_{\{\mathbf{Y}_{\mathbf{a}}\}} \text{Var}[\mathbf{Y}_{\mathbf{b}} | \mathbf{Y}_{\mathbf{a}}] \\ & \leq \mathbb{E}_{s \sim \Pi_{\ell_1, \ell_2}} [\text{Var}[\mathbf{Y}_{\mathbf{b}}]] - c_5 \cdot \frac{\varepsilon^4}{256 \cdot k^4 \cdot q^{4k} \cdot r}. \end{aligned}$$

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#### REFERENCES

- [AJQ<sup>+</sup>19] Vedat Levi Alev, Fernando Granha Jeronimo, Dylan Quintana, Shashank Srivastava, and Madhur Tulsiani. List decoding of direct sum codes. Manuscript, 2019. 3
- [AJT19] Vedat Levi Alev, Fernando Granha Jeronimo, and Madhur Tulsiani. Approximating constraint satisfaction problems on high-dimensional expanders. *CoRR*, abs/1907.07833, 2019. URL: <https://arxiv.org/abs/1907.07833>. 2, 11, 18, 20

- [AKK<sup>+</sup>08] Sanjeev Arora, Subhash Khot, Alexandra Kolla, David Steurer, Madhur Tulsiani, and Nisheeth Vishnoi. Unique games on expanding constraint graphs are easy. In *Proceedings of the 40th ACM Symposium on Theory of Computing*, 2008. 1
- [ALGV18] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials II: High-dimensional walks and an FPRAS for counting bases of a matroid. *arXiv preprint arXiv:1811.01816*, 2018. 3
- [BH13] Fernando G. S. L. Brandão and Aram Wettroth Harrow. Product-state approximations to quantum ground states. In *Proceedings of the 45th ACM Symposium on Theory of Computing*, pages 871–880, 2013. 3
- [BRS11] Boaz Barak, Prasad Raghavendra, and David Steurer. Rounding semidefinite programming hierarchies via global correlation. In *FOCS*, pages 472–481, 2011. 1, 2, 3, 5, 6, 7, 8, 17, 18, 19, 20
- [BS02] M. Bernstein and N. J. A. Sloane. Some Canonical Sequences of Integers. *arXiv Mathematics e-prints*, page math/0205301, May 2002. [arXiv:math/0205301](https://arxiv.org/abs/math/0205301). 10
- [BSHR05] Eli Ben-Sasson, Prahladh Harsha, and Sofya Raskhodnikova. Some 3CNF properties are hard to test. *SIAM Journal on Computing*, 35(1):1–21, 2005. 1
- [CTZ18] David Conlon, Jonathan Tidor, and Yufei Zhao. Hypergraph expanders of all uniformities from Cayley graphs. *arXiv e-prints*, page arXiv:1809.06342, September 2018. [arXiv:1809.06342](https://arxiv.org/abs/1809.06342). 2
- [DD19] Yotam Dikstein and Irit Dinur. Agreement testing theorems on layered set systems. In *Proceedings of the 60th IEEE Symposium on Foundations of Computer Science*, 2019. URL: <https://arxiv.org/abs/1909.00638>. 3
- [DDFH18] Yotam Dikstein, Irit Dinur, Yuval Filmus, and Prahladh Harsha. Boolean function analysis on high-dimensional expanders. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2018, August 20–22, 2018 - Princeton, NJ, USA*, pages 38:1–38:20, 2018. 2, 3, 4, 8, 11, 12, 13, 14
- [DHK<sup>+</sup>18] Irit Dinur, Prahladh Harsha, Tali Kaufman, Inbal Livni Navon, and Amnon Ta-Shma. List decoding with double samplers. *Electronic Colloquium on Computational Complexity (ECCC)*, 25:136, 2018. 1, 2
- [DK12] Irit Dinur and Tali Kaufman. Locally testable codes and expanders. Manuscript, 2012. 2
- [DK17] Irit Dinur and Tali Kaufman. High dimensional expanders imply agreement expanders. In *Proceedings of the 58th IEEE Symposium on Foundations of Computer Science*, pages 974–985, 2017. 1, 2, 3, 4, 5, 7
- [Fil16] Yuval Filmus. An orthogonal basis for functions over a slice of the boolean hypercube. *Electr. J. Comb.*, 23(1):P1.23, 2016. 16
- [FK96] A. Frieze and R. Kannan. The regularity lemma and approximation schemes for dense problems. In *Proceedings of the 37th IEEE Symposium on Foundations of Computer Science*, 1996. 3
- [GM15] Christopher Godsil and Karen Meagher. *Erdős-Ko-Rado Theorems: Algebraic Approaches*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2015. 11, 16
- [GS11] Venkatesan Guruswami and Ali Kemal Sinop. Lasserre hierarchy, higher eigenvalues, and approximation schemes for graph partitioning and quadratic integer programming with PSD objectives. In *Proceedings of the 52nd IEEE Symposium on Foundations of Computer Science*, pages 482–491, 2011. 1, 3
- [GS12] Venkatesan Guruswami and Ali Kemal Sinop. Faster SDP hierarchy solvers for local rounding algorithms. In *Proceedings of the 53rd IEEE Symposium on Foundations of Computer Science*, pages 197–206. IEEE, 2012. 2
- [Kho02] Subhash Khot. On the power of unique 2-prover 1-round games. In *Proceedings of the 34th ACM Symposium on Theory of Computing*, pages 767–775, 2002. 1
- [KKL16] Tali Kaufman, David Kazhdan, and Alexander Lubotzky. Isoperimetric inequalities for ramanujan complexes and topological expanders. *Geometric and Functional Analysis*, 26(1):250–287, Feb 2016. 2
- [KM17] Tali Kaufman and David Mass. High dimensional random walks and colorful expansion. In *Proceedings of the 8th Conference on Innovations in Theoretical Computer Science*, pages 4:1–4:27, 2017. 2, 3
- [KM18] Tali Kaufman and David Mass. Good distance lattices from high dimensional expanders. *CoRR*, abs/1803.02849, 2018. URL: <http://arxiv.org/abs/1803.02849>, [arXiv:1803.02849](https://arxiv.org/abs/1803.02849). 1
- [KMOW17] Pravesh Kothari, Ryuhei Mori, Ryan O’Donnell, and David Witmer. Sum of squares lower bounds for refuting any CSP. In *Proceedings of the 49th ACM Symposium on Theory of Computing*, 2017. 1
- [KO18a] Tali Kaufman and Izhar Oppenheim. Construction of new local spectral high dimensional expanders. In *Proceedings of the 50th ACM Symposium on Theory of Computing*, STOC 2018, pages 773–786. ACM, 2018. 2

- [KO18b] Tali Kaufman and Izhar Oppenheim. High order random walks: Beyond spectral gap. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2018, August 20-22, 2018 - Princeton, NJ, USA*, pages 47:1–47:17, 2018. 2, 3, 4
- [LLP17] Eyal Lubetzky, Alex Lubotzky, and Ori Parzanchevski. Random walks on Ramanujan complexes and digraphs. *arXiv e-prints*, page arXiv:1702.05452, Feb 2017. arXiv:1702.05452. 3
- [LSV05a] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Explicit constructions of ramanujan complexes of type ad. *Eur. J. Comb.*, 26(6):965–993, August 2005. 2
- [LSV05b] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Ramanujan complexes of type ad. *Israel Journal of Mathematics*, 149(1):267–299, Dec 2005. 2
- [MM11] Konstantin Makarychev and Yury Makarychev. How to play unique games on expanders. In *Approximation and Online Algorithms*, pages 190–200. Springer Berlin Heidelberg, 2011. 1
- [MR17] Pasin Manurangsi and Prasad Raghavendra. A birthday repetition theorem and complexity of approximating dense csps. In *Proceedings of the 44th International Colloquium on Automata, Languages and Programming. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik*, 2017. 3
- [OGT15] Shayan Oveis Gharan and Luca Trevisan. A new regularity lemma and faster approximation algorithms for low threshold rank graphs. *Theory of Computing*, 11(9):241–256, 2015. URL: <http://www.theoryofcomputing.org/articles/v011a009>, doi:10.4086/toc.2015.v011a009. 1, 3
- [PRT16] Ori Parzanchevski, Ron Rosenthal, and Ran J. Tessler. Isoperimetric inequalities in simplicial complexes. *Combinatorica*, 36(2):195–227, Apr 2016. 2
- [RW17] Prasad Raghavendra and Benjamin Weitz. On the bit complexity of sum-of-squares proofs. In *Proceedings of the 44th International Colloquium on Automata, Languages and Programming. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik*, 2017. 2
- [Sag13] B.E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. Graduate Texts in Mathematics. Springer New York, 2013. 16
- [WJ04] Martin J Wainwright and Michael I Jordan. Treewidth-based conditions for exactness of the Sherali-Adams and Lasserre relaxations. Technical report, Technical Report 671, University of California, Berkeley, 2004. 3
- [YZ14] Yuichi Yoshida and Yuan Zhou. Approximation schemes via Sherali-Adams hierarchy for dense constraint satisfaction problems and assignment problems. In *Proceedings of the 5th Conference on Innovations in Theoretical Computer Science*, pages 423–438. ACM, 2014. 3