

## Minor-free graphs have light spanners

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**Abstract**—We show that every  $H$ -minor-free graph has a light  $(1 + \epsilon)$ -spanner, resolving an open problem of Grigni and Sissokho [13] and proving a conjecture of Grigni and Hung [12]. Our lightness bound is

$$O\left(\frac{\sigma_H}{\epsilon^3} \log \frac{1}{\epsilon}\right)$$

where  $\sigma_H = |V(H)|\sqrt{\log |V(H)|}$  is the sparsity coefficient of  $H$ -minor-free graphs. That is, it has a practical dependency on the size of the minor  $H$ . Our result also implies that the polynomial time approximation scheme (PTAS) for the Travelling Salesperson Problem (TSP) in  $H$ -minor-free graphs by Demaine, Hajiaghayi and Kawarabayashi [7] is an efficient PTAS whose running time is  $2^{O_H\left(\frac{1}{\epsilon^4} \log \frac{1}{\epsilon}\right)} n^{O(1)}$  where  $O_H$  ignores dependencies on the size of  $H$ . Our techniques significantly deviate from existing lines of research on spanners for  $H$ -minor-free graphs, but build upon the work of Chechik and Wulff-Nilsen for spanners of general graphs [6].

**Keywords**—light spanners, Polynomial Time Approximation Scheme (PTAS), minor-free graphs, Traveling Salesperson Problem (TSP)

### I. INTRODUCTION

Peleg and Schäffer [18] introduced  $t$ -spanners of graphs as a way to sparsify graphs while approximately preserving pairwise distances between vertices. A  $t$ -spanner of a graph  $G$  is a subgraph  $S$  of  $G$  such that  $d_S(x, y) \leq t \cdot d_G(x, y)$  for all vertices  $x, y$ <sup>1</sup>. Two parameters of  $t$ -spanners that are of interest are their *sparsity* and *lightness*. The *sparsity* of  $S$  is the ratio of the number of edges to the number of vertices of  $S$ . The *lightness* of  $S$  is the ratio of the total weight of the edges of  $S$  to the weight of an MST of  $G$ ; generally, we assume that  $\text{MST}(G) \subseteq S$  (and so  $\text{MST}(S) = \text{MST}(G)$ ). Here, we are concerned with the lightness of  $(1 + \epsilon)$ -spanners, where  $\epsilon < 1$ , and so we refer to  $(1 + \epsilon)$ -spanners simply as *spanners*.

We say that a spanner is *light* if the lightness does not depend on the number of vertices in the graph. Grigni and Sissokho [13] showed that  $H$ -minor-free graphs have spanners of lightness

$$O\left(\frac{1}{\epsilon} \sigma_H \log n\right). \quad (1)$$

<sup>1</sup>We use standard graph terminology, which can be found in Appendix A.

where  $\sigma_H = |V(H)|\sqrt{\log |V(H)|}$  is the sparsity coefficient of  $H$ -minor-free graphs; namely that an  $H$ -minor-free graph of  $n$  vertices has  $O(|V(H)|\sqrt{\log |V(H)|}n)$  edges<sup>2</sup> [17]. Later Grigni and Hung [12], in showing that graphs of bounded pathwidth have light spanners, conjectured that  $H$ -minor-free graphs also have light spanners; that is, that the dependence on  $n$  can be removed from the lightness above. In this paper, we resolve this conjecture positively, proving:

**Theorem 1.** *Every  $H$ -minor-free graph  $G$  has a  $(1 + \epsilon)$ -spanner of lightness*

$$O\left(\frac{\sigma_H}{\epsilon^3} \log \frac{1}{\epsilon}\right). \quad (2)$$

Our algorithm consists of a reduction phase and a greedy phase. In the reduction phase, we adopt a technique of Chechik and Wulff-Nilsen [6]: edges of the graph are subdivided and their weights are rounded and scaled to guarantee that every MST-edge has unit weight and we include all very low weight edges in the spanner (Appendix C). In the greedy phase, we use the standard greedy algorithm for constructing a spanner to select edges from edges of the graph not included in the reduction phase (Appendix B).

As a result of the reduction phase, our spanner is not the ubiquitous *greedy spanner*. However, since Filtser and Solomon have shown that greedy spanners are (nearly) optimal in their lightness [10], our result implies that the greedy spanner for  $H$ -minor-free graphs is also light.

#### A. Implication: Approximating TSP

Light spanners have been used to give PTASes, and in some cases efficient PTASes, for the traveling salesperson problem (TSP) on various classes of graphs. A PTAS, or polynomial-time approximation scheme, is an algorithm which, for a fixed error parameter  $\epsilon$ , finds a solution whose value is within  $1 \pm \epsilon$  of optimal in polynomial time. A PTAS is *efficient* if its running time is  $f(\epsilon)n^{O(1)}$  where  $f(\epsilon)$  is a function of  $\epsilon$ . Rao and Smith [19] used light spanners of Euclidean graphs to give an EPTAS for Euclidean TSP. Arora, Grigni, Karger, Klein and Woloszyn [2] used light

<sup>2</sup>This bound is tight [21].

spanners of planar graphs, given by Althöfer, Das, Dobkin, Joseph and Soares [1], to design a PTAS for TSP in planar graphs with running time  $n^{O(\frac{1}{\epsilon^2})}$ . Klein [15] improved upon this running time to  $2^{O(\frac{1}{\epsilon^2})}n$  by modifying the PTAS framework, using the same light spanner. Borradaile, Demaine and Tazari generalized Klein’s EPTAS to bounded genus graphs [4].

In fact, it was in pursuit of a PTAS for TSP in  $H$ -minor-free graphs that Grigni and Sissokho discovered the logarithmic bound on lightness (Equation (1)); however, the logarithmic bound implies only a quasi-polynomial time approximation scheme (QPTAS) for TSP [13]. Demaine, Hajiaghayi and Kawarabayashi [7] used Grigni and Sissokho’s spanner to give a PTAS for TSP in  $H$ -minor-free graphs with running time  $n^{O(\text{poly}(\frac{1}{\epsilon}))}$ ; that is, *not* an efficient PTAS. However, Demaine, Hajiaghayi and Kawarabayashi’s PTAS is *efficient if the spanner used is light*. Thus, the main result of this paper implies an efficient PTAS for TSP in  $H$ -minor-free graphs.

### B. Techniques

In proving the lightness of spanners in planar graphs [1] and bounded genus graphs [11], the embedding of the graph was heavily used. Thus, it is natural to expect that showing minor-free graphs have light spanners would rely on the decomposition theorem of minor-free graphs by Robertson and Seymour [20], which shows that graphs excluding a fixed minor can be decomposed into the clique-sum of graphs *nearly* embedded on surfaces of fixed genus. Borradaile and Le [5] use this decomposition theorem to show that if graphs of bounded treewidth have light spanners, then  $H$ -minor-free graphs also have light spanners. As graphs of bounded treewidth are generally regarded as *easy instances* of  $H$ -minor-free graphs, it may be possible to give a simpler proof of lightness of spanners for  $H$ -minor-free graphs using this implication.

However, relying on the Robertson and Seymour decomposition theorem generally results in constants which are galactic in the size of the the minor [16], [14]. In this work, we take a different approach which avoids this problem. Our method is inspired from the recent work of Chechik and Wulff-Nilsen [6] on spanners for general graphs which uses an iterative super-clustering technique [3], [8]. Using the same technique in combination with amortized analysis, we show that  $H$ -minor-free graphs not only have light spanners, but also that the dependency of the lightness on  $\epsilon$  and  $|V(H)|$  is practical (Equation (2)).

At a high level, our proof shares several ideas with the work of Chechik and Wulff-Nilsen [6] who prove that (general) graphs have  $(2k-1) \cdot (1+\epsilon)$ -spanners with lightness  $O_\epsilon(n^{1/k})$ , removing a factor of  $k/\log k$  from the previous best-known bound and matching Erdős’s girth conjecture [9] up to a  $1 + \epsilon$  factor. Our work differs from Chechik and

Wulff-Nilsen in two major aspects. First, Chechik and Wulff-Nilsen reduce their problem down to a single hard case where the edges of the graph have weight at most  $g^k$  for some constant  $g$ . In our problem, we must partition the edges according to their weight along a logarithmic scale and deal with each class of edges separately. Second, we must employ the fact that  $H$ -minor-free graphs (and their minors) are sparse in order to get a lightness bound that does not depend on  $n$ .

### C. Future directions

Since we avoid relying on Robertson and Seymour’s decomposition theorem and derive bounds using only the sparsity of graphs excluding a fixed minor, it is possible this technique could be extended to related *spanner-like* constructions that are used in the design of PTASes for connectivity problems. Except for TSP, many connectivity problems [4] have PTASes for bounded genus graphs but are not known to have PTASes for  $H$ -minor-free graphs – for example, subset TSP and Steiner tree. The PTASes for these problems rely on having a light subgraph that approximates the optimal solution within  $1 + \epsilon$  (and hence is *spanner-like*). The construction of these subgraphs, though, rely heavily on the embedding of the graph on a surface and since the Robertson and Seymour decomposition gives only a weak notion of embedding for  $H$ -minor-free graphs, pushing these PTASes beyond surface embedded-graphs does not seem likely. The work of this paper may be regarded as a first step toward designing *spanner-like* graphs for problems such as subset TSP and Steiner tree that do not rely on the embedding.

## II. BOUNDING THE LIGHTNESS OF A $(1 + \epsilon)$ -SPANNER

As we already indicated, we start with a reduction that allows us to assume that the edges of the MST of the graph each have unit weight. (For details, see Appendix C.) For simplicity of presentation, we will also assume that the spanner is a greedy  $(1 + s \cdot \epsilon)$ -spanner for a sufficiently large constant  $s$ ; this does not change the asymptotics of our lightness bound.

Herein, we let  $S$  be the edges of a greedy  $(1 + s \cdot \epsilon)$ -spanner of graph  $G$  with an MST having edges all of unit weight. We simply refer to  $S$  as the *spanner*. The greedy spanner considers the edges in non-decreasing order of weights and adds an edge  $xy$  if  $(1 + s \cdot \epsilon)w(xy)$  is at most the  $x$ -to- $y$  distance in the current spanner.

We partition the edges of  $S$  according to their weight as it will be simpler to bound the weight of subsets of  $S$ . Let  $J_0$  be the edges of  $S$  of weight in the range  $[1, \frac{1}{\epsilon})$ ; note that  $\text{MST} \subseteq J_0$  and, since  $G$  has  $O(\sigma_H n)$  edges and  $w(\text{MST}) = n - 1$ ,

$$w(J_0) = O(\sigma_H n / \epsilon) = O\left(\frac{\sigma_H}{\epsilon} w(\text{MST})\right) \quad (3)$$

Let  $\Pi_j^i$  be the edges of  $S$  of weight in the range  $[\frac{2^j}{\epsilon^i}, \frac{2^{j+1}}{\epsilon^i})$  for every  $i \in \mathbb{Z}^+$  and  $j \in \{0, 1, \dots, \lceil \log \frac{1}{\epsilon} \rceil\}$ . Let  $J_j = \cup_i \Pi_j^i$ . We will prove that

**Lemma 1.** *There exists a set of spanner edges  $B$  such that  $w(B) = O(\frac{1}{\epsilon^2} w(\text{MST}))$  and for every  $j \in \{0, \dots, \lceil \log \frac{1}{\epsilon} \rceil\}$ ,*

$$w(\text{MST} \cup (J_j \setminus B)) = O\left(\frac{\sigma_H}{\epsilon^3}\right) w(\text{MST}).$$

Combined with Equation (3), Lemma 1 gives us

$$w(S) = w(B) + \sum_{j=0}^{\lceil \log \frac{1}{\epsilon} \rceil} w(J_j \setminus B) = O\left(\frac{\sigma_H}{\epsilon^3} \log \frac{1}{\epsilon}\right) w(\text{MST})$$

which, combined with the reduction to unit-weight MST-edges, proves Theorem 1 (noting that the stretch condition of  $S$  is satisfied since  $S$  is a greedy spanner of  $G$ ).

In the remainder, we prove Lemma 1 for a fixed  $j \geq 0$ . Let  $E_i = \Pi_j^i$  for this fixed  $j$  and some  $i \in \mathbb{Z}^+$ . Let  $\ell_i = \frac{2^{j+1}}{\epsilon^i}$ ; then, the weight of the edges in  $E_i$  are in the range  $[\ell_i/2, \ell_i)$ . Let  $E_0 = \text{MST}$ . We refer to the indices  $0, 1, 2, \dots$  of the edge partition as levels.

#### A. Proof overview

To prove Lemma 1, we use an amortized analysis, initially assigning each edge of  $E_0 = \text{MST}$  a credit of  $c = O(\frac{\sigma_H}{\epsilon^3})$ . For each level, we partition the vertices of the spanner into *clusters* where each cluster is defined by a subgraph of the graph formed by the edges in levels 0 through  $i$ . (Note that not every edge of level 0 through  $i$  may belong to a cluster; some edges may go between clusters.) Level  $i-1$  clusters are a refinements of level  $i$  clusters. We prove (by induction over the levels), that the clusters for each level satisfy the following *diameter-credit invariants*:

- DC1 A cluster in level  $i$  of diameter  $k$  has at least  $c \cdot \max\{k, \frac{\ell_i}{2}\}$  credits.
- DC2 A cluster in level  $i$  has diameter at most  $g\ell_i$  for some constant  $g > 2$  (specified later).

We achieve the diameter-credit invariants for the base case (level 0) as follows. Although a simpler proof could be given, the following method we use will be revisited in later, more complex, constructions. Recall that  $E_0 = \text{MST}$  and that, in a greedy spanner, the shortest path between endpoints of any edge is the edge itself. If the diameter of  $E_0$  is  $< \ell_0/2 = O(1)$ , edges in the spanner have length at most  $\ell_0/2$ . Thus, it is trivial to bound the weight of all the spanner edges across all levels using the sparsity of  $H$ -minor-free graphs. Assuming a higher diameter, let  $\mathcal{T}$  be a maximal collection of vertex-disjoint subtrees of  $E_0$ , each having diameter  $\lceil \ell_0/2 \rceil$  (chosen, for example, greedily). Delete  $\mathcal{T}$  from  $E_0$ . What is left is a set of trees  $\mathcal{T}'$ , each of diameter  $< \ell_0/2$ . For each tree  $T \in \mathcal{T}$ , let  $C_T$  be the union of  $T$  with any neighboring trees in  $\mathcal{T}'$  (connected to  $T$  by a single edge of  $E_0$ ). By construction,  $C_T$  has diameter at most  $3\ell_0/2 + 1 \leq 2\ell_0$  (giving DC2).  $C_T$  is assigned the

credits of all the edges in the cluster each of which have credit  $c$  (giving DC1).

We build the clusters for level  $i$  from the clusters of level  $i-1$  in a series of four phases (Section III). We call the clusters of level  $i-1$   $\epsilon$ -clusters, since the diameter of clusters in level  $i-1$  are an  $\epsilon$ -fraction of the diameters of clusters in level  $i$ . A cluster in level  $i$  is induced by a group of  $\epsilon$ -clusters.

We try to group the  $\epsilon$ -clusters so that the diameter of the group is smaller than the sum of the diameters of the  $\epsilon$ -clusters in the group (Phases 1 to 3). This *diameter reduction* will give us an excess of credit beyond what is needed to maintain DC1 which allows us to pay for the edges of  $E_i$ . We will use the sparsity of  $H$ -minor free graphs to argue that each  $\epsilon$ -cluster needs to pay for, on average, a constant number of edges of  $E_i$ . In Phase 4, we further grow existing clusters via MST edges and unpaid edges of  $E_i$ . Showing that the clusters for level  $i$  satisfy invariant DC2 will be seen directly from the construction. However, satisfying invariant DC1 is trickier. Consider a path  $D$  witnessing the diameter of a level- $i$  cluster  $\mathcal{B}$ . Let  $\mathcal{D}$  be the graph obtained from  $D$  by contracting  $\epsilon$ -clusters; we call  $\mathcal{D}$  the *cluster-diameter path*. The edges of  $\mathcal{D}$  are a subset of  $\text{MST} \cup E_i$ . If  $\mathcal{D}$  does not contain an edge of  $E_i$ , the credits from the  $\epsilon$ -clusters and MST edges of  $\mathcal{D}$  are sufficient for satisfying invariant DC1 for  $\mathcal{B}$ . However, since edges of  $E_i$  are not initialized with any credit, when  $\mathcal{D}$  contains an edge of  $E_i$ , we must use credits of the  $\epsilon$ -clusters of  $\mathcal{B}$  outside  $\mathcal{D}$  to satisfy DC1 as well as pay for  $E_i$ . Finally, we need to pay for edges of  $E_i$  that go between clusters. We do so in two ways. First, some edges of  $E_i$  will be paid for *by this level* by using credit leftover after satisfying DC1. Second, the remaining edges will be paid for at the end of the entire process (over all levels); we show that there are few such edges over all levels (the edges  $B$  of Lemma 1).

In our proof below, the fixed constant  $g$  required in DC2 is roughly 100 and  $\epsilon$  is sufficiently smaller than  $\frac{1}{g}$ . For simplicity of presentation, we make no attempt to optimize  $g$ . We note that a  $(1+\epsilon)$ -spanner is also a  $(1+2\epsilon)$ -spanner for any constant  $\epsilon$  and the asymptotic dependency of the lightness on  $\epsilon$  remains unchanged. That is, requiring that  $\epsilon$  is sufficiently small is not a limitation on the range of the parameter  $\epsilon$ .

### III. ACHIEVING DIAMETER-CREDIT INVARIANTS

In this section, we construct clusters for level  $i$  that satisfy DC2 using the induction hypothesis that  $\epsilon$ -clusters (clusters of level  $i-1$ ) satisfy the diameter-credit invariants (DC1 and DC2). Since  $\ell_{i-1} = \epsilon\ell_i$ , we let  $\ell = \ell_i$ , and drop the subscript in the remainder. For DC2, we need to group  $\epsilon$ -clusters into clusters of diameter  $\Theta(\ell)$ . Let  $\mathcal{C}_\epsilon$  be the collection of  $\epsilon$ -clusters and  $\mathcal{C}$  be the set of clusters that we construct for level  $i$ . Initially,  $\mathcal{C} = \emptyset$ . We define a *cluster graph*  $\mathcal{K}(\mathcal{C}_\epsilon, E_i)$  whose vertices are the  $\epsilon$ -clusters and edges

are the edges of  $E_i$ .  $\mathcal{K}(\mathcal{C}_\epsilon, E_i)$  can be obtained from the subgraph of  $G$  formed by the edges of the  $\epsilon$ -clusters and  $E_i$  by contracting each  $\epsilon$ -cluster to a single vertex. Recall each  $\epsilon$ -cluster is a subgraph of the graph formed by the edges in levels 0 through  $i - 1$ .

**Observation 1.**  $\mathcal{K}(\mathcal{C}_\epsilon, E_i)$  is a simple graph.

*Proof:* Since  $g\epsilon\ell \leq \frac{\ell}{2}$  when  $\epsilon$  is sufficiently smaller than  $\frac{1}{g}$ , there are no self-loops in  $\mathcal{K}(\mathcal{C}_\epsilon, E_i)$ . Suppose that there are parallel edges  $x_1y_1$  and  $x_2y_2$  where  $x_1, x_2 \in X \in \mathcal{C}_\epsilon$  and  $y_1, y_2 \in Y \in \mathcal{C}_\epsilon$ . Let  $w(x_2y_2) \leq w(x_1y_1)$ , w.l.o.g.. Then, the path  $P$  consisting of the shortest  $x_1$ -to- $x_2$  path in  $X$ , edge  $x_2y_2$  and the shortest  $y_1$ -to- $y_2$  path in  $Y$  has length at most  $w(x_2y_2) + 2g\epsilon\ell$  by DC2. Since  $w(x_2y_2) \leq w(x_1y_1)$  and  $w(x_1y_1) \geq \ell/2$ ,  $P$  has length at most  $(1 + 4g\epsilon)w(x_1y_1)$ . Therefore, if our spanner is a greedy  $(1 + 4g\epsilon)$ -spanner,  $x_1y_1$  would not be added to the spanner. ■

We call an  $\epsilon$ -cluster  $X$  *high-degree* if its degree in the cluster graph is at least  $\frac{20}{\epsilon}$ , and *low-degree* otherwise. For each  $\epsilon$ -cluster  $X$ , we use  $\mathcal{C}(X)$  to denote the cluster in  $\mathcal{C}$  that contains  $X$ . To both maintaining diameter-credit invariants and buying edges of  $E_i$ , we use credits of  $\epsilon$ -clusters in  $\mathcal{C}(X)$  and MST edges connecting  $\epsilon$ -clusters in  $\mathcal{C}(X)$ . We *save* credits of a subset  $\mathcal{S}(X)$  of  $\epsilon$ -clusters of  $\mathcal{C}(X)$  and MST-edges connecting  $\epsilon$ -clusters in  $\mathcal{S}(X)$  for maintaining invariant DC1. We then *reserve* credits of another subset  $\mathcal{R}(X)$  of  $\epsilon$ -clusters to pay for edges of  $E_i$  incident to  $\epsilon$ -clusters in  $\mathcal{S}(X) \cup \mathcal{R}(X)$ . We let other  $\epsilon$ -clusters in  $\mathcal{C}(X) \setminus (\mathcal{S}(X) \cup \mathcal{R}(X))$  *release* their credits to pay for their incident edges of  $E_i$ ; we call such  $\epsilon$ -clusters *releasing*  $\epsilon$ -clusters. We designate an  $\epsilon$ -cluster in  $\mathcal{C}(X)$  to be its *center* and let the center collect the credits of  $\epsilon$ -clusters in  $\mathcal{R}(X)$ . The credits collected by the center are used to pay for edges of  $E_i$  incident to non-releasing  $\epsilon$ -clusters.

#### A. Phase 1: High-degree $\epsilon$ -clusters

In this phase, we group high-degree  $\epsilon$ -clusters. The goal is to ensure that any edge of  $E_i$  not incident to a low-degree  $\epsilon$ -cluster has both endpoints in the new clusters formed (possibly in distinct clusters). Then we can use sparsity of the subgraph of  $\mathcal{K}(\mathcal{C}_\epsilon, E_i)$  induced by the  $\epsilon$ -clusters that were clustered to argue that the clusters can pay for all such edges; this is possible since this subgraph is a minor of  $G$ . The remaining edges that have not been paid for are all incident to low-degree  $\epsilon$ -clusters which we deal with in later phases.

With all  $\epsilon$ -clusters initially *unmarked*, we apply Step 1 until it no longer applies and then apply Step 2 to all remaining high-degree  $\epsilon$ -clusters at once and breaking ties arbitrarily:

**Step 1:** If there is a high-degree  $\epsilon$ -cluster  $X$  such that all of its neighbor  $\epsilon$ -clusters in  $\mathcal{K}$  are unmarked, we group  $X$ , edges in  $E_i$  incident to  $X$  and its neighboring  $\epsilon$ -cluster into

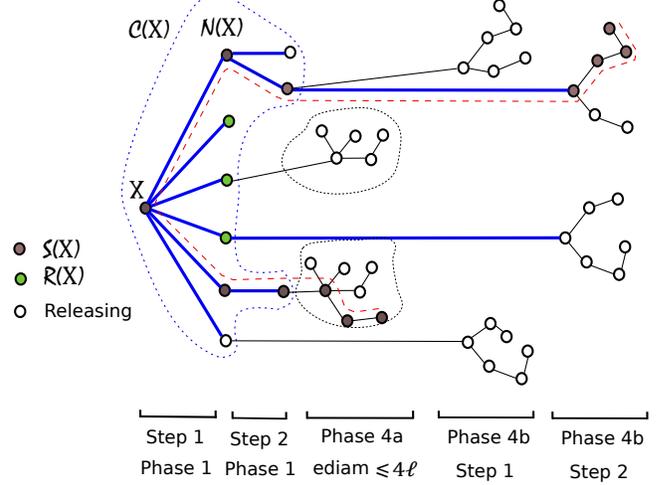


Figure 1. A cluster  $\mathcal{C}(X)$  formed in Phase 1 is enclosed in the dotted blue curve. The set  $\mathcal{S}(X)$  consists of five gray  $\epsilon$ -clusters inside the dotted blue closed curve and  $\mathcal{S}(X)$  consists of green-shaded  $\epsilon$ -clusters. Remaining hollow  $\epsilon$ -clusters are releasing. Cluster  $\mathcal{C}(X)$  will be augmented further in Phase 4 and augmenting  $\epsilon$ -clusters are outside the dotted blue curve. Solid blue edges are in  $E_i$  and thin black edges are in MST. The diameter path  $\mathcal{D}$  is highlighted by the dashed red curve and  $\epsilon$ -clusters in  $\mathcal{D}$  are gray-shaded.

a new cluster  $\mathcal{C}(X)$ . We then mark all  $\epsilon$ -clusters in  $\mathcal{C}(X)$ . We call  $X$  the *center*  $\epsilon$ -cluster of  $\mathcal{C}(X)$ .

**Step 2:** After Step 1, any unmarked high-degree  $\epsilon$ -cluster, say  $Y$ , must have at least one marked neighboring  $\epsilon$ -cluster, say  $Z$ . We add  $Y$  and the edge of  $E_i$  between  $Y$  and  $Z$  to  $\mathcal{C}(X)$  and mark  $Y$ .

In the following, the upper bound is used to guarantee DC2 and the lower bound will be used to guarantee DC1.

**Claim 1.** *The diameter of each cluster added in Phase 1 is at least  $\ell$  and at most  $(4 + 5g\epsilon)\ell$ .*

*Proof:* Since the clusters formed are trees each containing at least two edges of  $E_i$  and since each edge of  $E_i$  has weight at least  $\ell/2$ , the resulting clusters have diameter at least  $\ell$ .

Consider an  $\epsilon$ -cluster  $X$  that is the center of a cluster  $\mathcal{C}$  in Step 1 that is augmented to  $\widehat{\mathcal{C}}$  in Step 2 (where, possibly  $\mathcal{C} = \widehat{\mathcal{C}}$ ). The upper bound on the diameter of  $\widehat{\mathcal{C}}$  comes from observing that any two vertices in  $\widehat{\mathcal{C}}$  are connected via at most 5  $\epsilon$ -clusters and via at most 4 edges of  $E_i$  (each  $\epsilon$ -cluster that is clustered in Step 2 is the neighbor of a marked  $\epsilon$ -cluster from Step 1). Since  $\epsilon$ -clusters have diameter at most  $g\epsilon\ell$  and edges of  $E_i$  have weight at most  $\ell$ , the diameter of  $\widehat{\mathcal{C}}$  is at most  $(4 + 5g\epsilon)\ell$ . ■

Let  $\mathcal{C}(X)$  be a cluster in Phase 1 with the center  $X$ . Let  $\mathcal{N}(X)$  be the set of  $X$ 's neighbors in the cluster graph  $\mathcal{K}(\mathcal{C}_\epsilon, E_i)$ . By construction,  $\mathcal{C}(X)$  is a tree of  $\epsilon$ -clusters. Thus, at most five  $\epsilon$ -clusters in  $\mathcal{C}(X)$  would be in the cluster-diameter path  $\mathcal{D}$  while at most three of them are in  $\mathcal{N}(X) \cup \{X\}$ . We use the credit of  $X$  and of two  $\epsilon$ -clusters in

$\mathcal{N}(X)$  for maintaining DC1. Let this set of three  $\epsilon$ -clusters be  $\mathcal{S}(X)$ . Since  $X$  is high-degree and  $\epsilon < 1$ ,  $\mathcal{N}(X) \setminus \mathcal{S}(X)$  has at least  $\frac{20}{\epsilon} - 2 \geq \frac{18}{\epsilon}$   $\epsilon$ -clusters. Let  $\mathcal{R}(X)$  be any subset of  $\frac{18}{\epsilon}$   $\epsilon$ -clusters in  $\mathcal{N}(X) \setminus \mathcal{S}(X)$ . The center  $X$  collects the credits of  $\epsilon$ -clusters in  $\mathcal{R}(X)$ . We let other  $\epsilon$ -clusters in  $\mathcal{C} \setminus (\mathcal{R}(X) \cup \mathcal{S}(X))$  release their own credits; we call such  $\epsilon$ -clusters *releasing  $\epsilon$ -clusters*. By diameter-credit invariants for level  $i - 1$ , each  $\epsilon$ -cluster has at least  $\frac{c\ell}{2}$  credits. Thus, we have:

**Observation 2.** *The center  $X$  of  $\mathcal{C}(X)$  collects at least  $9c\ell$  credits.*

Let  $A_1$  be the set of edges of  $E_i$  that have both endpoints in marked  $\epsilon$ -clusters.

**Claim 2.** *If  $c = \Omega(\frac{\sigma_H}{\epsilon})$ , we can buy edges of  $A_1$  using  $c\ell$  credits deposited in the centers and credit of releasing  $\epsilon$ -clusters.*

*Proof:* Since the subgraph of  $\mathcal{K}$  induced by marked  $\epsilon$ -clusters and edges of  $A_1$  is  $H$ -minor-free, each marked  $\epsilon$ -cluster, on average, is incident to at most  $O(\sigma_H)$  edges of  $A_1$ . Thus, each  $\epsilon$ -cluster must be responsible for buying  $\Omega(\sigma_H)$  edges of  $A_1$ .

Consider a cluster  $\mathcal{C}(X)$ . The total credits of each releasing  $\epsilon$ -clusters is at least  $\frac{c\ell}{2}$ , which is  $\Omega(\sigma_H)\ell$  when  $c = \Omega(\frac{\sigma_H}{\epsilon})$ . For non-releasing  $\epsilon$ -clusters, we use  $c\ell$  credits from their center  $X$  to pay for incident edges of  $A_1$ . Recall that non-releasing  $\epsilon$ -clusters are in  $\mathcal{R}(X) \cup \mathcal{S}(X)$  and:

$$|\mathcal{R}(X) \cup \mathcal{S}(X)| \leq 5 + \frac{18}{\epsilon} \quad (4)$$

Thus, non-releasing  $\epsilon$ -cluster are responsible for paying at most  $O(\frac{\sigma_H}{\epsilon})$  edges of  $A_1$  and  $c\ell$  credits suffice if  $c = \Omega(\frac{\sigma_H}{\epsilon})$ . ■

By Claim 2, each center  $\epsilon$ -cluster has at least  $8c\ell$  credits remaining after paying for  $A_1$ . We note that clusters in Phase 1 could be augmented further in Phase 4. We will use these remaining credits at the centers to pay for edges of  $E_i$  in Phase 4.

### B. Phase 2: Low-degree, branching $\epsilon$ -clusters

Let  $\mathcal{F}$  be a maximal forest whose nodes are the  $\epsilon$ -clusters that remain unmarked after Phase 1 and whose edges are MST edges between pairs of such  $\epsilon$ -clusters.

Let  $\text{diam}(\mathcal{P})$  be the diameter of a path  $\mathcal{P}$  in  $\mathcal{F}$ , which is the diameter of the subgraph of  $G$  formed by edges inside  $\epsilon$ -clusters and MST edges connecting  $\epsilon$ -clusters of  $\mathcal{P}$ . We define the *effective diameter*  $\text{ediam}(\mathcal{P})$  to be the sum of the diameters of the  $\epsilon$ -clusters in  $\mathcal{P}$ . Since the edges of  $\mathcal{F}$  have unit weight (since they are MST edges), the true diameter of a path in  $\mathcal{F}$  is bounded by the effective diameter of  $\mathcal{P}$  plus the number of MST edges in the path. Since each  $\epsilon$ -cluster has diameter at least 1 (by construction of the base case), we have:

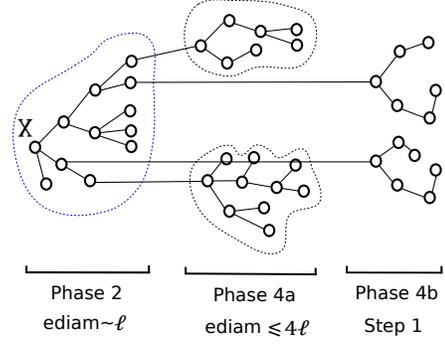


Figure 2. A cluster  $\mathcal{C}(X)$  formed in Phase 2 is enclosed in the dotted blue curve.  $\mathcal{C}(X)$  will be augmented further in Phase 4 and augmenting  $\epsilon$ -clusters are outside the dotted blue curve. Edges connecting  $\epsilon$ -clusters are MST edges.

**Observation 3.**  $\text{diam}(\mathcal{P}) \leq 2\text{ediam}(\mathcal{P})$ .

We define the effective diameter of a tree (in  $\mathcal{F}$ ) to be the maximum effective diameter over all paths of the tree. Let  $\mathcal{T}$  be a tree in  $\mathcal{F}$  that is not a path and such that  $\text{ediam}(\mathcal{T}) \geq 2\ell$ . Let  $X$  be a *branching vertex* of  $\mathcal{T}$ , i.e., a vertex of  $\mathcal{T}$  of degree is at least 3, and let  $\mathcal{C}(X)$  be a minimal subtree of  $\mathcal{T}$  that contains  $X$  and  $X$ 's neighbors and such that  $\text{ediam}(\mathcal{C}(X)) \geq 2\ell$ . We add  $\mathcal{C}(X)$  to  $\mathcal{C}$  and delete  $\mathcal{C}(X)$  from  $\mathcal{T}$  (see Figure 2); this process is repeated until no such tree exists in  $\mathcal{F}$ . We refer to  $X$  as the *center  $\epsilon$ -cluster* of  $\mathcal{C}(X)$ .

**Claim 3.** *The diameter of each cluster added in Phase 2 is at most  $(4 + 2g\epsilon)\ell$ .*

*Proof:* Since  $\mathcal{C}(X)$  is minimal, its effective diameter is at most  $2\ell + g\ell$ . The claim follows from Observation 3. ■

Let  $\mathcal{X}$  be a set of  $\epsilon$ -clusters. We define a subset of  $\mathcal{X}$  as follows:

$$\lfloor \mathcal{X} \rfloor_{2g/\epsilon} = \begin{cases} \mathcal{X} & \text{if } |\mathcal{X}| \leq 2g/\epsilon \\ \text{any subset of } 2g/\epsilon \text{ of } \mathcal{X} & \text{otherwise} \end{cases}$$

By definition, we have:

$$|\lfloor \mathcal{X} \rfloor_{2g/\epsilon}| \leq \frac{2g}{\epsilon} \quad (5)$$

Let  $\mathcal{S}(X) = \lfloor \mathcal{C}(X) \cap \mathcal{D} \rfloor_{2g/\epsilon}$  where  $\mathcal{D}$  is the diameter path of  $\mathcal{C}(X)$ . We save credits of  $\epsilon$ -clusters in  $\mathcal{S}(X)$  for maintaining DC1 and we use credits of  $\epsilon$ -clusters in  $\mathcal{C} \setminus \mathcal{S}(X)$  to buy edges of  $E_i$  incident to  $\epsilon$ -clusters in  $\mathcal{C}(X)$ . Since  $X$  is branching, at least one neighbor  $\epsilon$ -cluster of  $X$ , say  $Y$ , is not in  $\mathcal{S}(X)$ . Let  $\mathcal{R}(X) = \{Y\}$ . The center collects credits of clusters in  $\mathcal{R}(X)$ ; other  $\epsilon$ -clusters in  $\mathcal{C}(X) \setminus \{\mathcal{S}(X) \cup \mathcal{R}(X)\}$  release their credits.

Let  $A_2$  be the set of unpaid edges of  $E_i$  incident to  $\epsilon$ -clusters grouped in Phase 2.

**Claim 4.** If  $c = \Omega(\frac{g}{\epsilon^3})$ , we can buy edges of  $A_2$  using  $\frac{c\epsilon\ell}{6}$  credits from the center  $\epsilon$ -clusters and half the credit from releasing  $\epsilon$ -clusters.

*Proof:* Consider a cluster  $\mathcal{C}(X)$  formed in Phase 2. Recall  $\epsilon$ -clusters in Phase 2 are low-degree. Thus, each  $\epsilon$ -cluster in  $\mathcal{C}(X)$  is incident to at most  $\frac{20}{\epsilon}$  edges of  $A_2$ . We need to argue that each  $\epsilon$ -cluster has at least  $\frac{20\ell}{\epsilon} = \Omega(\frac{\ell}{\epsilon})$  credits to pay for edges of  $A_2$ . By invariant DC1 for level  $i - 1$ , half credits of releasing  $\epsilon$ -clusters are at least  $\frac{c\epsilon\ell}{4}$ , which is  $\Omega(\frac{1}{\epsilon})\ell$  when  $c = \Omega(\frac{1}{\epsilon^2})$ .

Since  $|\mathcal{R}(X)| = 1$ , the center  $X$  collects at least  $\frac{c\epsilon\ell}{2}$  credits by invariant DC1 for level  $i - 1$ . Recall non-releasing  $\epsilon$ -clusters are all in  $\mathcal{S}(X)$ . Thus, by Equation 5, the total number of edges of  $A_2$  incident to  $\epsilon$ -clusters in  $\mathcal{S}(X) \cup \mathcal{R}(X)$  is at most:

$$\left(\frac{2g}{\epsilon} + 1\right) \frac{20}{\epsilon} = O\left(\frac{g}{\epsilon^2}\right)$$

Since  $c = \Omega(\frac{g}{\epsilon^3})$ ,  $\frac{c\epsilon\ell}{6}$  credits of the center  $X$  is at least  $\Omega(\frac{g\ell}{\epsilon^2})$  which suffices to buy all edges of  $A_2$  incident to  $\epsilon$ -clusters in  $\mathcal{S}(X) \cup \mathcal{R}(X)$ . ■

We use remaining *half the credit* of releasing  $\epsilon$ -clusters to achieve invariant DC1. More details will be given later when we show diameter-credit invariants of  $\mathcal{C}(X)$ .

### C. Phase 3: Grouping $\epsilon$ -clusters in high-diameter paths

In this phase, we consider components of  $\mathcal{F}$  that are paths with high effective diameter. To that end, we partition the components of  $\mathcal{F}$  into HD-components (equiv. HD-paths), those with (high) effective diameter at least  $4\ell$  (which are all paths) and LD-components, those with (low) effective diameter less than  $4\ell$  (which may be paths or trees).

*Phase 3a: Edges of  $E_i$  within an HD-path:* Consider an HD-path  $\mathcal{P}$  that has an edge  $e \in E_i$  with endpoints in  $\epsilon$ -clusters  $X$  and  $Y$  of  $\mathcal{P}$  such that the two disjoint affices ending at  $X$  and  $Y$  both have effective diameter at least  $2\ell$ . We choose  $e$  such that there is no other edge with the same property on the  $X$ -to- $Y$  subpath of  $\mathcal{P}$  (By Observation 1, there is no edge of  $E_i$  parallel to  $e$ ). Let  $\mathcal{P}_{X,Y}$  be the  $X$ -to- $Y$  subpath of  $\mathcal{P}$ . By the stretch guarantee of the spanner,  $\text{diam}(\mathcal{P}_{X,Y}) \geq (1 + s\epsilon)w(e)$ . Let  $\mathcal{P}_X$  and  $\mathcal{P}_Y$  be *minimal subpaths* of the disjoint affices of  $\mathcal{P}$  that end at  $X$  and  $Y$ , respectively, such that the effective diameters of  $\mathcal{P}_X$  and  $\mathcal{P}_Y$  are at least  $2\ell$ .  $\mathcal{P}_X$  and  $\mathcal{P}_Y$  exist by the way we choose  $e$ .

*Case 1:  $\text{ediam}(\mathcal{P}_{X,Y}) \leq 2\ell$ :* We construct a new cluster consisting of (the  $\epsilon$ -clusters and MST edges of)  $\mathcal{P}_{X,Y}$ ,  $\mathcal{P}_X$ ,  $\mathcal{P}_Y$  and edge  $e$  (see Figure 3). We refer to, w.l.o.g,  $X$  as the center  $\epsilon$ -cluster of the new cluster.

**Claim 5.** The diameter of each cluster added in Case 1 of Phase 3a is at least  $\frac{\ell}{2}$  and at most  $(12 + 4\epsilon g)\ell$ .

*Proof:* Since the new cluster contains edge  $e$  of  $E_i$  and, in spanner  $S$ , the shortest path between endpoints of

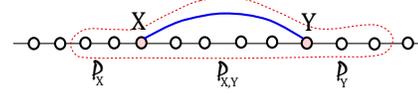


Figure 3. (A cluster of  $\mathcal{C}$  in Case 1 of Phase 3a. Thin edges are edges of MST, solid blue edges are edges of  $E_i$  and vertices are  $\epsilon$ -clusters. Edges and vertices inside the dashed red curves are grouped into a new cluster.

any edge is the edge itself, we get the lower bound of the claim. The effective diameters of  $\mathcal{P}_X$  and  $\mathcal{P}_Y$  are each at most  $(2 + \epsilon g)\ell$  since they are minimal. By Observation 3, we get that the diameter is at most:

$$\begin{aligned} & 2(\text{ediam}(\mathcal{P}_X) + \text{ediam}(\mathcal{P}_Y) + \text{ediam}(\mathcal{P}_{X,Y})) \\ & \leq 4(2 + \epsilon g)\ell + 4\ell = (12 + 4\epsilon g)\ell \end{aligned}$$

**Claim 6.** Let  $x, y$  be any two vertices of  $G$  in a cluster  $\mathcal{C}(X)$  added in Case 1 of Phase 3a. Let  $P_{x,y}$  be the shortest  $x$ -to- $y$  path in  $\mathcal{C}(X)$  as a subgraph of  $G$ . Let  $\mathcal{P}_{x,y}$  be obtained from  $P_{x,y}$  by contracting  $\epsilon$ -clusters into a single vertex. Then,  $\mathcal{P}_{x,y}$  is a simple path.

*Proof:* By construction, the only cycle of  $\epsilon$ -clusters in  $\mathcal{C}(X)$  is  $\mathcal{P}_{X,Y} \cup \{e\}$  (see Figure 3). Therefore, if  $\mathcal{P}_{x,y}$  is not simple,  $e \in \mathcal{P}_{x,y}$  and  $\mathcal{P}_{x,y}$  must enter and leave  $\mathcal{P}_{X,Y}$  at some  $\epsilon$ -cluster  $Z$ . In this case,  $\mathcal{D}$  could be short-cut through  $Z$ , reducing the weight of the path by at least  $w(e) \geq \ell/2$  and increasing its weight by at most  $\text{diam}(Z) \leq g\epsilon\ell$ . This contradicts the shortness of  $P_{x,y}$  for  $\epsilon$  sufficiently smaller than  $\frac{1}{g}$  ( $g\epsilon < \frac{1}{2}$ ). ■

Since  $\mathcal{P}_{X,Y} \cup \{e\}$  is the only cycle of  $\epsilon$ -clusters, by Claim 6,  $\epsilon$ -clusters in  $\mathcal{D} \cap \mathcal{C}(X)$  form a simple subpath of  $\mathcal{D}$  where  $\mathcal{D}$  is the diameter path of  $\mathcal{C}(X)$ . We have:

**Observation 4.**  $\mathcal{P}_{X,Y} \not\subseteq \mathcal{D}$ .

*Proof:* For otherwise,  $\mathcal{D}$  could be shortcut through  $e$  at a cost of

$$\begin{aligned} & \leq \underbrace{\text{diam}(X) + \text{diam}(Y) + w(e)}_{\text{cost of shortcut}} \\ & - \underbrace{(\text{diam}(\mathcal{P}_{X,Y}) - \text{diam}(X) - \text{diam}(Y))}_{\text{lower bound on diameter}} \\ & \leq +w(e) + 4g\epsilon\ell - (1 + s\epsilon)w(e) \\ & \leq 4g\epsilon\ell - s\epsilon\ell/2 \quad (\text{since } w(e) \geq \ell/2) \end{aligned}$$

The second equation is due to the stretch condition for  $e$ . This change in cost is negative for  $s \geq 8g + 1$ . ■

Let  $\mathcal{S}(X) = [\mathcal{D} \cap \mathcal{C}(X)]_{2g/\epsilon}$  and  $\mathcal{R}(X) = [\mathcal{C}(X) \setminus \mathcal{D}]_{2g/\epsilon}$ . The center  $X$  collects the credits of  $\epsilon$ -clusters in  $\mathcal{R}(X)$  and MST edges outside  $\mathcal{D}$  connecting  $\epsilon$ -clusters of  $\mathcal{C}(X)$ . We let other  $\epsilon$ -clusters in  $\mathcal{C}(X) \setminus (\mathcal{R}(X) \cup \mathcal{S}(X))$  release their credits.

**Claim 7.** If  $\mathcal{D}$  does not contain  $e$ , then  $X$  has at least  $\frac{c\epsilon\ell}{2}$  credits. Otherwise,  $X$  has at least  $cw(e) + \frac{c\epsilon\ell}{2}$  credits.

*Proof:* If  $\mathcal{C}(X) \setminus \mathcal{D}$  contains at least  $\frac{2g}{\epsilon}$   $\epsilon$ -clusters, then  $|\mathcal{R}(X)| = \frac{2g}{\epsilon}$ . Thus, by invariant DC1 for level  $i - 1$ , the total credit of  $\epsilon$ -clusters in  $\mathcal{R}(X)$  is at least:

$$\begin{aligned} \frac{2g}{\epsilon} \cdot \frac{c\epsilon\ell}{2} &= gcl \geq c\ell + \frac{c\epsilon\ell}{2} \quad (\text{for } g \geq 2 \text{ and } \epsilon < 1) \\ &\geq cw(e) + \frac{c\epsilon\ell}{2} \quad (\text{since } w(e) \leq \ell) \end{aligned}$$

Thus, we can assume that  $\mathcal{C}(X) \setminus \mathcal{D}$  contains less than  $\frac{2g}{\epsilon}$   $\epsilon$ -clusters. In this case,  $\mathcal{R}(X) = \mathcal{C}(X) \setminus \mathcal{D}$ . Since  $\mathcal{P}_{X,Y} \not\subseteq \mathcal{D}$  by Observation 4,  $\mathcal{D}$  does not contain, w.l.o.g.,  $\mathcal{P}_X$ . Thus,  $\mathcal{R}(X)$  contains at least one  $\epsilon$ -cluster and the claim holds for the case that  $e \notin \mathcal{D}$ .

Suppose that  $\mathcal{D}$  contains  $e$  and an internal  $\epsilon$ -clusters of  $\mathcal{P}_{X,Y}$ , then w.l.o.g.,  $\mathcal{D}$  does not contain  $\mathcal{P}_X \setminus X$ .  $\mathcal{P}_X \setminus X$  has credit  $2c\ell - g\epsilon\ell$ . Since  $\ell \geq w(e)$  and  $\ell - g\epsilon\ell \geq \frac{\epsilon\ell}{2}$  when  $\epsilon$  is sufficiently small ( $\epsilon \leq \frac{2}{1+2g}$ ), the claim holds.

If  $\mathcal{D}$  contains  $e$  but no internal  $\epsilon$ -clusters of  $\mathcal{P}_{X,Y}$ , then

$$\begin{aligned} \text{diam}(\mathcal{P}_{X,Y} \setminus \{X, Y\}) &\geq \text{diam}(\mathcal{P}_{X,Y}) - \text{diam}(X) - \text{diam}(Y) \\ &\geq (1 + s\epsilon)w(e) - \text{diam}(X) - \text{diam}(Y) \\ &\geq w_e + s\epsilon\ell/2 - 2g\epsilon\ell \quad (\text{by bounds on } w(e) \text{ and DC2}) \\ &\geq w_e + \epsilon\ell/2 \quad (\text{for } s \geq 8g + 1, \text{ as previously required}) \end{aligned} \tag{6}$$

The credit of the MST edges and  $\epsilon$ -clusters of  $\mathcal{P}_{X,Y} \setminus \{X, Y\}$  is at least:

$$\begin{aligned} c \cdot (\text{MST}(\mathcal{P}_{X,Y} \setminus \{X, Y\}) + \text{ediam}(\mathcal{P}_{X,Y} \setminus \{X, Y\})) \\ \geq c \cdot \text{diam}(\mathcal{P}_{X,Y} \setminus \{X, Y\}) \\ \geq c(w_e + \epsilon\ell/2) \end{aligned} \tag{7}$$

■

Let  $A_3$  be the set of *unpaid edges* of  $E_i$  incident to  $\epsilon$ -clusters of clusters in Case 1 of Phase 3a.

**Claim 8.** If  $c = \Omega(\frac{g}{\epsilon^3})$ , we can buy edges of  $A_3$  using  $\frac{c\epsilon\ell}{6}$  credits from each center and credits of releasing  $\epsilon$ -clusters.

*Proof:* Consider a cluster  $\mathcal{C}(X)$  in Phase 3. Similar to Claim 4, releasing  $\epsilon$ -clusters can pay for their incident edges in  $A_3$  when  $c = \Omega(\frac{g}{\epsilon^2})$ . By construction, non-releasing clusters of  $\mathcal{C}(X)$  are in  $\mathcal{S}(X) \cup \mathcal{R}(X)$ . Since  $|\mathcal{R}(X)| \leq \frac{2g}{\epsilon}$  and  $|\mathcal{S}(X)| \leq \frac{2g}{\epsilon}$  by Equation (5) and since clusters now we are considering have low degree, there are at most

$$\frac{4g}{\epsilon} \cdot \frac{20}{\epsilon} = O\left(\frac{g}{\epsilon^2}\right)$$

edges of  $A_3$  incident to non-releasing  $\epsilon$ -clusters. Thus, if  $c = \Omega(\frac{g}{\epsilon^3})$ ,  $\frac{c\epsilon\ell}{6} \geq \Omega(\frac{g}{\epsilon^2})\ell$ . That implies  $\frac{c\epsilon\ell}{4}$  credits of  $X$  suffice to pay for all edges of  $A_3$  incident to non-releasing  $\epsilon$ -clusters. ■

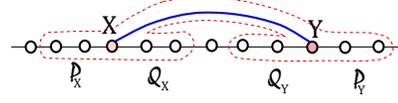


Figure 4. A cluster of  $\mathcal{C}$  in Case 2 of Phase 3a. Thin edges are edges of MST, solid blue edges are edges of  $E_i$  and vertices are  $\epsilon$ -clusters. Edges and vertices inside the dashed red curves are grouped into a new cluster.

**Case 2:  $\text{ediam}(\mathcal{P}_{X,Y}) > 2\ell$ :** Refer to Figure 4. Let  $Q_X$  and  $Q_Y$  be *minimal* affices of  $\mathcal{P}_{X,Y}$  such that each has effective diameter at least  $\ell$ . We construct a new cluster consisting of (the  $\epsilon$ -clusters and MST edges of)  $\mathcal{P}_X$ ,  $\mathcal{P}_Y$ ,  $Q_X$  and  $Q_Y$  and edge  $e$ . We refer to  $X$  as the center of the new cluster.

We apply Case 1 to all edges of  $E_i$  satisfying the condition of Case 1 until no such edges exist. We then apply Case 2 to all remaining edges of  $E_i$  satisfying the conditions of Case 2. After each new cluster is created (by Case 1 or 2), we delete the  $\epsilon$ -clusters in the new cluster from  $\mathcal{P}$ , reassign the resulting components of  $\mathcal{P}$  to the sets of HD- and LD-components. At the end, any edge of  $E_i$  with both endpoints in the same HD-path have both endpoints in two disjoint affices of effective diameter less than  $2\ell$ .

We bound the diameter and credit of the centers of clusters in Case 2 of Phase 3a in Phase 3b.

**Phase 3b: Edges of  $E_i$  between HD-paths:** Let  $e$  be an edge of  $E_i$  that connects  $\epsilon$ -cluster  $X$  of HD-path  $\mathcal{P}$  to  $\epsilon$ -cluster  $Y$  of different HD-path  $\mathcal{Q}$  such that none affix of effective diameter less than  $2\ell$  of  $\mathcal{P}$  contains  $X$  and none affix of effective diameter less than  $2\ell$  of  $\mathcal{Q}$  contains  $Y$ . Such edge  $e$  is said to have both endpoints *far* from endpoint  $\epsilon$ -clusters of  $\mathcal{P}$  and  $\mathcal{Q}$ .

Let  $\mathcal{P}_X$  and  $\mathcal{Q}_X$  be *minimal edge-disjoint* subpaths of  $\mathcal{P}$  that end at  $X$  and each having effective diameter at least  $2\ell$ . ( $\mathcal{P}_X$  and  $\mathcal{Q}_X$  exist by the way we choose edge  $e$ .) Similarly, define  $\mathcal{P}_Y$  and  $\mathcal{Q}_Y$ . We construct a new cluster consisting of (the  $\epsilon$ -clusters and MST edges of)  $\mathcal{P}_X$ ,  $\mathcal{P}_Y$ ,  $\mathcal{Q}_X$ ,  $\mathcal{Q}_Y$  and edge  $e$  (see Figure 5). We refer to  $X$  as the center of the new cluster. We then delete the  $\epsilon$ -clusters in the new cluster from  $\mathcal{P}$  and  $\mathcal{Q}$ , reassign the resulting components of  $\mathcal{P}$  and  $\mathcal{Q}$  to the sets of HD- and LD-components. We continue to create such new clusters until there are no edges of  $E_i$  connecting HD-paths with far endpoints.

We now bound the diameter and credits of the center of a cluster, say  $\mathcal{C}(X)$ , that is formed in Case 2 of Phase 3a or in Phase 3b. By construction in both cases,  $\mathcal{C}(X)$  consists of two paths  $\mathcal{P}_X \cup \mathcal{Q}_X$  and  $\mathcal{P}_Y \cup \mathcal{Q}_Y$  connected by an edge  $e$ .

**Claim 9.** The diameter of each cluster in Case 2 of Phase 3a and Phase 3b is at least  $\frac{\ell}{2}$  and at most  $(9 + 4\epsilon g)\ell$ .

*Proof:* The lower bound follows from the same argument as in the proof of Claim 5. Since the effective diameters of  $\mathcal{Q}_X$  and  $\mathcal{Q}_Y$  are smaller than the effective diameters of

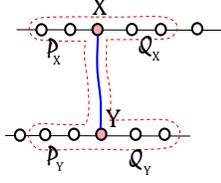


Figure 5. A cluster of  $\mathcal{C}$  in **Phase 4a**. Thin edges are edges of MST, solid blue edges are edges of  $E_i$  and vertices are  $\epsilon$ -clusters. Edges and vertices inside the dashed red curves are grouped into a new cluster.

$\mathcal{P}_X$  and  $\mathcal{P}_Y$ , the diameter of the new cluster is bounded by the sum of the diameters of  $\mathcal{P}_X$  and  $\mathcal{P}_Y$  and  $w(e)$ . The upper bound follows from the upper bounds on these diameters as given in the proof of Claim 5. ■

We show how to pay for unpaid edges of  $E_i$  incident to  $\epsilon$ -clusters in Case 2 of Phase 3a and Phase 3b. W.l.o.g, we refer to  $X$  as the center  $\epsilon$ -cluster of  $\mathcal{C}(X)$ . Let  $\mathcal{S}(X) = \lfloor \mathcal{D} \cap \mathcal{C}(X) \rfloor_{2g/\epsilon}$  and  $\mathcal{R}(X) = \lfloor \mathcal{C}(X) \setminus \mathcal{D} \rfloor_{2g/\epsilon}$  where  $\mathcal{D}$  is the cluster-diameter path of  $\mathcal{C}(X)$ . We save credits of  $\epsilon$ -clusters in  $\mathcal{S}(X)$  for maintaining invariant DC1. The center  $X$  collects credits of  $\epsilon$ -clusters in  $\mathcal{R}(X)$ . We let other  $\epsilon$ -clusters in  $\mathcal{C}(X) \setminus (\mathcal{S}(X) \cup \mathcal{R}(X))$  to release their credits.

**Claim 10.** *The center of a cluster in Case 2 of Phase 3a or in Phase 3b has at least  $2c(1 - g\epsilon)\ell$  credits.*

*Proof:* If  $|\mathcal{C}(X) \setminus \mathcal{D}| \geq \frac{2g}{\epsilon}$ ,  $\mathcal{R}(X)$  has  $\frac{2g}{\epsilon}$   $\epsilon$ -clusters which have at least  $g\epsilon\ell$  total credits by invariant DC1 for level  $i - 1$ . Since  $g\epsilon\ell > 2c(1 - g\epsilon)\ell$  when  $g > 2$ , the claim holds. Thus, we assume that  $|\mathcal{C}(X) \setminus \mathcal{D}| < \frac{2g}{\epsilon}$  which implies  $\mathcal{R}(X) = \mathcal{C}(X) \setminus \mathcal{D}$ . By construction,  $\mathcal{S}(X)$  contains  $\epsilon$ -clusters of at most two of four paths  $\mathcal{P}_X, \mathcal{P}_Y, \mathcal{Q}_X, \mathcal{Q}_Y$ . Since each path has effective diameter at least  $\ell$ , the  $\epsilon$ -clusters of each path in  $\mathcal{R}(X)$  have total diameter at least  $\ell - g\epsilon\ell$ . By invariant DC1 for level  $i - 1$ , each path in  $\mathcal{R}(X)$  has at least  $c(1 - g\epsilon)\ell$  credits that implies the claim. ■

Let  $A_4$  be the set of *unpaid edges* of  $E_i$  incident to  $\epsilon$ -clusters of clusters in Case 2 of Phase 3a and clusters in Phase 3b.

**Claim 11.** *If  $c = \Omega(\frac{g}{\epsilon^2})$ , we can buy edges of  $A_4$  using  $c(1 - 3g\epsilon)\ell$  credits of the centers of clusters in Case 2 of Phase 3a and Phase 3b and credits of releasing  $\epsilon$ -clusters.*

*Proof:* Similar to the proof of Claim 4, releasing  $\epsilon$ -clusters of  $\mathcal{C}(X)$  can buy their incident edges in  $A_4$  when  $c = \Omega(\frac{1}{\epsilon^2})$ . By construction, non-releasing  $\epsilon$ -clusters are in  $\mathcal{S}(X) \cup \mathcal{R}(X)$ . Since  $|\mathcal{R}(X)| \leq \frac{2g}{\epsilon}$  and  $|\mathcal{S}(X)| \leq \frac{2g}{\epsilon}$ , there are at most  $O(\frac{g}{\epsilon})$  edges of  $A_4$  incident to non-releasing  $\epsilon$ -clusters. When  $\epsilon$  is sufficiently small ( $\epsilon < \frac{1}{6g}$ ),  $c(1 - 3g\epsilon)\ell > \frac{c\ell}{2}$ . Thus, if  $c = \Omega(\frac{g}{\epsilon^2})$ ,  $\frac{c\ell}{2} = \Omega(\frac{g}{\epsilon^2})\ell$  and hence,  $c(1 - 3g\epsilon)\ell$  credits suffice to pay for all edges of  $A_4$  incident to non-releasing  $\epsilon$ -clusters of  $\mathcal{C}(X)$ . ■

#### D. Phase 4: Remaining HD-paths and LD-components

We assume that  $\mathcal{C} \neq \emptyset$  after Phase 3. The case when  $\mathcal{C} = \emptyset$  will be handled at the end of this section.

*Phase 4a: LD-components:* Consider a LD-component  $\mathcal{T}$ , that has effective diameter less than  $4\ell$ . By construction,  $\mathcal{T}$  must have an MST edge to a cluster, say  $\mathcal{C}(X)$ , in  $\mathcal{C}$  formed in a previous phase. We include  $\mathcal{T}$  and an MST edge connecting  $\mathcal{T}$  and  $\mathcal{C}(X)$  to  $\mathcal{C}(X)$ . Let  $A_5$  be the set of unpaid edges of  $E_i$  that incident to  $\epsilon$ -clusters merged into new clusters in this phase. We use credit of the center  $X$  and  $\epsilon$ -clusters in this phase to pay for  $A_5$ . More details will be given in Phase 4b.

*Phase 4b: Remaining HD-paths:* Let  $\mathcal{P}$  be a HD-path. By construction, there is at least one MST edge connecting  $\mathcal{P}$  to an existing cluster in  $\mathcal{C}$ . Let  $e$  be one of them. Greedily break  $\mathcal{P}$  into subpaths such that each subpath has effective diameter at least  $2\ell$  and at most  $4\ell$ . We call a subpath of  $\mathcal{P}$  a *long subpath* if it contains at least  $\frac{2g}{\epsilon} + 1$   $\epsilon$ -clusters and *short subpath* otherwise. We process subpaths of  $\mathcal{P}$  in two steps. In Step 1, we process affixes of  $\mathcal{P}$ , long subpaths of  $\mathcal{P}$  and the subpath of  $\mathcal{P}$  containing an endpoint  $\epsilon$ -cluster of  $e$ . In Step 2, we process remaining subpaths of  $\mathcal{P}$ .

*Step 1:* If a subpath  $\mathcal{P}'$  of  $\mathcal{P}$  contain an  $\epsilon$ -cluster that is incident to  $e$ , we merge  $\mathcal{P}'$  to the cluster in  $\mathcal{C}$  that contains another endpoint  $\epsilon$ -cluster of  $e$ . We call  $\mathcal{P}'$  the *augmenting subpath* of  $\mathcal{P}$ . We form a new cluster from each long subpath of  $\mathcal{P}$  and each affix of  $\mathcal{P}$ . It could be that one of two affixes of  $\mathcal{P}$  is augmenting. We repeatedly apply Step 1 for all HD-paths. The remaining cluster paths which are short subpaths of HD-paths would be handled in Step 2. We then pay for every unpaid edges of  $E_i$  incident to  $\epsilon$ -clusters in this step. We call a cluster a *long cluster* if it is a long subpath of  $\mathcal{P}$  and a *short cluster* if it is a short subpath of  $\mathcal{P}$ .

Let  $A_6$  be the set of unpaid edges of  $E_i$  incident to  $\epsilon$ -clusters of long clusters. We show below that each long cluster can both maintain diameter-credit invariant and pay for its incident edges in  $A_6$  using credits of its  $\epsilon$ -clusters.

Let  $A_7$  be the set of unpaid edges of  $E_i$  incident to remaining  $\epsilon$ -clusters involved in this step; those belong to augmenting subpaths and short affices of HD-paths. We can pay for edges of  $A_7$  incident to  $\epsilon$ -clusters in augmenting subpaths using the similar argument in previous phases. However, we must be careful when paying for other edges of  $A_7$  that are incident to  $\epsilon$ -clusters in short affices of  $\mathcal{P}$ . Since short affices of  $\mathcal{P}$  spend all credits of their children  $\epsilon$ -clusters to maintain invariant DC1, we need to use credits of  $\epsilon$ -clusters in  $\mathcal{P}'$  to pay for edges of  $A_7$  incident to short affices of  $\mathcal{P}$ .

*Step 2:* Let  $\mathcal{P}'$  be a short subpath of  $\mathcal{P}$ . If edges of  $E_i$  incident to  $\epsilon$ -clusters of  $\mathcal{P}'$  are all paid, we let  $\mathcal{P}'$  become a new cluster. Suppose that  $\epsilon$ -clusters in  $\mathcal{P}'$  are incident to at least one unpaid edge of  $E_i$ , say  $e$ . We have:

**Observation 5.** Edge  $e$  must be incident to an  $\epsilon$ -cluster merged in Phase 1.

We defer the proof of Observation 5 to the full version.

We merge  $\epsilon$ -clusters, MST edges of  $\mathcal{P}'$  and  $e$  to the cluster in  $\mathcal{C}$  that contains another endpoint of  $e$ . This completes the clustering process. Let  $A_8$  be the set of remaining unpaid edges of  $E_i$  incident to  $\epsilon$ -clusters involved in Step 2. We now analyze clusters of  $\mathcal{C}$  which are formed or modified in Phase 4.

**Claim 12.** Let  $\mathcal{B}$  be a short cluster. Then,  $\text{diam}(\mathcal{B}) \leq 8\ell$  and credits of  $\epsilon$ -clusters and MST edges connecting  $\epsilon$ -clusters in  $\mathcal{B}$  suffice to maintain invariant DC1 for  $\mathcal{B}$ .

*Proof:* Since  $\text{ediam}(\mathcal{B}) \leq 4\ell$ , by Observation 3,  $\text{diam}(\mathcal{B}) \leq 8\ell$ . The total credit of  $\epsilon$ -clusters and MST edges in  $\mathcal{B}$  is at least:

$$c(|\text{MST}(\mathcal{B})| + \text{ediam}(\mathcal{B})) \geq c \cdot \text{diam}(\mathcal{B})$$

Since  $\text{ediam}(\mathcal{B}) \geq 2\ell$ ,  $\mathcal{B}$  has at least  $2c\ell$  credits. Thus,  $\mathcal{B}$  has at least  $c \cdot \max(\text{diam}(\mathcal{B}), \ell/2)$  credits. ■

We show how to pay for edges of  $E_i$  in  $A_6$  and maintain diameter-credit invariants of long clusters. We use  $cr(\mathcal{X})$  to denote the total credit of  $\epsilon$ -clusters of a set of  $\epsilon$ -clusters  $\mathcal{X}$ .

**Claim 13.** Let  $\mathcal{B}$  be a long cluster. If  $c = \Omega(\frac{g}{\epsilon^3})$  and  $g \geq 8$ , we can maintain diameter-credit invariants of  $\mathcal{B}$  and pay for edges in  $A_6$  incident to  $\epsilon$ -clusters in  $\mathcal{B}$  using credits of  $\epsilon$ -clusters in  $\mathcal{B}$ .

*Proof:* By construction,  $\mathcal{B}$  has effective diameter at most  $4\ell$ . By Observation 3,  $\mathcal{B}$  has diameter at most  $8\ell$ . Thus,  $\mathcal{B}$  satisfies invariant DC2 if  $g \geq 8$ . Since  $\mathcal{B}$  is a long cluster, it has at least  $\frac{2g}{\epsilon} + 1$   $\epsilon$ -clusters. Let  $\mathcal{S}$  be a set of  $\frac{2g}{\epsilon}$   $\epsilon$ -clusters in  $\mathcal{B}$  and  $X$  be an  $\epsilon$ -cluster in  $\mathcal{B} \setminus \mathcal{S}$ . Let  $\mathcal{R} = \{X\}$ . We save credits of  $\mathcal{S}$  for maintaining invariant DC1 of  $\mathcal{B}$  and use credits of  $\mathcal{R}$  to pay for edges of  $A_6$  incident to  $\epsilon$ -clusters in  $\mathcal{S} \cup \mathcal{R}$ . Since  $|\mathcal{S} \cup \mathcal{R}| = \frac{2g}{\epsilon} + 1$  and  $\epsilon$ -clusters in  $\mathcal{S} \cup \mathcal{R}$  are low-degree, there are at most  $O(\frac{g}{\epsilon^2})$  edges of  $A_6$  incident to  $\epsilon$ -clusters in  $\mathcal{S} \cup \mathcal{R}$ . By invariant DC1 for level  $i - 1$ ,  $\mathcal{R}$  has at least  $\frac{c\ell}{2}$  credits which is sufficient to pay for  $O(\frac{g}{\epsilon^2})$  edges of  $A_6$  when  $c = \Omega(\frac{g}{\epsilon^3})$ . We let other  $\epsilon$ -clusters in  $\mathcal{B} \setminus (\mathcal{S} \cup \mathcal{R})$  pay for their incident edges of  $A_6$  using their credits. This is sufficient when  $c = \Omega(\frac{1}{\epsilon^2})$  since each  $\epsilon$ -cluster is incident to at most  $\frac{20}{\epsilon}$  edges and has at least  $\frac{c\ell}{2}$  credits.

We use credits of  $\mathcal{S}$  to maintain invariant DC1. Since  $|\mathcal{S}| = \frac{2g}{\epsilon}$  and each  $\epsilon$ -clusters has at least  $\frac{c\ell}{2}$  credits,  $cr(\mathcal{S}) \geq g\ell$ . Since  $\text{diam}(\mathcal{B}) \leq g\ell$  by DC2,  $cr(\mathcal{S}) \geq c\text{diam}(\mathcal{B})$ . Thus,  $cr(\mathcal{S}) \geq c\max(\text{diam}(\mathcal{B}), \ell/2)$ ; invariant DC1 is satisfied. ■

Let  $\mathcal{C}(X)$  be a cluster in  $\mathcal{C}$  before Phase 4. Let  $\mathcal{C}'(X)$ ,  $\mathcal{C}''(X)$  and  $\mathcal{C}'''(X)$  be the corresponding clusters that are augmented from  $\mathcal{C}(X)$  in Phase 4a, Step 1 of Phase 4b and Step 2 of Phase 4b, respectively. It could be that any two of three clusters are the same.

By construction in Phase 4a, LD-components are attached to  $\mathcal{C}(X)$  via MST edges. Recall each LD-component has

effective diameter at most  $4\ell$  and hence, diameter at most  $8\ell$  by Observation 3. Thus,  $\text{diam}(\mathcal{C}'(X)) - \text{diam}(\mathcal{C}(X)) \leq 16\ell + 2$ . By construction in Step 1 of Phase 4b, subpaths of effective diameter at most  $4b\ell$  are attached to  $\mathcal{C}'(X)$  via MST edges. Thus,  $\text{diam}(\mathcal{C}''(X)) - \text{diam}(\mathcal{C}'(X)) \leq 16\ell + 2$ . We have:

**Claim 14.**  $\text{diam}(\mathcal{C}''(X)) - \text{diam}(\mathcal{C}(X)) \leq 32\ell + 4$ .

By construction in Step 2 of Phase 4b, subpaths of HD-paths are attached to  $\mathcal{C}''(X)$  via edges of  $E_i$ . Since attached subpaths have effective diameter at most  $4\ell$ , by Observation 3, we have:

**Claim 15.**  $\text{diam}(\mathcal{C}'''(X)) - \text{diam}(\mathcal{C}''(X)) \leq 18\ell$

We are now ready to show invariant DC2 for  $\mathcal{C}'''(X)$ .

**Claim 16.**  $\text{diam}(\mathcal{C}'''(X)) \leq g\ell$  when  $g \geq 70$ .

*Proof:* The claim follows directly from Claim 1, Claim 3, Claim 5 and Claim 9. We defer details to the full version. ■

Recall we show how to pay for edges in  $A_1, A_2, A_3, A_4, A_6$  before. It remains to show how to pay for edges in  $A_5 \cup A_7 \cup A_8$ . We first consider edges in  $A_5 \cup A_7$ . Recall  $\mathcal{S}(X) = \lfloor (\mathcal{D} \cap \mathcal{C}(X)) \rfloor_{2g/\epsilon}$  where  $\mathcal{D}$  is the cluster-diameter path. We call  $\epsilon$ -clusters in  $\mathcal{C}''(X) \setminus \mathcal{C}(X)$  *augmenting*  $\epsilon$ -clusters. Let  $\mathcal{S}''(X) = \lfloor (\mathcal{D} \cap (\mathcal{C}''(X) \setminus \mathcal{C}(X))) \rfloor_{2g/\epsilon}$  be the set of augmenting  $\epsilon$ -clusters that are in the diameter path  $\mathcal{D}$ . We save credits of  $\epsilon$ -clusters in  $\mathcal{S}''(X)$  for maintaining DC1 and let other augmenting  $\epsilon$ -clusters release their credits.

**Claim 17.** If  $c = \Omega(\frac{g}{\epsilon^2})$ , we can buy edges in  $A_5 \cup A_7$  using  $\frac{c\ell}{3}$  credits of the cluster centers and credits of releasing augmenting  $\epsilon$ -clusters.

*Proof:* We use  $\frac{c\ell}{6}$  credits of  $X$  to pay for edges of  $A_5 \cup A_7$  incident to  $\epsilon$ -clusters in  $\mathcal{S}''(X)$ . Recall each  $\epsilon$ -cluster is incident to at most  $\frac{20}{\epsilon}$  edges of  $E_i$  since it is low-degree. Thus,  $\epsilon$ -clusters in  $\mathcal{S}''(X)$  are incident to most  $O(\frac{g}{\epsilon^2})$  edges of  $A_5 \cup A_7$ . Hence,  $\frac{c\ell}{6}$  credits suffice when  $c = \Omega(\frac{g}{\epsilon^2})$ . We let releasing augmenting  $\epsilon$ -clusters of LD-components to pay for their incident edges of  $A_5$ . This is sufficient when  $c = \Omega(\frac{1}{\epsilon^2})$ . Thus, all edges of  $A_5$  are paid. We now turn to edges of  $A_7$ .

Let  $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$  be three segments of a HD-path  $\mathcal{P}$  in Step 1 where  $\mathcal{P}_1, \mathcal{P}_2$  are affixes of  $\mathcal{P}$  and  $\mathcal{P}_3$  is the augmenting subpath of  $\mathcal{P}$ . It could be that  $\mathcal{P}_1 = \mathcal{P}_3$  or  $\mathcal{P}_2 = \mathcal{P}_3$ . Since edges of  $E_i$  incident to long clusters are paid in Claim 13,  $\epsilon$ -clusters of  $\mathcal{P}_i$ ,  $1 \leq i \leq 2$ , are incident to unpaid edges of  $E_i$  only when  $\mathcal{P}_i$  is a short cluster and thus, incident to at most  $O(\frac{g}{\epsilon^2})$  edges of  $A_7$ . Note that in Claim 12, we use all credits of  $\epsilon$ -clusters and MST edges of  $\mathcal{P}_i$  to maintain diameter-credit invariants and we need to pay for edges of  $A_7$  incident to  $\mathcal{P}_i$ . We consider two cases:

1) If  $\mathcal{P}_3 \cap \mathcal{S}''(X) = \emptyset$ , then  $\epsilon$ -clusters in  $\mathcal{P}_3$  are releasing.

Recall  $\mathcal{P}_3$  has effective diameter at least  $2\ell$ . We let each  $\epsilon$ -cluster in  $\mathcal{P}_3$  pay for its incident edges of  $A_7$  using half of its credits, which is at least  $\frac{c\ell}{4}$  by invariant DC1 for level  $i-1$ . This amount of credits is enough when  $c = \Omega(\frac{1}{\epsilon^2})$ . The total remaining credit from  $\epsilon$ -clusters of  $\mathcal{P}_3$  is at least  $c\ell$ , that is sufficient to pay for  $O(\frac{g}{\epsilon^2})$  edges of  $A_7$  incident to  $\epsilon$ -clusters of  $\mathcal{P}_1 \cup \mathcal{P}_2$  when  $c = \Omega(\frac{g}{\epsilon^2})$ .

- 2) If  $\mathcal{P}_3 \cap \mathcal{S}''(X) \neq \emptyset$ , we use  $c\ell/6$  credits of the center  $X$  of  $\mathcal{C}''(X)$  to pay for edges of  $A_7$  incident to  $\epsilon$ -clusters in  $\mathcal{X} = (\mathcal{P}_1 \cap \mathcal{S}''(X)) \cup \mathcal{P}_2 \cup \mathcal{P}_3$ . Recall  $\mathcal{S}''(X)$  has at most  $\frac{2g}{\epsilon}$   $\epsilon$ -clusters,  $\mathcal{X}$  has at most  $\frac{6g}{\epsilon}$   $\epsilon$ -clusters. Thus,  $\epsilon$ -clusters in  $\mathcal{X}$  are incident to at most  $\frac{120g}{\epsilon^2}$  edges in  $A_7$ . Since there are at most two augmenting subpaths that contain  $\epsilon$ -clusters of the cluster diameter path  $\mathcal{D}$ ,  $X$  only need to pay for at most  $\frac{240g}{\epsilon^2} = O(\frac{g}{\epsilon^2})$  edges. Thus,  $\frac{c\ell}{6}$  credits are sufficient if  $c = \Omega(\frac{g}{\epsilon^2})$ . Other  $\epsilon$ -clusters of  $\mathcal{P}_3 \setminus \mathcal{S}''(X)$  are releasing and we can use their released credits to pay for their incident edges of  $A_7$ .  $\blacksquare$

We now show how to pay for edges of  $A_8$  which consists of edges of  $E_i$  incident to  $\epsilon$ -clusters in Step 2 of Phase 4b. By Observation 5,  $\mathcal{C}(X)$  is formed in Phase 1. Let  $\mathcal{S}'''(X)$  be augmenting  $\epsilon$ -clusters in  $\mathcal{D}$  of  $\mathcal{C}'''(X)$  that are not in  $\mathcal{S}(X) \cup \mathcal{S}''(X)$ . We save credit of  $\mathcal{S}'''(X)$  for maintaining DC1 and let other augmenting  $\epsilon$ -clusters release their credits.

**Claim 18.** *If  $c = \Omega(\frac{g}{\epsilon^2})$ , we can pay for edges of  $A_8$  incident to  $\epsilon$ -clusters in  $\mathcal{C}'''(X)$  using credits of releasing  $\epsilon$ -clusters and  $c\ell$  credits of the center  $X$ .*

*Proof:* Since augmenting  $\epsilon$ -clusters are low-degree, each augmenting  $\epsilon$ -cluster is incident to at most  $\frac{20}{\epsilon}$  edges of  $A_8$ . When  $c = \Omega(\frac{1}{\epsilon^2})$ ,  $\frac{c\ell}{2}$  credits of each releasing  $\epsilon$ -cluster suffice to buy their incident edges of  $A_8$ .

By construction, the augmenting subpath  $\mathcal{P}'$  in Step 1 of Phase 4b is a short path. Since the cluster-diameter path  $\mathcal{D}$  contains  $\epsilon$ -clusters of at most two short subpaths of HD-paths,  $|\mathcal{S}'''(X)| \leq \frac{4g}{\epsilon}$ . Thus, there are at most  $O(\frac{g}{\epsilon^2})$  edges of  $A_8$  incident to non-releasing  $\epsilon$ -clusters. Hence,  $c\ell$  credits of  $X$  suffice to pay for such edges when  $c = \Omega(\frac{g}{\epsilon^2})$ .  $\blacksquare$

It remains to maintain invariant DC1 for clusters in  $\mathcal{C}$ . We have:

**Claim 19.** *If any of the sets  $\mathcal{S}(X)$ ,  $\mathcal{S}''(X)$  and  $\mathcal{S}'''(X)$  has at least  $\frac{2g}{\epsilon}$   $\epsilon$ -clusters, then  $\mathcal{C}'''(X)$  satisfies invariant DC1.*

*Proof:* Suppose, w.l.o.g, say  $\mathcal{S}(X)$  has at least  $\frac{2g}{\epsilon}$   $\epsilon$ -clusters. Then, by DC1 for level  $i-1$ , the total credits of  $\epsilon$ -clusters in  $\mathcal{S}(X)$  is at least:

$$\frac{2g}{\epsilon} \cdot \frac{c\ell}{2} = g\ell$$

which is at least  $c \cdot \max(\text{diam}(\mathcal{C}'''(X)), \ell/2)$  since  $\text{diam}(\mathcal{C}'''(X)) \leq g\ell$  by Claim 16 and  $g > 1$ .  $\blacksquare$

**Claim 20.** *If  $c = \Omega(\frac{g}{\epsilon^3})$ , we can maintain invariant DC1 of  $\mathcal{C}'''(X)$  using credits of  $\epsilon$ -clusters and MST edges in  $\mathcal{D}$  and the credits of the cluster center  $X$ .*

*Proof:* By Claim 19, credits of all  $\epsilon$ -clusters and MST edges of  $\mathcal{D}$  are saved for maintaining DC1. We prove the claim by case analysis.

**Case 1:  $\mathcal{C}(X)$  is formed in Phase 1.** Recall  $\mathcal{D}$  contains at most six edges of  $E_i$  where four edges of  $E_i$  are in  $\mathcal{C}(X)$  and two more edges of  $E_i$  are by the augmentation in Step 2 of Phase 4b. We use  $6c\ell$  credits from  $X$  and credits of  $\epsilon$ -clusters and MST edges in  $\mathcal{D}$ . The total credit is:

$$\begin{aligned} 6c\ell + c(|\text{MST}(\mathcal{D})| + \text{ediam}(\mathcal{D})) &\geq c \cdot \text{diam}(\mathcal{D}) \\ &= c \cdot \text{diam}(\mathcal{C}'''(X)) \end{aligned}$$

Since  $\mathcal{C}'''(X)$  contains an edge in  $E_i$ ,  $\text{diam}(\mathcal{C}'''(X)) \geq \ell/2$ . Thus,  $c \cdot \text{diam}(\mathcal{C}'''(X)) \geq c\ell/2$ .

To complete the proof, we need to argue that  $X$  has non-negative credits after paying for edges of  $E_i$  and maintaining invariant DC1 of  $\mathcal{C}'''(X)$ . Recall  $X$  initially has  $9c\ell$  credits by Observation 2 and loses: (a)  $c\ell$  credits in Claim 2, (b)  $\frac{c\ell}{3}$  credits in Claim 17, (c)  $c\ell$  credits to pay for the edges of  $A_8$  incident to non-releasing augmenting  $\epsilon$ -clusters in Step 2 of Phase 4b and (d)  $6c\ell$  credits for maintaining DC1 of  $\mathcal{C}'''(X)$ . Thus,  $X$  still has:

$$9c\ell - 8c\ell - \frac{c\ell}{3} = c(1 - \frac{\epsilon}{3})\ell$$

which is non-negative since  $\epsilon < 1$ .

**Case 2:  $\mathcal{C}(X)$  is formed in Phase 2.** Recall the center  $X$  collects at least  $\frac{c\ell}{2}$  from a neighbor  $Y$  of  $X$  ( $\mathcal{R}(X) = \{Y\}$ ). We observe that credits in  $X$  is taken totally by at most  $\frac{c\ell}{2}$  in Claim 4 and Claim 17. Thus, the center still has non-negative credits after buying incident edges  $E_i$  when  $c = \Omega(\frac{g}{\epsilon^3})$ .

Since credits of  $\epsilon$ -clusters and MST edges in  $\mathcal{D}$  are reserved and  $\mathcal{D}$  does not contain any edge of  $E_i$ , the total reserved credit is:

$$c(|\text{MST}(\mathcal{D})| + \text{ediam}(\mathcal{D})) \geq c \cdot \text{diam}(\mathcal{D}) = c \cdot \text{diam}(\mathcal{C}''(X)) \quad (8)$$

It remains to argue that  $\mathcal{C}''(X)$  has at least  $\frac{c\ell}{2}$  credits. Note that we do not have lower bound on the diameter of  $\mathcal{C}''(X)$  as in other cases. Let  $\mathcal{X}$  be the set of releasing  $\epsilon$ -clusters of  $\mathcal{C}(X)$  and  $cr(\mathcal{X})$  be the total credits of  $\epsilon$ -clusters in  $\mathcal{X}$ . Since  $\text{ediam}(\mathcal{C}(X)) \geq 2\ell$ , we have:

$$cr(\mathcal{X}) + cr(\mathcal{S}(X)) \geq 2c\ell \quad (9)$$

Recall half credit of  $\mathcal{X}$  is taken in Claim 4. We use the remaining half to guarantee that the credit of  $\mathcal{C}''(X)$  is at least  $c\ell/2$ .

**Case 3:  $\mathcal{C}(X)$  is formed in Case 1 of Phase 3a.** Recall (in Claim 7) the center  $X$  collects at least  $\frac{c\ell}{2}$  credits if  $\mathcal{D}$  does not contain  $e$  (we are using notation in Case 1 Phase 3a) and at least  $cw(e) + \frac{c\ell}{2}$  credits if  $\mathcal{D}$  contains  $e$ . We observe that credits in  $X$  is taken totally by at most  $\frac{c\ell}{2}$  in Claim 8

and Claim 14. By construction,  $\mathcal{D}$  contains at most one edge of  $E_i$  which is  $e$  (in this case  $X$  has at least  $c \cdot w(e) + \frac{\epsilon \ell}{2}$  credits). Thus, the remaining credits of  $X$  and credits from reserved  $\epsilon$ -clusters and MST edges in  $\mathcal{D}$  are sufficient for maintaining invariant DC1. Since  $\text{diam}(\mathcal{C}''(X)) \geq \ell/2$  by Claim 5,  $c \cdot \text{diam}(\mathcal{C}''(X)) \geq c\ell/2$ .

**Case 4:  $\mathcal{C}(X)$  is formed in Case 2 of Phase 3a or in Phase 3b.** Recall in Claim 10, we argue that the center of cluster  $X$  collects at least  $2c(1 - g\epsilon)\ell$  credits. By construction,  $\mathcal{D}$  can contain at most one edge of  $E_i$ , which connects two cluster paths in Case 2 of Phase 3a or Phase 3b. We observe that credits in  $X$  is taken totally by at most  $c(1 - 3g\epsilon - \epsilon/3)\ell$  in Claim 11 and Claim 17. Thus,  $X$  has at least:

$$c(2 - 2g\epsilon)\ell - c(1 - 3g\epsilon - \epsilon/3)\ell > c\ell$$

remaining credits. That implies the remaining credits of  $X$  and credits from reserved  $\epsilon$ -clusters and MST edges in  $\mathcal{D}$  are sufficient for maintaining invariant DC1. Since  $\text{diam}(\mathcal{C}''(X)) \geq \ell/2$  by Claim 9,  $c \cdot \text{diam}(\mathcal{C}''(X)) \geq c\ell/2$ . ■

*Proof of Lemma 1:* Recall in the beginning of Phase 4, we assume that  $\mathcal{C} \neq \emptyset$  after Phase 3 and in this case, we already paid for every edges of  $E_i$  with:

$$c = \max\left(\frac{\Theta(g)}{\epsilon^3}, \frac{\Theta(\sigma_H)}{\epsilon}\right) = O\left(\frac{\sigma_H}{\epsilon^3}\right)$$

and  $\epsilon$  sufficiently small.

We only need to consider the case when  $\mathcal{C} = \emptyset$  after Phase 3. We have:

**Observation 6.** *The case when  $\mathcal{C} = \emptyset$  after Phase 3 only happens when: (i) there is a single cluster-path  $\mathcal{P}$  that contains all  $\epsilon$ -clusters, (ii) every edge of  $E_i$  is incident to an  $\epsilon$ -cluster in an affix of  $\mathcal{P}$  of effective diameter at most  $2\ell$  and (iii)  $\epsilon$ -clusters of  $\mathcal{P}$  are low-degree in  $\mathcal{K}(\mathcal{C}_\epsilon, E_i)$ .*

We greedily break  $\mathcal{P}$  into subpath of  $\epsilon$ -clusters of effective diameter at least  $2\ell$  and at most  $4\ell$  as in Phase 4b and form a new cluster from each subpath. Recall a long cluster is formed from a subpath containing at least  $\frac{2g}{\epsilon} + 1$   $\epsilon$ -clusters. Let  $\mathcal{P}'$  be a subpath of  $\mathcal{P}$ . If  $\mathcal{P}'$  is long, we can both buy edges of  $E_i$  incident to  $\epsilon$ -clusters of  $\mathcal{P}'$  and maintain two diameter-credit invariants as in Claim 13. If  $\mathcal{P}'$  is short, we use credits of  $\epsilon$  and MST edges of  $\mathcal{P}'$  to maintain DC1. Recall  $\mathcal{P}'$  has effective diameter at least  $\ell$ , thus, has at least  $c\ell$  credits by DC1 for level  $i - 1$ . That implies  $c \cdot \text{diam}(\mathcal{P}') \geq c \max(\frac{\ell}{2}, \text{diam}(\mathcal{P}'))$ .

We put remaining unpaid edges of  $E_i$  to the holding bag  $B$ . Recall unpaid edges of  $E_i$  must be incident to  $\epsilon$ -clusters of short clusters, which are affixes of  $\mathcal{P}$ . By Observation 6,  $B$  holds at most  $O(\frac{g}{2\epsilon}) = O(\frac{1}{2\epsilon^2})$  edges of  $E_i$ . Thus, the

total weight of edges of  $B$  in all levels is at most:

$$\begin{aligned} O\left(\frac{1}{\epsilon^2}\right) \sum_i \ell_i &\leq O\left(\frac{1}{\epsilon^2}\right) \ell_{\max} \sum_i \epsilon^i \\ &\leq O\left(\frac{1}{\epsilon^2}\right) w(\text{MST}) \sum_i \epsilon^i \\ &\leq O\left(\frac{1}{\epsilon^2}\right) w(\text{MST}) \frac{1}{1 - \epsilon} \\ &= O\left(\frac{1}{\epsilon^2} w(\text{MST})\right) \end{aligned} \quad (10)$$

In the above equation,  $\ell_{\max} = \max_{e \in S} \{w(e)\}$  ■

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## APPENDIX

### A. Notation and definitions

Let  $G(V(G), E(G))$  be a connected and undirected graph with a positive edge weight function  $w : E(G) \rightarrow \mathbb{R}^+ \setminus \{0\}$ . We denote  $|V(G)|$  and  $|E(G)|$  by  $n$  and  $m$ , respectively. Let  $\text{MST}(G)$  be a minimum spanning tree of  $G$ ; when the graph is clear from the context, we simply write  $\text{MST}$ . A walk of length  $p$  is a sequence of alternating vertices and edges  $\{v_0, e_0, v_1, e_1, \dots, e_{p-1}, v_p\}$  such that  $e_i = v_i v_{i+1}$

for every  $i$  such that  $1 \leq i \leq p-1$ . A path is a *simple walk* where every vertex appears exactly once in the walk. For two vertices  $x, y$  of  $G$ , we use  $d_G(x, y)$  to denote the shortest distance between  $x$  and  $y$ . Let  $S$  be a subgraph of  $G$ . We define  $w(S) = \sum_{e \in E(S)} w(e)$ . Let  $X \subseteq V(G)$  be a set of vertices. We use  $G[X]$  to denote the subgraph of  $G$  induced by  $X$ . Let  $Y \subseteq E(G)$  be a subset of edges of  $G$ . We denote the graph with vertex set  $V(G)$  and edge set  $Y$  by  $G[Y]$ . We call a graph  $K$  a *minor* of  $G$  if  $K$  can be obtained from  $G$  from a sequences of edge contraction, edge deletion and vertex deletion operations. A graph  $G$  is  $H$ -minor-free if it excludes a fixed graph  $H$  as a minor. If  $G$  excludes a fixed graph  $H$  as a minor, it also excludes the complete  $h$ -vertex graph  $K_h$  as a minor where  $h = |V(H)|$ .

**Observation 7.** *If a graph  $G$  excludes  $K_h$  as a minor for  $h \geq 3$ , then any graph obtained from  $G$  by subdividing an edge of  $G$  also excludes  $K_h$  as a minor.*

### B. Greedy spanners

A subgraph  $S$  of  $G$  is a  $(1 + \epsilon)$ -spanner of  $G$  if  $V(S) = V(G)$  and  $d_S(x, y) \leq (1 + \epsilon)d_G(x, y)$  for all  $x, y \in V(G)$ . In the greedy algorithm by Althöfer et al. [1] to find a  $(1 + \epsilon)$ -spanner of  $G$ , edges are added in order of increasing weight while doing so decreases the distance between their endpoints by a  $1 + \epsilon$  factor. Since the greedy algorithm is a relaxation of Kruskal's algorithm,  $\text{MST}(G) = \text{MST}(S)$ .

### C. Reduction to unit-weight MST edges

We adapt the reduction technique of Chechik and Wulff-Nilsen [6] to analyze the increase in lightness due to this simplification for  $H$ -minor-free graphs. Let  $G$  be the input graph and let  $w : E(G) \rightarrow \mathbb{R}^+$  be the edge weight function for  $G$ . Let  $\bar{w} = \frac{w(\text{MST})}{n-1}$  be the average weight of the MST edges. We do the following. First, we round up the weight of each edge of  $E(G)$  to an integral multiple of  $\bar{w}$ . Second, we subdivide each MST edge so that each resulting edge has weight exactly  $\bar{w}$ . Let  $G'$  be the resulting graph. Third, we scale down the weight of every edge by  $\bar{w}$ . Let  $w'$  be the resulting edge weights of  $G'$ .  $G'$  is minor-free by Observation 7. Fourth, we find a  $(1 + \epsilon)$ -spanner  $S'$  of  $G'$ . Finally, let  $S$  be a graph on  $V(G)$  with edge set equal to the union of  $E(S') \cap E(G)$ , the edges of  $\text{MST}(G)$ , and every edge  $e$  in  $G$  of weight  $w(e) \leq \frac{\bar{w}}{\epsilon}$ .

**Lemma 2.** *If  $S'$  is a  $(1 + \epsilon)$ -spanner of  $G'$  with lightness  $f(\epsilon)$ , then  $S$  is a  $(1 + O(\epsilon))$ -spanner of  $G$  with lightness  $2f(\epsilon) + O(\sigma_H/\epsilon)$ .*

*Proof:* The proof is an adaption of the proof of Chechik and Wulff-Nilsen [6] that we defer to the full version. ■

By Lemma 2, we may assume that all edges of  $\text{MST}(G)$  have weight 1. We find the  $(1 + \epsilon)$ -spanner  $S$  of  $G$  by using the greedy algorithm. Thus, the stretch condition of  $S$  is satisfied.