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High dimensional expanders imply agreement expanders

Irit Dinur
Department of Computer Science and Applied Mathematics
Weizmann Institute
Rehovot, Israel
Email: irit.dinur@weizmann.ac.il

Tali Kaufman
Department of Computer Science
Bar-Ilan University
Ramat Gan, Israel
Email: kaufmant@mit.edu

***Abstract*—We show that high dimensional expanders imply derandomized direct product tests, with a number of subsets that is *linear* in the size of the universe.**

Direct product tests belong to a family of tests called *agreement tests* that are important components in PCP constructions and include, for example, low degree tests such as line vs. line and plane vs. plane.

For a generic hypergraph, we introduce the notion of *agreement expansion*, which captures the usefulness of the hypergraph for an agreement test. We show that explicit bounded degree agreement expanders exist, based on Ramanujan complexes.

to Oded Goldreich, with love and admiration, on the occasion of his 60th birthday

***Keywords*-high dimensional expanders; agreement tests; direct product; direct sum**

I. Introduction

This paper shows that derandomized direct product tests can be obtained from high dimensional expanders. Direct product tests fit into a more general family of tests called *agreement tests* which include low degree agreement tests such as the plane vs. plane [RS97] and line vs. line test [AS97], and were first abstracted by Goldreich and Safra in [GS97]. These are important components in the construction of nearly all probabilistically checkable proofs (PCPs) and capture a certain local to global behavior.

PCPs and agreement tests: In all efficient PCP constructions we break a proof into small pieces, use inefficient PCPs (i.e. PCP encodings that incur a large blowup) to encode each small piece, and then through an *agreement test* put the pieces back together. The agreement test is needed because given the collection of encoded pieces, there is no guarantee that the different pieces come from the same underlying global proof, i.e. that the proofs of each piece can be "put back together again". The PCP system must ensure this through *agreement testing*: take two pieces that have some overlap, and check that they agree. For this idea to work we must be able to pass from good pairwise (local) agreement to consistency with a single global proof.

That is, the scheme should have two features,

1) "Sampling property": the collection of subsets $X = \{s \subset [n]\}$ should be a good sampler, so that any set of μn elements are seen by almost all sets $s \in X$ with the correct proportion (i.e. each s should see roughly $\mu|s|$ elements). We want the subsets in X to be small, and we want the number of subsets to be not too large.
2) "Agreement expansion": There should be an agreement test for X. An agreement test is a distribution over say pairs of subsets such that, roughly speaking, if a given collection has high pairwise agreement on average, then it is close to being consistent with some global string.

We initiate a study of the following general question: which collections of subsets X satisfy the two above properties? We formulate this as a type of high dimensional expansion of X which we term *agreement expansion*, and show a construction of such an X that has only $O(n)$ subsets.

A. Agreement Expansion - definition and main theorem

Let $[n]$ be a ground set and let $X(d)$ be a collection of subsets of $[n]$ which, for concreteness, can be the set of all d-dimensional faces of a simplicial complex X on n vertices.

A *local assignment* is a collection $f = \{f_s\}$ of local functions $f_s \in \{0,1\}^s$, one per subset $s \in X(d)$. To be clear, f_s specifies a $0/1$ value for each $x \in s$. It has no information about elements $x \notin s$ so it is "local". A local assignment is called *global* if there is a global function $g : [n] \to \{0,1\}$ such that

$$\forall s \in X, \qquad f_s \equiv g|_s$$

We denote by $\mathrm{Global} = \mathrm{Global}(X(d))$ the set of global assignments over $X(d)$.

An *agreement-check* for a pair of subsets s_1, s_2 checks whether their local functions agree, denoted $f_{s_1} \sim f_{s_2}$. Formally,

$$f_{s_1} \sim f_{s_2} \qquad \Leftrightarrow \qquad \forall x \in s_1 \cap s_2, \quad f_{s_1}(x) = f_{s_2}(x).$$

It is easy to see that any local assignment that is global passes all agreement checks. The converse is also true: a local assignment that passes all agreement checks must be global.

An *agreement test* is specified by giving a distribution $\mathcal{D}$ over pairs of subsets s_1, s_2. We define the agreement of a

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local assignment to be the probability of agreement,

$$\mathrm{agree}_{\mathcal{D}}(f) \doteq \Pr_{s_1,s_2\sim D}\left[f_{s_1} \sim f_{s_2}\right] .$$

An agreement theorem shows that if f is a local assignment with $\mathrm{agree}_{\mathcal{D}}(f) > 1-\varepsilon$ then f is $1-O(\varepsilon)$ close to a global assignment. Such a theorem relates two ways of measuring the closeness of f to being global: the actual distance $\mathrm{dist}(f, \mathrm{Global})$ and the distance we observe when looking at the "boundary", namely the checks that fail. The latter we denote by $\mathrm{disagree}_{\mathcal{D}}(f) \doteq 1 - \mathrm{agree}_{\mathcal{D}}(f)$. This gives rise to the following definition of *agreement expansion* of X and $\mathcal{D}$ as a type of "Rayleigh quotient",

$$\Upsilon(X, \mathcal{D}) \;=\; \inf_{f} \frac{\mathrm{disagree}_{\mathcal{D}}(f)}{\mathrm{dist}(f, \mathrm{Global}(X))}, \tag{1}$$

where the infimum is over all possible non-global assignments f. A lower bound on Υ implies that when the disagreement is small then the distance to global is also small. This means that the test "works" in that it provides a good approximation to the actual distance of f from being global. We are now ready to provide the formal definition of an agreement expander,

Definition I.1 (Agreement expander)**.** A d-dimensional complex X is a *c-agreement-expander* if its underlying graph[1] is connected and if there exists a distribution $\mathcal{D}$ such that

$$\Upsilon(X, \mathcal{D}) \geq c.$$

In other words, for every $f = \{f_s\}_{s\in X(d)}$, if

$$\mathrm{agree}_{\mathcal{D}}(f) \geq 1-\varepsilon,$$

then there is $g : X(0) \to \{0,1\}$ such that

$$\Pr_s[f_s = g|_s] \geq 1 - \varepsilon/c.$$

The "in other words" part of the definition is a statement that is the bread and butter of property testing: if a test passes with probability at least $1-\varepsilon$ then the object is $1-\varepsilon'$ to the property. Thus, proving that a certain pair $(X, \mathcal{D})$ is an agreement expander is equivalent to showing that the property $\mathrm{Global}(X)$ is testable with $\mathcal{D}$ as the test distribution.

For a $d+1$ dimensional complex, there is one arguably most natural distribution $\mathcal{D}_{\uparrow}$ over pairs of subsets in $X(d)$, which we shall call the *one-up distribution*. It is the distribution obtained by choosing a random $d{+}1$ dimensional face r, and then two random d-faces in it $s_1, s_2 \subset r$ independently. (The name is explained from the point of view of s_1: we move "one-dimension-up" towards r and then to s_2).

[1]The graph underlying a complex has an edge between u and v whenever they belong to a common face.

Definition I.2 (One-up agreement-expander)**.** A $d+1$-dimensional complex X is a *c-one-up agreement-expander* if its underlying graph is connected and if

$$\Upsilon(X, \mathcal{D}_{\uparrow}) \geq c \cdot \frac{1}{d}$$

In other words, for every $f = \{f_s\}_{s\in X(d)}$, if $\mathrm{agree}_{\mathcal{D}_{\uparrow}}(f) \geq 1-\varepsilon/d$, then there is a global $g : X(0) \to \{0,1\}$ such that

$$\Pr_s[f_s = g|_s] \geq 1 - \varepsilon/c.$$

For the one-up distribution $\mathcal{D}_{\uparrow}$ the factor $\frac{1}{d}$ is necessary as can be seen from its presence also in the complete d-dimensional complex on n vertices (whose d-faces are all $d+1$ element subsets of the vertices). We prove,

Theorem I.3 (Main)**.** *There exists a constant $c > 0$ and an explicit infinite family of bounded degree complexes that are c-agreement expanders, and c-one-up agreement expanders.*

This theorem implies a very strong derandomization of direct product tests. Previously, the only known agreement test with comparable parameters was known for the complete d-dimensional complex [DS14] which has $\approx n^{d+1}$ subsets. In comparison, the construction here has only $O_d(n)$ subsets. There are some known derandomizations of direct product tests [GS97], [IKW09] (but none have a *linear* number of subsets) which we discuss later in the introduction.

B. Agreement expansion from high dimensional expanders

Our main theorem shows that high dimensional expansion implies agreement expansion. We begin by introducing high dimensional expanders.

High dimensional expansion of simplicial complexes: A d-dimensional simplicial complex X is a hypergraph on n vertices such that for every hyperedge s that belongs to the hypergraph, all of its subsets also belong to the hypergraph. Hyperedges of a simplicial complex are also called *faces*, and the dimension of a face is one less than its cardinality. Simplicial complexes are viewed as higher dimensional analogs of graphs. It is standard to denote the vertices of the complex by $X(0)$, the edges by $X(1)$ and in general $X(i)$ is the collection of i-dimensional faces, which are subsets of cardinality $i{+}1$. The following two definitions are important,

- The *graph underlying* a complex is simply the graph obtained by keeping only the vertices and the edges of the complex.
- The *link* of a face $s \in X(i)$ in the complex, for $i < d-1$, is itself a complex that is the neighborhood of s, formally defined as

$$X_s = \{t \setminus s \mid \; s \subset t \in X\}.$$

In recent years several distinct notions of high dimensional expansion (of simplicial complexes) have been explored. Coboundary expansion, introduced by Linial and

Meshulam [LM06] and by Gromov [Gro10], is an extension of graph expansion to higher dimensions, from a cohomological perspective. A relaxation of the notion of coboundary expansion which is called *cosystolic* expansion was introduced by [EK16]. Cosystolic expansion was shown [KKL14], [DKW16] to imply the topological overlapping property defined by Gromov [Gro10]. In [KM17] a combinatorial "random-walk" type of expansion was defined. This notion is concerned with the convergence speed of high dimensional random walks to the stationary distribution. Our work is most related to the notion studied in [KM17], since we essentially prove that agreement expansion is implied by high order random walks with *optimal* convergence rate. The work of [KM17] showed that high order random walks in Ramanujan complexes converge rapidly to their stationary distribution, and in this work we derive optimal bounds on the convergence rate.

Marvelous Ramanujan Complexes: Much of the work on high dimensional expanders is motivated by the existence of the Ramanujan complexes whose properties seem to be nearly impossible. More than ten years ago Lubotzky, Samuels, and Vishne [LSV05b] constructed higher-dimensional analogs to the celebrated LPS Ramanujan expander graphs [LPS88]. The LPS graphs come from quotients of the infinite tree. In the algebraic world there is a higher dimensional version of the infinite tree called the Bruhat-Tits building. This lead [LSV05b] to study quotients of this infinite object as a generalization of [LPS88] (both [LPS88] and [LSV05b] rely on deep number theoretic theorems establishing the Ramanujan conjectures for GL_2 by Drinfeld and for GL_d by L. Lafforgue). In [LSV05b] the authors describe an explicit construction of a family of quotients and show that they are simplicial complexes with uniformly bounded degree (i.e. every vertex participates in a bounded number of faces) that look locally exactly like the infinite building.

The technical tools for reasoning about their construction are representation theoretic, and the local similarity to the infinite building. A typical argument would first analyze what's going on in the infinite building and then proceed to prove that the same holds for the quotient. Thus the infinite building is used as a "model" for understanding the quotient. In contrast, we use the complete complex as a model. The advantage is that the complete complex is a finite and simple combinatorial object that is easier to analyze than is the infinite building.

Previous works on the Ramanujan complex [KKL14], [EK16] developed a combinatorial property called there λ-skeleton expansion that these complexes enjoy, and that is much easier to reason about. The power of this property is that on one hand it is easy to understand combinatorially, and on the other hand it is powerful enough to imply interesting results. It is also quite baffling in that except for the [LSV05b] construction there seems to be 'no way' to satisfy the property.

Indeed this property was shown by [KKL14], [EK16] to imply co-systolic expansion which implies the topological overlapping property. Additionally in [KM17] it was shown that for a complex with the λ-skeleton expansion property it holds that all its high order random walks converge rapidly to their stationary distribution.

In this work we continue this approach of trying to capture a simple combinatorial property of simplicial complexes and using that in order to understand further properties of the complex. We introduce an arguably cleaner variant of the λ-skeleton expansion which we term λ-*HD expansion*.

Definition I.4 (λ-HD expander)**.** A d dimensional simplicial complex is a λ-HD expander if for every $i < d-1$ and every $s \in X(i)$, the underlying graph of the link X_s is a λ-spectral expander graph, namely its second largest normalized eigenvalue is bounded in absolute value by λ.

This definition is nice in that the graphs underlying each link are expanding in the most convenient way, namely spectrally. Previous work used a different and more subtle definition (namely, the λ-skeleton expansion) because the LSV complexes are not λ-linik expanders: they only have "one sided" spectral expansion. This is because the links of LSV complexes are d-partite, which means that even though all eigenvalues are at most some small λ, there is a *negative* eigenvalue with magnitude $1/d$. We observe however that it is easy to derive λ-HD expanders from LSV complexes by taking an appropriately small-dimensional skeleton. Relying on the work of [LSV05b], [EK16] we prove

Lemma I.5 (λ-HD expanders exist)**.** *For every $\lambda > 0$ and every $d \in \mathbb{N}$ there exists an explicit infinite family of bounded degree d-dimensional complexes which are λ-HD expanders.*

We remark that for $d > 1$ we know of only one way to obtain such complexes, and in particular there is *no known random construction* that is a λ-HD expander, even for $d = 2$. In contrast, for $d = 1$ they are in abundance.

Returning to agreement expansion: We show that every λ-HD expander has a lower-dimensional skeleton that is an agreement expander. Recall that the k-skeleton of a d-dimensional complex X is the k-dimensional complex obtained by keeping only faces of X of dimensions at most k.

Theorem I.6 (λ-HD expanders give agreement expanders)**.** *There is some constant $c > 0$ such that for every $1 < d \in \mathbb{N}$ and every d^2-dimensional complex that is a λ-HD expander, its k-skeleton for $k \leq d$ is a k-dimensional complex that is a c-agreement expander.*

Our main theorem, Theorem I.3, is an immediate corollary of Lemma I.5 and Theorem I.6. Thus, the bulk of this paper is devoted to proving Theorem I.6.

C. Technical results on the way to proving the criterion for agreement expansion

Our proof of Theorem I.6 has two main components. First, we analyze high order random walks on a λ-HD expander, namely walks that move from k-face to k-face if they belong together to a $(k+t)$-face (see Section III for precise definitions). We show that these walks are strongly mixing in the sense that their spectral behavior is just like that of the analogous random walks on the complete complex, up to an error term bounded by λ.

The second component is a proof of agreement expansion that proceeds by reduction to the agreement expansion of the complete complex. The reduction crucially uses the strong mixing of the high order random walks: we essentially prove that strong mixing of high order random walks suffices for inheriting the agreement expansion of the complete complex.

1) Optimal high order random walks from decreasing differences: The key to our proof is an analysis of random walks that move from k-face to k-face if they belong together to a $k+1$ face.

Theorem I.7 (Spectral gap of one-up random walk)**.** *Let X be a d-dimensional λ-HD expander. For any $k < d$ consider the random walk distribution $\mathcal{D}_\uparrow$ that moves from a k-face s_1 to a random $k+1$ face $r \supset s_1$, and then to a random k-face $s_2 \subset r$. Let $A_{k,k+1}$ be its transition matrix. Then the second largest eigenvalue of $A_{k,k+1}$ is at most $1 - \frac{1}{k+1} + O(k\lambda)$.*

If X is the complete complex, then it is not hard to see that the second largest eigenvalue is $1 - \frac{1}{k+1} - o_n(1)$. So this theorem is "best possible" in the sense that the loss in comparison to the complete complex is negligible. This random walk was analyzed also in [KM17] who proved that the second largest eigenvalue is at most $1 - O(1/k^2)$. However, we will see below that for our application this bound is insufficient and it is crucial to have the spectral gap close to that of the complete complex.

Our proof of Theorem I.7 introduces a method of decreasing differences. We study the variance of a random walk simultaneously in multiple dimensions. It is easy to see that the variance decreases as we go down in dimension, but in fact a stronger property holds. If we look at the difference between the variance of successive dimensions, this difference itself turns out to be (λ-approximately) decreasing as the dimension decreases from k to 0.

2) Samplers from optimal high order random walks: One can write the transition matrix $A_{k,k+1}$ of the one-up distribution as $A_{k,k+1} = M^\dagger M$, where M is the transition matrix taking us from a k face to a random $k+1$ face that contains it. This matrix is denoted $M_{k\nearrow k+1}$ in Section III. It turns out that the adjoint operator $M^\dagger$ is the reverse transition matrix, moving us from a $k+1$ face to a random k face contained in it. By multiplying these matrices for increasing dimensions one after the other we get a description of the t-step random walk: $A_{k,k+t} = B^\dagger B$ where

$$B = M_{k+t-1\nearrow k+1} M_{k+t-2\nearrow k+t-1} \cdots M_{k\nearrow k+1}.$$

Once we have this description of B as a product of the M's, the proof of the next theorem follows directly from the previous one through a telescoping product of the eigenvalue bounds.

Theorem I.8 (Spectral gap of t-up random walk)**.** *Let X be a d-dimensional λ-HD expander. Consider the random walk that moves from a k-face s_1 to a random $k+t$ face $r \supset s_1$, and then to a random k-face $s_2 \subset r$. Let $A_{k,k+t}$ be the transition matrix of this random walk, then the second largest eigenvalue of $A_{k,k+t}$ is at most $\frac{k+1}{t+k+1} + O(tk\lambda)$.*

Recall that we think of λ as arbitrarily small so this means that the eigenvalue above is nearly $\frac{k+1}{t+k+1}$. The fact that $\frac{k+1}{t+k+1}$ can be arbitrarily small as t increases is crucial. If $t \gg k$ then $\lambda(A_{k,k+t})$ is small and this implies that the bipartite graph whose left vertices are the k-faces and whose right vertices are the $(k+t)$-faces is a good sampler. This sampling property drives our proof of agreement expansion. We remark that for the argument above to hold it is crucial that we have a bound on the spectral gap of $\mathcal{D}_\uparrow$ of at most $1 - O(1/k)$. The spectral gap proven in [KM17] which is $1 - O(\frac{1}{k^2})$ only gives a constant bound on $\lambda(A_{k,k+t})$, and not one that tends to zero as $t \to \infty$, and this is insufficient for sampling.

3) Double Sampler: We wish to highlight the combinatorial object that we now have in our hands, and in particular its strong double sampling property. Combining Theorem I.8 with Lemma I.5, we get the following theorem,

Theorem I.9 (Double sampler)**.** *For every $1 < k < d$ and $\gamma > 0$, there is an infinite family of three-partite incidence graphs $\{G(U,V,W,E)\}_n$ with three sets of vertices $U = [n]$, $V \subset \binom{[n]}{k}$, and $W \subset \binom{[n]}{d}$ and non-negative weights on the vertices such that there is an edge between $x \in [n]$ and $s \in V$ iff $x \in s$, and there is an edge between $s \in V$ and $r \in W$ iff $s \subset r$, and such that the following properties hold*

- *$|V| + |W| + |E| = O(n)$ where the constant depends on k, d, γ.*
- *G has the following* **double expansion** *property,*

$$\lambda(G(U,V))^2 \le 1/k + \gamma \quad \text{and} \quad \lambda(G(V,W))^2 \le k/d + \gamma \tag{2}$$

where $G(U,V)$ and $G(V,W)$ are the respective bipartite graphs and λ is the second largest normalized singular value of the appropriate transition matrix.

We refer to (2) as a double sampling property because if $1 \ll k \ll d$ then both spectral gaps are small and this implies good sampling properties: every set $V' \subset V$ is seen with the correct proportion by almost all $w \in W$, and at the same time, every set $U' \subset U$ is seen with the correct proportion by almost all $v \in V$. We know of no other way of obtaining such an *incidence* graph. In fact, it is interesting to compare this to the complete and to the random construction:

- The *complete construction* is a construction as above for which $V = \binom{[n]}{k}$ and $W = \binom{[n]}{d}$. The complete construction has the same spectral gap but $|W| = \binom{n}{d} \gg \Omega(n)$.
- Every *random construction* that is obtained by choosing a sparsification parameter p and then leaving alive edges or vertices with probability p, is easily seen to fail to give these properties. For example, if we choose to keep each $r \in \binom{[n]}{d}$ with probability $O(n/\binom{n}{d})$ so as to leave a linear number of subsets in W, the induced graph will be highly disconnected.

4) Reduction to agreement expansion on the complete complex, using the double sampler: Given a d dimensional complex X we move to a lower dimensional skeleton $X(k)$ consisting of all the k-faces of X, and prove agreement expansion for $X(k)$. Our proof capitalizes on the fact that $X(k)$ contains many copies of the complete complex: one for every $a \in X(d)$ consisting of all sets $\{s \in X(k) \mid s \subset a\}$. (In fact $X(k)$ can be viewed as a "convex combination" of complete complexes). On each complete sub-complex we can apply the agreement expansion theorem of [DS14] to deduce that the sets $s \subset a$ must usually agree with one function $g_a : a \to \{0, 1\}$. We crucially use the double sampling property as follows,

- Sampling from d-sets to k-sets is used to prove that for many d-faces $a \in X(d)$ we have high agreement inside the complete sub-complex of sets contained in a.
- Sampling from k-sets to points is used to move from distance ε between the global majority and g_a on the level of k sets, to a distance of ε/k on the level of points. This shrinkage in distance allows us to deduce that f_s agrees with the majority for *every* $x \in s$.

D. Derandomized direct products and sums

The study of agreement tests continues a line of work on direct product tests which are combinatorial analogs of parallel repetition (a PCP transformation that obtains strong gap amplification). Parallel repetition has a high cost in terms of the blow up which is exactly analogous to the fact that the complete complex on n vertices has $\approx n^k$ k-faces. This lead researchers to look for "derandomized parallel repetition", and unfortunately this has hit a wall in that there are known limitations to generic derandomization [FK95].

Nevertheless, in the world of direct product tests which are the combinatorial analog of parallel repetition derandomization is not ruled out and [IKW09] have come up with a derandomization for which they proved an agreement testing theorem (i.e., in our terms, agreement expansion). This construction was later used [DM11] for a bona fide PCP construction. The difficulty in moving from an agreement test to a PCP construction is in incorporating the arbitrary PCP query structure into the test. In [DM11] this was done by modifying the PCP itself to fit into the agreement expander.

This raises the question of whether a PCP test can be made to fit into the high dimensional expanders that we study here. This would potentially allow using the agreement expansion in a PCP construction. Whether or not this is possible is left to future work, but in the meantime, in this work we show for the first time a derandomized direct product test with a mere *linear* number of subsets. We define, for every simplicial complex X, the *direct product* encoding corresponding to X (see Definition IV.2). Our main theorem can be rephrased as a theorem about the two-query testability of this encoding, see Lemma IV.3.

The direct product encoding has been used for hardness amplification in settings outside of PCPs, and it is possible that this derandomization would be useful there as well.

The direct sum encoding is very related to the direct product one: for every subset s we replace f_s by $\sum_{x \in s} f_s(x) \mod 2$, i.e. we simply take the XOR of the bits. This gives an encoding from n bits to $|X(k)|$ bits, i.e. when we map a function on the vertices to a "cochain" which is a Boolean function on the k-faces. Concretely, we define for every simplicial complex X, the *direct sum* encoding corresponding to X (see Definition IV.1). When $X(k)$ is bounded-degree this encoding has "constant rate" since it maps n bits to $O(n)$ bits. We show in Lemma IV.4 that if $X(k)$ is an agreement expander then this encoding is testable with the minimal number of 3 queries.

Distance amplification code: Note that this encoding is far from an error correcting code because of its poor relative distance, which is about $\frac{k}{n}$, but nevertheless it has the interesting distance amplification property: the distance between every two message strings w, w' grows roughly k-fold. This gives the first construction, to the best of our knowledge, of a distance amplification code with constant rate that is locally testable with a constant number of queries, independent of k.

One can view the set $\{0, 1\}^n$ of possible functions on the vertices as a code of distance $1/n$ that is transformed, through the direct sum encoding, to a new code whose distance is $\Omega(k/n)$. If we begin with a restricted set of functions, say a code $C \subset \{0, 1\}^n$ whose distance is δ, then this transformation results in a new code whose distance is $\Omega(k\delta)$ (as long as $\delta < 1/k$), see Lemma IV.5. However, even if C is locally testable to begin with, it is not clear how to retain the local testability of the amplified code.

E. More related work

Works on PCP agreement tests: Agreement tests were initially studied as a type of low degree test, e.g. the line vs. line test of [RS92a], [RS92b], [AS97] and the plane vs.

plane test of [RS97]. Goldreich and Safra [GS97] were the first to consider the more general question of agreement tests and listed a set of axioms that imply agreement expansion (in current terminology). They were interested in finding a smaller collection of subsets on which such a theorem holds and proved agreement expansion of a certain (derandomized) collection of subsets. However, their result employs a weaker notion of *approximate* global consistency, namely that $f_s \approx g|_s$ instead of $f_s \equiv g|_s$. Further works [DR06], [DG08], [IKW09] adopted this approximate consistency notion which is in fact inherent in the small acceptance regime. The only setting where an agreement test is known to have the (more natural) *exact* global consistency is in the work of [DS14] on the complete complex.

For the approximate global consistency notion, Impagliazzo et. al. [IKW09] suggested to look at a collection of subsets that corresponds to affine subspaces inside a high-dimensional vector space. The collection has size that is polynomial in the size of the ground set which is much better than the exponential size of the complete complex, but still far from linear and certainly at least quadratic. The [IKW09] agreement test theorem holds also in the so-called small acceptance regime, also known as the 1% regime. Extending our results on the bounded-degree complexes to this regime is an intriguing open question. In particular, we conjecture the following to be true

Conjecture I.10 (Derandomization in the 1% regime)**.** *For every $\delta > 0$ there exists a $d \in \mathbb{N}$ and an infinite family $X_1, X_2, \dots$ of* **sparse** *d-dimensional complexes, and for each X_n a distribution $\mathcal{D}_n$ over pairs of subsets, such that for each $X = X_n$ the following holds. For every $f = \{f_s\}_{s\in X(d)}$, if $\mathrm{agree}_{\mathcal{D}}(f) > \delta$ then there is a global $g : X(0) \to \{0,1\}$ such that $\Pr[f_s \approx g|_s] \geq \Omega(\delta)$.*

Such a result holds for the complete complex [DG08], and the subspaces complex [IKW09].

So far in the PCP literature essentially two constructions are known that give non trivial agreement tests. The first, called the direct product construction, is where X is the collection of all subsets of size d, i.e. the complete complex. The second, called the subspaces construction, is where the ground set $[n]$ is identified with the points of some vector space F^m and the subsets correspond to all fixed-dimensional linear (or affine) sub-spaces of F^m. Apart from these two constructions (and some very similar variants) no other construction is known and certainly not one with linear or nearly-linear size that so much as comes close to results cited above.

Recently agreement tests on the subspaces complex (Grassmann) were studied [KMS16], [DKK$^+$16] towards proving strong inapproximability results and in particular the so-called 2-to-1 conjecture. This may be taken as further indication of the importance of agreement tests inside PCP constructions.

We remark that although finding a smaller collection of subsets is called a derandomization task (and this can be justified because we want to use fewer random bits to choose a random subset in the collection), it is unlike most other derandomization questions studied in the context of pseudorandom generators or extractors. The difference is that in standard derandomization a random object with the correct size almost surely has the desired property, and the difficulty is coming up with an explicit construction that imitates the random object. Here, in contrast, a random collection of linearly many subsets, also called the random sparse complex, is very far from having the desired agreement behavior. This is for a very similar reason to the fact that a random sparse simplicial complex is not at all a good high-dimensional expander.

Works on high order random walks: Combinatorial high order random walks on high dimensional expanders were first defined and analysed by [KM17], who showed that these walks are rapidly mixing. However the second largest eigenvalue bound obtained by [KM17] is $1 - O(\frac{1}{k^2})$ and not $1 - O(\frac{1}{k})$. This innocent looking difference is quite important since only the optimal gap of $1/k$ (that we end up showing in Theorem I.7) suffices for proving the strong sampling properties that underly our proof.

First [Fir16] studies a broad collection of high order random walks and shows that their spectral behavior is the same as that of the infinite Affine Building. This could potentially lead to an alternate way of calculating the spectral gap of these walks: understand them on the infinite building and then transfer the results to the finite quotient. However, this path has so far not been carried out.

We refer the reader to the work [LLP17] and the references therein for a broader discussion of high order random walks.

F. Discussion

High dimensional expanders and PCPs: We believe that there is a true connection between high dimensional expanders and PCPs. These objects posses a mixture of pseudo-randomness and structure that can not be obtained by any known random construction. This is in striking contrast to the one dimensional case, where random graphs easily give nearly optimal expanders.

We think that further exploration of the relations between these two objects could be beneficial. It could well be the case that known high dimensional expanders can be used to construct better PCPs, either towards the sliding scale conjecture or towards linear size PCPs and locally testable codes. Additionally, it can be the case that known PCP constructions based on composition can be used to obtain new constructions of high dimensional expanders that are not algebraic. Remark: although some limitations are known regarding constructing high dimensional expanders (under some conditions only number theoretic constructions can be Ramanujan) there is no limitation for constructing a generic λ-HD high dimensional expander. It would be very interesting to construct such an object without using representation theory; This could possibly be achieved through PCP techniques.

Agreement expansion is a kind of approximate cohomology with local coefficients: Functions on a topological space are sometimes easier to specify by giving them as a collection of local functions, one per small part of the space. It is required of course that the different local functions agree on the intersection of their domains. This is called the *sheaf* condition, and corresponds exactly to our notion of agreement. If the collection of local functions satisfies agreement perfectly then it is a *global section* (or a *cohomology with local coefficients*). In this language what we are studying is a notion of "approximate sections".

The fact that agreement testing has a natural countepart in topology (although exact and not approximate), hints towards promising relations between these seemingly different areas.

G. Organization

The rest of the paper contains some preliminaries followed by two sections. The analysis of higher order random walks is given in Section III. We skip the proofs of the main theorems (Theorem I.3 and Theorem I.6) and of existence of sparse λ-HD-expanders of all dimensions (Lemma I.5). These can be found in the full version of the manuscript. We conclude with Section IV about application to direct sum and direct product.

II. Preliminaries

A. Markov operators and singular values

The following is a slight generalization of the theory of spectral decomposition of graphs to the case of bipartite graphs with nonnegative weights. By normalizing the weights to sum up to one we can always think of such a bipartite graph as a probability distribution over pairs of vertices $(u, v) \in U \times V$.

Throughout the paper we will be working with Markov operators that are defined via a distribution. We define this next,

Definition II.1 (Markov operator of a bipartite graph)**.** Let $G = (U, V, E)$ be a bipartite graph, and assume that each edge carries a non-negative weight p_{uv} such that $\sum_{u,v} p_{uv} = 1$.

- The probability distribution $\{p_{uv}\}$ induces a marginal probability distribution on U and similarly on V given by
$$p_u = \sum_{v\in V} p_{uv}, \quad p_v = \sum_{u\in U} p_{uv}$$
All expectations on U, V are with respect to these distributions. Moreover, we define an inner product on the space $L^2(U)$ of functions $f : U \to \mathbb{R}$ by
$$\langle f, f'\rangle \doteq \mathbf{E}_u[f(u)f'(u)] = \sum_{u\in U} p_u f(u)f'(u).$$
and similarly on the space $L^2(V)$.
- There are two natural linear operators $A : L^2(U) \to L^2(V)$ and $A^\dagger : L^2(V) \to L^2(U)$ that are associated with G. These are the conditional expectation operators given by,
$$\forall f \in L^2(U),\ Af(v) \doteq \mathbf{E}_{u|v}[f(u)],$$
$$\forall g \in L^2(V),\ A^\dagger g(u) \doteq \mathbf{E}_{v|u}[g(v)],$$
or, in terms of the normalized adjacency matrix, $A_{uv} = \frac{p_{uv}}{p_u}$ and $(A^\dagger)_{vu} = \frac{p_{uv}}{p_v}$.

In case $U = V$ and the distribution is symmetric (i.e. $p_{uv} = p_{vu}$ which corresponds to an undirected graph), then it is more natural to view G simply as an undirected graph instead of connecting two copies of V. Indeed there will be only one marginal distribution on the vertices and only one (self-adjoint) operator $A = A^\dagger$, and so this definition coincides with that of a Markov operator for undirected non-bipartite graphs.

One can check that for every $f \in L^2(U), g \in L^2(V)$
$$\langle Af, g\rangle = \mathbf{E}_{xy}[f(x)g(y)] = \langle f, A^\dagger g\rangle$$
justifying the notation. Note that the inner product on the right is over the space $L^2(U)$ whereas the inner product on the left is over the space $L^2(V)$. The following claim justifies the use of the term *Markov* operator,

Claim II.2. Let $A : L^2(U) \to L^2(V)$ be a Markov operator as defined above. Then for every $f \in L^2(U)$, $\|Af\|^2 \le \|f\|^2$ and also $A\mathbf{1} = \mathbf{1}$.

It now makes sense to consider the space of functions orthogonal to $\mathbf{1}$ and upper bound $\|Af\|/\|f\|$ in this space,

Definition II.3 (Second largest singular value)**.** Let A be a Markov operator. Define
$$\lambda(A) = \sup_{f\perp\mathbf{1}} \frac{\|Af\|}{\|f\|}.$$
We remark that it also holds that
$$\lambda(A) = \sup_{f,g\perp\mathbf{1}} \frac{\langle Af, g\rangle}{\|f\| \cdot \|g\|}. \tag{3}$$
Clearly this second definition is only larger because one can plug in $g = Af$, observing that if $f \perp \mathbf{1}$ then also $Af \perp \mathbf{1}$. For the other direction use Cauchy Schwartz.

The following definition coincides with the standard definition, but is slightly more general as it pertains to general and not necessarily uniform edge distribution,

Definition II.4 (λ-expander)**.** A bipartite graph $G = (U, V, E)$ is called a λ-expander if $\lambda(A) \le \lambda$, where $A : L^2(U) \to L^2(V)$ is the associated Markov operator.

A non-bipartite graph $G = (V, E)$ is called a λ-expander if $\lambda(A) \leq \lambda$, where $A : L^2(V) \to L^2(V)$ is the associated Markov operator.

1) Concatenation of two Markov operators: Let U, V, W be three vertex sets. Let $G' = (U, V, E')$ and $G'' = (V, W, E'')$ each be a bipartite graph with a probability distribution P' and P'' on the respective sets of edges E' and E''. Assume further that the marginal distribution that P' induces on V is identical to the marginal distribution that P'' induces on V. Define the bipartite graph $G = (U, W, E)$ with edge distribution P defined by

$$p_{uw} := \sum_v p'_{uv} p''_{vw} = \Pr_{u_1, v_1, w_1}[u_1 = u, w_1 = w].$$

Lemma II.5. *Let A, A', A'' be the Markov operators associated with G, G', G'' respectively. Then $A = A''A'$ and $\lambda(A) \leq \lambda(A') \cdot \lambda(A'')$.*

Let $G = (U, V, E)$ be a weighted bipartite graph and let $A : L^2(U) \to L^2(V)$ be its Markov operator. The operator $A^\dagger A : L^2(U) \to L^2(U)$ is self adjoint, i.e. $(A^\dagger A)^\dagger = A^\dagger A$, as is the operator $AA^\dagger : L^2(V) \to L^2(V)$. Every self-adjoint operator M on an n-dimensional space has a spectral decomposition, namely there is a basis of eigenfunctions $f_1, \ldots, f_n \in L^2(U)$ and real eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ such that $Mf_i = \lambda_i f_i$. Clearly if M is self-adjoint then $\lambda(M) = \max(|\lambda_2|, |\lambda_n|)$.

Claim II.6. Let $A : L^2(U) \to L^2(V)$ and let $A^\dagger : L^2(V) \to L^2(U)$. Then $\lambda(A^\dagger A) = \lambda(A)\lambda(A^\dagger) = \lambda(A)^2$. □

B. Simplicial complexes and high dimensional expansion

A simplicial complex $X = X(0) \cup X(1) \cup \cdots \cup X(d)$ of dimension d is a hypergraph on vertex set $X(0)$ such that for all $0 < i \leq d$, $X(i)$ is a collection of i-faces, which are subsets of $X(0)$ of size $i + 1$. The complex has the property that if $s \in X$ then for every $s' \subset s$, also $s' \in X$. We also denote by $X(-1)$ the set containing the single empty set face.

We will consider a slight generalization where the complex comes with a distribution over the top (i.e. d-dimensional) faces that is not necessarily uniform. This distribution naturally extends to a distribution over $X(i)$ by letting the probability of $s \in X(i)$ be proportional to the probability of the set $\{r \in X(d) \mid r \supset s\}$, see a more detailed description in Section III-A.

For a face $s \in X(i)$, the *link* X_s is a simplicial complex of dimension $d - i - 1$ defined by

$$X_s = \{t \setminus s \mid s \subset t \in X\}.$$

More accurately, we will give each top face in X_s a probability proportional to its probability in X (but renormalized so that the probabilities sum to 1).

The *one-dimensional skeleton* of a complex X is the graph whose vertices are $X(0)$ and whose edges are $X(1)$. The *k-dimensional skeleton* of a complex X is the k-dimensional complex whose i-faces are $X(i)$ for every $0 \leq i \leq k$.

High dimensional expanders:

Proposition II.7 (Ramanujan complexes of [LSV05b], [LSV05a])**.** *For every $d \in \mathbb{N}$ and every $\gamma > 0$ there is a number $c = (\frac{1}{\gamma})^{O(d^2)}$ and an infinite sequence of explicitly constructible d-dimensional simplicial complexes $X^1, X^2, \ldots$ where X^t is on n_t vertices and $|X^t(d)| \leq c \cdot n^t$, and for each t, $X = X^t$ has the following properties. For each $i < d - 1$ and each face $v \in X(i)$, the vertices of the link X_v are colorable by $d - i$ colors such that:*

- *Every $d - i - 1$-dimensional face in X_v has one vertex from each color.*
- *Consider the 1-skeleton of X_v, namely the graph whose vertices are $X_v(0)$ and whose edges are $X_v(1)$. Then there are no edges inside a color class, and moreover, for every $1 \leq i < j \leq d$, the graph induced on vertices colored i and j is a bipartite graph that is a γ-expander.*

Proof: We choose a prime q whose size is at least $1/\gamma^2$. The work of [LSV05a] gives an infinite sequence of explicitly constructible d-dimensional simplicial complexes based on finite quotients of the Bruhat Tits building over a local field with characteristic q. These complexes have the claimed number of vertices and faces and moreover, the link of every vertex looks exactly like the link of the (infinite) affine building of dimension d. The link is a $d-1$-dimensional simplicial complex is known under the name "spherical building" or the "subspaces flag complex" and it is possible to analyze it by elementary combinatorial considerations. The two itemized properties are proven in [EK16, Section 5.2]. ■

Definition II.8 (λ-HD expander, restatement of Definition I.4)**.** A d-dimensional simplicial complex is a λ-HD expander if for every $i < d - 1$ and every $s \in X(i)$, the one dimensional skeleton of X_s is a λ-expander graph.

The advantage of this definition is that it tells us that the links of a complex have the most convenient expansion guarantee: their one-skeleton is a λ-expander (as per Definition II.4).

A potential explanation to why this definition did not show up before is that it *does not* hold for the LSV complexes. It turns out that sufficiently low-dimensional skeletons of an LSV complex are indeed λ-HD expanders, as we show in Section **??**.

III. Random walks on simplicial complexes

A. Random walks on simplicial complexes

Let X be a pure simplicial complex of dimension d, and let $\mathcal{D}_d$ be an arbitrary probability distribution on $X(d)$.

We extend $\mathcal{D}_d$ to a natural probability distribution $\mathcal{D}$ over sequences

$$s_d \supset s_{d-1} \supset \ldots \supset s_1 \supset s_0, \qquad s_i \in X(i)$$

Simply choose $s_d \in X(d)$ according to the distribution $\mathcal{D}_d$, and then $s_{d-1} \subset s_d$ by removing a random element from s_d, and inductively we choose $s_{i-1} \subset s_i$ by removing a random element from s_i. Since X is simplicial, $s_i \in X(i)$ for every i.

Let $\mathcal{D}_i$ be the probability distribution induced in this way on $X(i)$. It is easy to see that the probability of a face $r \in X(i)$ is directly proportional to the weight of top faces $s \in X(d)$ that contain it. Note that even if $\mathcal{D}_d$ happens to be uniform, for $i < d$, $\mathcal{D}_i$ is *not necessarily* uniform, because different $r \in X(i)$ may be contained in a different proportion of top faces in $X(d)$.

For each i we consider the space of functions $f : X(i) \to \mathbb{R}$ with inner product

$$\langle f, f' \rangle = \mathbf{E}_{s\sim\mathcal{D}_i}[f(s)f'(s)]$$

and we denote this space by $L^2(X(i))$. The norm on this space is $\|f\|^2 = \langle f, f \rangle = \mathbf{E}_{s\sim\mathcal{D}_i}[f(s)^2]$.

Fix $0 \le t < k \le d$. Let P_{tk} be the distribution over pairs $r \in X(t), s \in X(k)$ given by

$$\Pr_{P_{tk}}[(r,s)] := \Pr_{s_d\supset\ldots\supset s_0\sim\mathcal{D}}[s_k = s \text{ and } s_t = r].$$

We view P_{tk} as a bipartite graph whose vertices are $X(t)$ and $X(k)$ and where we connect r and s by an edge iff $r \subset s$. The weight on this edge is exactly $\Pr_{P_{tk}}[(r,s)]$. Observe that the sum of weights of edges adjacent to a vertex $r \in X(t)$ is exactly the probability of r under $\mathcal{D}_t$. Similarly the sum of weights of edges adjacent to a vertex $s \in X(k)$ is exactly the probability of s under $\mathcal{D}_k$.

As discussed above for a general bipartite graph with non-negative weights, there are two natural operators which we will denote $M_{t\nearrow k} : L^2(X(t)) \to L^2(X(k))$ defined by

$$\forall r \in X(k), \quad M_{t\nearrow k} f(r) := \mathbf{E}_{s|r} f(s)$$

and $M_{k\searrow t} : L^2(X(k)) \to L^2(X(t))$ defined by

$$\forall s \in X(t), \quad M_{k\searrow t} g(s) := \mathbf{E}_{r|s} g(r)$$

Easily check that

$$(M_{t\nearrow k})^\dagger = M_{k\searrow t},$$

namely for every $f : X(k) \to \mathbb{R}$ and $g : X(t) \to \mathbb{R}$

$$\langle M_{k\searrow t} f, g \rangle = \mathbf{E}_{(r,s)\sim P_{tk}}[g(r)f(s)] = \langle f, M_{t\nearrow k} g \rangle.$$

B. Random walks on $X(k)$

Define random walk distributions on pairs $(s_1, s_2) \in X(k)$ by defining their Markov operators:

$$M_{k\frown k} := M_{k+1\searrow k}\, M_{k\nearrow k+1} \quad \text{and}$$

$$M_{k\smile k} := M_{k-1\nearrow k}\, M_{k\searrow k-1}$$

We observe that from the definition, and since $(A^\dagger A)^\dagger = A^\dagger A$ both operators are self adjoint. Morever, since $\lambda(A^\dagger A) = \lambda(A)^2$ (see Claim II.6)

$$\lambda(M_{k\frown k}) = \lambda(M_{k\nearrow k+1})^2 \quad \text{and}$$

$$\lambda(M_{k\smile k}) = \lambda(M_{k-1\nearrow k})^2$$

- The distribution $\mathcal{D}_{k\smile k}$ corresponding to taking a random step according to the Markov operator $M_{k\smile k}$ can be described by choosing a random $r \in X(k-1)$ and then independently two random k-faces $s_1, s_2 \supset r$ and outputting s_1, s_2.
- The distribution $\mathcal{D}_{k\frown k}$ corresponding to taking a random step according to the Markov operator $M_{k\frown k}$ can be described by choosing a random $w \in X(k+1)$ and then independently two random k-faces $s_1, s_2 \subset w$ and outputting s_1, s_2.

It is easy to check that in each of these distributions s_1 is distributed according to $\mathcal{D}_k$. The same holds for s_2 since each distribution is symmetric with respect to s_1 and s_2.

In the next section we will prove,

Lemma III.1. *Assume that the complex X is a γ-HD expander. Then,*

$$\lambda(M_{k\smile k}) \le 1 - 1/(k+1) + O(k\gamma)$$

Let us first see how the lemma implies Theorem I.7.

Proof of Theorem I.7: Firstly, ignoring the $O(\gamma)$ term, observe that it implies

$$\lambda(M_{k\searrow k-1}) \le \left(1 - \frac{1}{k+1} + O(k\gamma)\right)^{1/2}. \tag{4}$$

Plugging $k \leftarrow k+1$ into the above equation and moving to the adjoint we get

$$\lambda(M_{k\nearrow k+1}) = \lambda(M_{k+1\searrow k}) \le \left(1 - \frac{1}{k+2} + O(k\gamma)\right)^{1/2} \tag{5}$$

which completes the proof. ■

Next, we show how the lemma implies Theorem I.8.

Proof of Theorem I.8:

$$M_{t\nearrow k} = M_{k-1\nearrow k} M_{k-2\nearrow k-1} \cdots M_{t\nearrow t+1},$$

so by relying on Lemma II.5 and plugging in the bound from (5),

$$\begin{aligned}\lambda(M_{t\nearrow k}) &\leq \lambda(M_{k-1\nearrow k})\cdot\lambda(M_{k-2\nearrow k-1})\cdots\lambda(M_{t\nearrow t+1})\\ &\leq \prod_{\ell=t+1}^{k}\left(1-\frac{1}{\ell}\right)^{1/2}+O(tk\gamma)\\ &= \left(\frac{t}{k}\right)^{1/2}+O(tk\gamma)\end{aligned}$$

where the last equality is because of a telescoping argument, and the previous inequality is true as long assuming that $\gamma < 1/k$. ■

C. Decreasing differences and proof of Lemma III.1

Let $f : X(k) \to \mathbb{R}$ be such that $\mathbf{E}[f] = 0$ and $\mathbf{E}[f^2] = 1$. Let us define, for $0 \leq i < k$ and a function $f : X(k) \to \mathbb{R}$, $f_i = M_{k\searrow i} f$, and the correlation quantity

$$\alpha_i(f) = \|f_i\|^2 = \|M_{k\searrow i}f\|^2 = \mathbf{E}_{s'\sim X(i),s_1,s_2\sim X_{s'}(k)}[f(s_1)f(s_2)]. \tag{6}$$

We also denote $\alpha_{-1}(f) = \mathbf{E}[f] = 0$ and $\alpha_k(f) = \|f\|^2 = 1$. By definition, $\alpha_{k-1}(f) = \|M_{k\searrow k-1}f\|^2$. This value is related to the spectral gap since

$$\frac{\|M_{k\searrow k-1}f\|^2}{\|f\|^2} \leq \lambda(M_{k\searrow k-1}f)^2 = \lambda(M_{k\smile k}).$$

It is clear that for any $i > j$ also $\alpha_i \geq \alpha_j$. We will be interested in the "second derivative" of this sequence. For each $-1 < i \leq k$

$$\Delta_i(f) \doteq \alpha_i(f) - \alpha_{i-1}(f)\,. \tag{7}$$

In particular $\Delta_k = 1 - \alpha_{k-1}(f)$, and our goal is to prove that $\Delta_k \geq 1/(k+1) - O(k\gamma)$.

Lemma III.2 (Decreasing Differences)**.** *Let $f : X(k) \to \mathbb{R}$. If X is a γ-HD expander, then for each $i > 0$*

$$\Delta_i(f) \geq \Delta_{i-1}(f)\cdot(1-\gamma) \geq \Delta_{i-1}(f) - \gamma\,.$$

The lemma directly implies that for each $i < k$, $\Delta_i \leq \Delta_k + (k-i)\gamma$. By assumption $\alpha_{-1}(f) = \mathbf{E}[f] = 0$ and $\alpha_k(f) = 1$, so $1 = \alpha_k(f) - \alpha_{-1}(f) = \Delta_k + \Delta_{k-1} + \Delta_{k-2} + \cdots + \Delta_1 + \Delta_0 \leq (k+1)\Delta_k + O(k^2\gamma)$. This implies that

$$1 - \lambda(M_{k\smile k}) \geq 1 - \alpha_{k-1}(f) = \Delta_k \geq \frac{1}{k+1} - O(k\gamma)$$

and completes the proof of Lemma III.1. □

The proof of Lemma III.2 relies on the following lemma on graphs whose proof can be found in the full version of this manuscript.

Lemma III.3. *Let $G = (V,E)$ be a graph with non-negative weights on the edges. Suppose that G is a γ-expander. Let $h : E \to \mathbb{R}$ be a function on the edges of G. Define $h_1 : V \to \mathbb{R}$ by setting for each vertex $i \in [n]$, $h_1(i) = \mathbf{E}_{j|i}h(i,j)$ and also let $h_0 = \mathbf{E}_i[h_1(i)]$. Define*

$$\delta_1 = \mathbf{E}_i[(h_1(i)-h_0)^2] \quad \textit{and} \quad \delta_2 = \mathbf{E}_{ij}[(h(i,j)-h_1(i))^2]$$

where all expectations above are with respect to the normalized edge and vertex distribution of G. Then $\delta_2 \geq \delta_1\cdot(1-\lambda)$.

IV. Derandomized direct product and direct sum encodings

The direct product and direct sum encodings are studied in various complexity settings espectially since they are very useful for hardness amplification. In the direct sum encoding, we map a string $w \in \{0,1\}^n$ to the string $\mathrm{DS}(w) \in \{0,1\}^{Y(k-1)}$ where $Y(k-1) = \binom{[n]}{k}$ is the set of all possible k-element subsets of $[n]$, namely, Y is the complete $(k-1)$-dimensional complex on n vertices. The encoding is defined by

$$\forall s \in \binom{[n]}{k}, \qquad \mathrm{DS}(w)(s) = \sum_{x\in s} w(x) \mod 2.$$

In the direct product encoding, we map a string $w \in \{0,1\}^n$ to a table $\mathrm{DP}(w) \in \{0,1\}^{k\times Y(k-1)}$ whose rows correspond to subsets $s \in Y(k-1)$. Every row in this table is a k bit string that is equal to $w|_s$,

$$\forall s \in \binom{[n]}{k}, \qquad \mathrm{DP}(w)(s) = w|_s.$$

These encodings are often very useful for hardness amplification, essentially because they are locally computable and provide good distance amplification. Two strings $w, w' \in \{0,1\}^n$ that differ on δ fraction of their coordinates, have encodings that are $k\delta$ apart (see Lemma IV.5).

One serious drawback of these encodings is that their length is $\binom{n}{k}$ which grows exponentially with k. This leads us to consider a so-called "derandomized" version of these encodings, that has shorter length while hopefully retaining the all of the good properties. The term "derandomized" comes from trying to minimize the amount of randomness needed to choose a single symbol in the encoding. Such ideas have been explored in the past and [IKW09] have showed how to obtain a derandomized encoding that maps n bit strings to $\mathrm{poly}(n)$ bit strings.

We suggest to use simplicial complexes for obtaining such derandomization. Given *any* $k-1$ dimensional complex $X(k-1)$, we now define the appropriate direct sum and direct product encodings with respect to X.

Definition IV.1 (Direct sum encoding with respect to a simplicial complex)**.** A simplicial complex $X(k-1)$ gives rise to the following encoding, called the direct sum encoding, that maps strings $w \in \{0,1\}^{X(0)}$ to strings $\mathrm{DS}(w) \in \{0,1\}^{X(k-1)}$ via

$$\forall s \in X(k-1), \quad \mathrm{DS}_X(w)(s) = \sum_{x\in s} w(x) \mod 2.$$

Definition IV.2 (Direct product encoding with respect to a simplicial complex)**.** A simplicial complex $X(k-1)$ on n vertices gives rise to the following encoding, called the direct product encoding, that maps strings $w \in \{0,1\}^{X(0)}$ to strings $\mathrm{DS}(w) \in \{0,1\}^{k \times X(k-1)}$ via

$$\forall s \in X(k-1), \qquad \mathrm{DP}_X(w)(s) = w|_s.$$

where we view $w|_s$ as a k-bit string using some fixed ordering on the vertex set $X(0)$.

The crucial point is that if X is a bounded degree complex, namely $|X(k-1)| = O(|X(0)|)$, then the encoding length is *linear* in the message length, quite a big savings compared to the non-derandomized situation. Agreement expansion of X impies quite directly that these encodings can be locally tested with 2 or 3 queries.

Lemma IV.3 (Derandomized Direct Product - two query test)**.** *Let $X(k-1)$ be a $k-1$ dimensional simplicial complex on n vertices that is an agreement expander. Let $\mathcal{D}$ be a distribution for which $\gamma(X, \mathcal{D}) \geq \Omega(1)$. Then $\mathcal{D}$ gives rise to a natural two-query agreement test :*

- *Choose $(s_1, s_2) \sim \mathcal{D}$*
- *Read the rows of f corresponding to s_1, s_2*
- *Accept iff for every $x \in s_1 \cap s_2$ the corresponding values agree: $f[s_1](x) = f[s_2](x)$.*

Namely, if $f = \mathrm{DS}(w)$ for some w then the test succeeds with probability 1; and if the test succeeds on f with probability $1-\varepsilon$ then there is some $w \in \{0,1\}^n$ such that for at least $1 - O(\varepsilon)$ of the sets $s \in X(k-1)$, $f[s] = \mathrm{DP}_X(w)(s)$.

The proof of this lemma is immediate from Theorem I.3.

Using a reduction from [DDG+15] from direct sum to direct product, and relying on the fact that inside an r-set we are exactly in the setting of the complete complex as studied in [DDG+15], we can prove

Lemma IV.4 (Derandomized Direct Sum - three query test)**.** *Let $X(d)$ be a d dimensional simplicial complex on n vertices that is an agreement expander and such that $\gamma(X, \mathcal{D}) \geq \Omega(1)$ for $\mathcal{D}$ the distribution $d - 2d - d$. Let $k = 2\lceil d/10 \rceil$ be an even integer, then $\mathrm{DS}_{X(k)}$ is locally testable with three queries with the following test*

- *Choose $r \sim X(k)$*
- *Choose $s_1, s_2, s_3 \subset r$ such that every element in r is covered by an even number sets out of s_1, s_2, s_3 and such that $s_1, s_2, s_3 \in X(k/2)$.*
- *Accept iff $f(s_1) + f(s_2) + f(s_3) = 0 \mod 2$.*

Namely, if $f = \mathrm{DS}(w)$ for some w then the test succeeds with probability 1; and if the test succeeds on f with probability $1-\varepsilon$ then there is some $w \in \{0,1\}^n$ such that for at least $1 - O(\varepsilon)$ of the sets $s \in X(k-1)$, $f[s] = \mathrm{DS}_{X(d/2)}(w)(s)$.

We omit the proof of this lemma, but let us explain the main idea. The idea is to rely on the testing result from [DDG+15] to show that for typical $r \in X(k)$, there is one function $h_r \in \{0,1\}^r$ whose DS encoding agrees with $1-\varepsilon$ fraction the sets $s \subset r$, $|s| = k$. This is enough to prove that the local function $\{h_r\}_{r \in X(d)}$ has agreement at least $agree_{\mathcal{D}}(h) \geq 1 - O(\varepsilon)$. We apply Theorem I.3 to deduce a global function g that agrees with most of h_r and therefore with most of f_s.

A. Distance amplification code

Note that the direct sum (and the direct product encoding) is far from an error correcting code because of its poor relative distance, which is about $\frac{k}{n}$, but nevertheless it has the interesting distance amplification property: the distance between every two message strings w, w' grows roughly k-fold. This gives the first construction, to the best of our knowledge, of a distance amplification code with constant rate that is locally testable with a constant number of queries that is independent of k.

One can view the set $\{0,1\}^n$ of possible functions on the vertices as a code of distance $1/n$ that is transformed, through the direct sum encoding, to a new code whose distance is $\Omega(k/n)$. If we begin with a restricted set of functions, say a code $C \subset \{0,1\}^n$ whose distance is δ, then this transformation results in a new code whose distance is $\Omega(k\delta)$ (as long as $\delta < 1/k$), see Lemma IV.5. However, even if C is locally testable to begin with, it is not clear how to retain the local testability of the amplified code.

We next prove a lemma showing distance amplification of the direct product encoding. This easily implies a similar result for the direct sum encoding as well, but we omit the details.

Lemma IV.5 (Distance Amplification)**.** *Let $X(d)$ be a β-HD expander d-dimensional simplicial complex on n vertices, and assume $\beta < 1/d$. Then for every $1 < k \leq d$ and every pair of strings $w, w' \in \{0,1\}^n$ whose Hamming distance is $\delta < 1/k$,*

$$\Pr_{s \sim X(k-1)}[w|_s \neq w'|_s] \geq k \cdot \delta/4.$$

The lemma follows essentially due to an expander missing lemma, see the full version for details.

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