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Incidences between points and lines in $\mathbb{R}^4$*

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Abstract

We show that the number of incidences between m distinct points and n distinct lines in $\mathbb{R}^4$ is $O\left(2^{c\sqrt{\log m}}(m^{2/5}n^{4/5}+m)+m^{1/2}n^{1/2}q^{1/4}+m^{2/3}n^{1/3}s^{1/3}+n\right)$, for a suitable absolute constant c, provided that no 2-plane contains more than s input lines, and no hyperplane or quadric contains more than q lines. The bound holds without the extra factor $2^{c\sqrt{\log m}}$ when $m \le n^{6/7}$ or $m \ge n^{5/3}$. Except for this possible factor, the bound is tight in the worst case.

The context of this work is incidence geometry, a topic that has been widely studied for more than three decades, with strong connections to a variety of topics, from range searching in computational geometry to the Kakeya problem in harmonic analysis and geometric measure theory. The area has picked up considerable momentum in the past seven years, following the seminal works of Guth and Katz [12, 13], where the later work solves the point-line incidence problem in three dimensions, using new tools and techniques from algebraic geometry. This work extends their result to four dimensions. In doing so, it had to overcome many new technical hurdles that arise from the higher-dimensional context, by developing and adapting more advanced tools from algebraic geometry.

Keywords. Combinatorial geometry, incidences, the polynomial method, algebraic geometry, ruled surfaces.

1 Introduction

Let P be a set of m distinct points and L a set of n distinct lines in $\mathbb{R}^4$. Let $I(P,L) := |\{(p,\ell) \in P \times L \mid p \in \ell\}|$ denote the number of incidences between the points of P and the lines of L. If all the points of P and all the lines of L lie in a common plane, then the classical Szemerédi–Trotter theorem [42] yields the worst-case tight bound

$$I(P,L) = O\left(m^{2/3}n^{2/3}+m+n\right). \quad (1)$$

This bound clearly also holds in four (or in any other) dimension, by projecting the given lines and points onto some generic plane. Moreover, the bound will continue to be worst-case tight by placing all the points and lines in a common plane, so that they yield the planar lower bound.

In the recent groundbreaking paper of Guth and Katz [13], an improved bound has been derived for $I(P,L)$, for a set P of m points and a set L of n lines in $\mathbb{R}^3$, provided that not too many lines of L lie in a common plane [1]. Specifically, they showed:

Theorem 1 (Guth and Katz [13])**.** *Let P be a set of m distinct points and L a set of n distinct lines in $\mathbb{R}^3$, and let $s \le n$ be a parameter, such that no plane contains more than s lines of L. Then*

$$I(P,L) = O\left(m^{1/2}n^{3/4}+m^{2/3}n^{1/3}s^{1/3}+m+n\right). \quad (2)$$

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[1] The additional requirement in [13], that no regulus contains too many lines, is not needed for the incidence bound given below.

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This bound is tight in the worst case.

In this paper, we establish the following analogous and sharper result in four dimensions.

Theorem 2. *Let P be a set of m distinct points and L a set of n distinct lines in $\mathbb{R}^4$, and let $q, s \le n$ be parameters, such that (i) each hyperplane or quadric contains at most q lines of L, and (ii) each 2-flat contains at most s lines of L. Then*

$$I(P,L) \le 2^{c\sqrt{\log m}}\left(m^{2/5}n^{4/5} + m\right) + A\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + n\right), \quad (3)$$

where A and c are suitable absolute constants. When $m \le n^{6/7}$ or $m \ge n^{5/3}$, we get the sharper bound

$$I(P,L) \le A\left(m^{2/5}n^{4/5} + m + m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + n\right). \quad (4)$$

In general, except for the factor $2^{c\sqrt{\log m}}$, the bound is tight in the worst case, for any values of m, n, with corresponding suitable ranges of q and s.

The proof of Theorem 2 will be by induction on m. To facilitate the inductive process, we extend the theorem as follows. We say that a hyperplane or a quadric H in $\mathbb{R}^4$ is *q-restricted* for a set of lines L and for an integer parameter q, if there exists a real 4-variate polynomial g_H of degree at most $O(\sqrt{q})$, such that each of the lines of L that is contained in H, except for at most q lines, is contained in some irreducible component of $H \cap Z(g_H)$ that is ruled by lines and is not a 2-flat (see below for details). In other words, a q-restricted hyperplane or quadric contains in principle at most q lines of L, but it can also contain an unspecified number of additional lines, all fully contained in ruled (non-planar) components of the zero set of some polynomial of degree $O(\sqrt{q})$. We then have the following more general result.

Theorem 3. *Let P be a set of m distinct points and L a set of n distinct lines in $\mathbb{R}^4$, and let q and $s \le n$ be parameters, such that (i') each hyperplane or quadric is q-restricted, and (ii) each 2-flat contains at most s lines of L. Then $I(P,L)$ satisfies the bound in (3), with the same A and c. It satisfies the sharper bound in (4) when $m \le n^{6/7}$ or $m \ge n^{5/3}$, and the bound is nearly tight in the worst case, as above.*

The requirement that a hyperplane or quadric H be q-restricted extends (i.e., is a weaker condition than) the simpler requirement that H contain at most q lines of L. Hence, Theorem 2 is an immediate corollary of Theorem 3.

In this abstract we only discuss the proof of the upper bound; the lower bound construction is given in the full version, available in [35].

Remarks. **(a)** Only the range $\sqrt{n} \le m \le n^2$ is of interest; outside this range, regardless of the dimension of the ambient space, we always have $I(P,L) = O(m+n)$, resulting from (1). **(b)** The term $m^{1/2}n^{1/2}q^{1/4}$ comes from the bound (2) of Guth and Katz in three dimensions, and is unavoidable, as it can be attained if we densely "pack" points and lines into hyperplanes, in patterns that realize the bound (2) within each hyperplane. **(c)** Likewise, the term $m^{2/3}n^{1/3}s^{1/3}$ comes from the planar Szemerédi–Trotter bound (1), and is too unavoidable, as it can be attained if we densely pack points and lines into 2-planes, in patterns that realize the bound in (1). **(d)** Ignoring these terms, and the term n, which is included only to cater for the case $m < \sqrt{n}$, the two terms $m^{2/5}n^{4/5}$ and m "compete" for dominance; the former dominates when $m = O(n^{4/3})$ and the latter when $m = \Omega(n^{4/3})$. Thus the bound in (3) (and in (4)) is qualitatively different within these two ranges. (The threshold $m = n^{4/3}$ also arises in the related problem of *joints* (points incident to at least four lines not in a common hyperplane) in a set of n lines in 4-space; see [20, 26].)

By a standard argument, the theorem implies the following corollary.

Corollary 4. *Let L be a set of n lines in $\mathbb{R}^4$, satisfying the assumptions (i)–(ii) in Theorem 3, for given parameters q and s. Then, for any $k = \Omega(2^{c\sqrt{\log n}})$, the number $m_{\ge k}$ of points incident to at least k lines of L satisfies*

$$m_{\ge k} = O\left(\frac{2^{\frac{4}{3}c\sqrt{\log n}}n^{4/3}}{k^{5/3}} + \frac{nq^{1/2}}{k^2} + \frac{ns}{k^3} + \frac{n}{k}\right).$$

Background. Incidence problems have been a major topic in combinatorial and computational geometry for more than thirty years, starting with the bound (1) of Szemerédi and Trotter [42] back in 1983. Several techniques, interesting in their own right, have been developed, or adapted, for the analysis of incidences, including the *crossing-lemma* technique of Székely [41], and the use of *cuttings* as a divide-and-conquer mechanism (e.g., see [3]). Connections with range searching and related problems in computational geometry have also been noted and exploited. In fact, the study of incidences and the design of many algorithms in computational geometry have "co-existed" for about 25 years, because of the close relationship between the methodologies used in both of these subareas. Finally, studies of the Kakeya problem (see, e.g., Tao [43] and Dvir [6]) indicate the connection between this problem and incidence problems. See Pach and Sharir [24] for a comprehensive (albeit a bit outdated) survey of the topic.

The landscape of incidence geometry has dramatically changed in the past seven years, due to the infusion, in the two groundbreaking papers by Guth and Katz [12, 13], inspired by earlier work of Dvir [6] in finite fields, of new tools and techniques drawn from algebraic geometry. Although their two direct goals have been to obtain a tight upper bound on the number of joints in a set of lines in three dimensions [12], and an almost tight lower bound for the classical distinct distances problem of Erdős [13], the new tools (most notably, the *polynomial partitioning* technique) have quickly been recognized as useful for incidence bounds of various sorts. See [8, 18, 19, 32, 39, 45, 46] for a sample of recent works on incidence problems that use the new algebraic machinery. It is fair to say that incidence geometry, and its related topics, have become a very "hot" and active area in the past seven years, where problems that have been deemed hopeless to tackle only a few years back, can now be confronted, with significant success. Equally interesting is the new methodology, which is based on tools and techniques from algebraic geometry, adding a new level of sophistication to this area.

The simplest instances of incidence problems involve points and lines. Szemerédi and Trotter completely solved this special case in the plane [42]. Guth and Katz's second paper [13] provides a worst-case tight bound in three dimensions, under the assumption that no plane contains too many lines (see Theorem 1), which is significantly smaller than the planar bound, and the intuition is that this phenomenon should also show up as we move to higher dimensions. Unfortunately, the analysis becomes considerably more involved in higher dimensions, and requires the development or adaptation of progressively more complex tools from algebraic geometry.

The present paper is a first (and significant) step in this direction, by considering the four-dimensional case. It does indeed derive a sharper bound, assuming that the configuration of points and lines is "truly four-dimensional", in the sense spelled out in Theorems 2 and 3.

We also note that studying incidence problems in four (or higher) dimensions has already taken place in several contemporary works, such as in Solymosi and Tao [39], Zahl [46], Basu and Sombra [1], and Solymosi and de Zeeuw [40]. These works, though, consider incidences with higher-dimensional varieties, and the study of incidences involving lines, presented in this paper, is new. (There are several ongoing studies, including a companion work joint with Sheffer [33], that aim to derive weaker but more general bounds involving incidences between points and curves in higher dimensions.)

Our study of point-line incidences in four dimensions has lead us to adapt and develop more advanced tools in algebraic geometry, such as tools involving surfaces that are ruled by lines or by flats, including Severi's 1901 work [31] (see below), as well as the more recent works of Landsberg [16, 22] on osculating lines and flats to algebraic surfaces in higher dimensions, and our own companion works [36, 37] that derive incidence-related properties of lines and points on algebraic varieties.

In a preliminary version of this study [34], we have obtained a weaker and more constrained bound. A discussion of the substantial differences between this preliminary work and the present one is given in the overview of the proof, which comes next.

The proof is long and technical, and all its details are spelled out in the full version of the paper [35]. We find this study fascinating, in its delicate interplay between sophisticated techniques in algebraic geometry, and more "conventional" combinatorial geometry tools. This extended abstract begins with an overview of the proof, and then provides details of some steps of the analysis, and sketches (or only mentions) the other steps. It concludes with a brief discussion of many interesting and open issues. The algebraic geometry tools that we need to use are described in detail in Section 2 of the full version [35], and are also presented, with less detail, in Section 2 below.

Overview of the proof.[2] The analysis follows the general high-level approach of Guth and Katz [13], albeit with many significant adaptations and modifications. We use induction on $m = |P|$, but we begin the description by ignoring this aspect (for a while).

Partitioning. We apply the polynomial partitioning technique of Guth and Katz [13], with some polynomial $f \in \mathbb{R}[x, y, z, w]$ of suitable degree D, and obtain a partition of $\mathbb{R}^4$ into $O(D^4)$ cells, each containing at most

[2]In this overview we assume some familiarity of the reader with the new "polynomial method" of Guth and Katz, and with subsequent applications thereof.

$O(m/D^4)$ points of P. In our first phase, we use

$$D = O(m^{2/5}/n^{1/5}), \quad \text{for } m = O(n^{4/3}), \qquad \text{and} \qquad D = O(n/m^{1/2}), \quad \text{for } m = \Omega(n^{4/3}). \tag{5}$$

(Note that $D^4 \le m$ for both ranges, and that $D \ge 1$ if we assume that $n^{1/2} \le m \le n^2$, which, as notes, is the only interesting range.) There are three types of incidences that may arise: an incidence between a point in some cell of the partition and a line crossing that cell, an incidence between a point on the zero set $Z(f)$ of f and a line not fully contained in $Z(f)$, and an incidence between a point on $Z(f)$ and a line fully contained in $Z(f)$. The above choices of D make it a fairly easy task (similar to many preceding works; see below) to bound the number of incidences of the first two types, and the hard part is to estimate the number of incidences of the third kind, as the polynomial partitioning technique has no control on the number of points and lines contained in $Z(f)$—in the worst case all the points and lines could be of this kind.

Large degree vs. constant degree. At the "other end of the spectrum," choosing D to be a constant (as done in our preliminary aforementioned study of this problem [34] and in other recent studies of related problems [11, 32, 39]) simplifies considerably the handling of incidences on $Z(f)$ (informally, "not too much can happen on a constant-degree surface"), but then the analysis of incidences within the cells of the partition becomes more involved, as the sizes of the subproblems within each cell are too large. In the works just cited (as well as in this paper), this is handled via induction, but the price of a naive induction is three-fold: **First**, the bound becomes weaker, involving additional factors of the form $O(m^{\varepsilon})$, for any $\varepsilon > 0$ (with a constant of proportionality that depends on ε). **Second**, the requirement that no hyperplane or quadric contains too many input lines (or similar requirements in related problems) has to be replaced by a much more restrictive assumption, that no algebraic surface of some (constant but large) maximum degree c_{ε} contains too many lines (the degree depends on the parameter ε just mentioned, and increases exponentially as ε decreases). **Finally**, the sharp "lower-dimensional" terms, such as $m^{1/2}n^{1/2}q^{1/4}$ and $m^{2/3}n^{1/3}s^{1/3}$ in our case (recall that both are worst-case tight), do not pass through the induction successfully, so they have to be replaced by weaker terms; see the preliminary version [34] for such weaker terms, and [32] for a similar phenomenon in a different incidence problem in three dimensions. A recent study by Guth [11] applies a constant-degree partitioning to the point-line incidence problem in $\mathbb{R}^3$, which suffers from the first and second drawbacks mentioned above, but manages to overcome the third. In a companion work [37], we provide yet another simpler derivation (which is sharper than Guth's) of such an incidence bound in three dimensions.

Using polynomials with large and with small degrees. Our approach is to use two different choices of the degree of the partitioning polynomial. We first choose a large degree D, as specified in (5), and show that the bound in the right-hand side of (4) accounts for the incidences within the partition cells, and for most of the cases involving incidences between points and lines on the zero set $Z(f)$. We are then left with "problematic" subsets of points and lines on $Z(f)$, which are difficult to analyze when the degree is large. (Informally, this happens when the lines lie in certain two-dimensional subvarieties of $Z(f)$ that are ruled by lines.) To handle them, we retain only these subsets, discard the partitioning, and start afresh with a new partitioning polynomial of a much smaller, albeit still non-constant degree. As the degree is now too small, we need induction to bound the number of incidences within the (new) partition cells. A major feature that makes the induction work well is that the first partitioning step ensures that the surviving set L_1 of lines that is passed to the induction is such that each hyperplane or quadric is now $O(D^2)$-restricted with respect to L_1, and each 2-flat contains at most $O(D)$ lines of L_1 (where D is the large degree used in (5)). As a consequence, the induction works better, and "retains" the lower-dimensional terms $m^{1/2}n^{1/2}q^{1/4}$ and $m^{2/3}n^{1/3}s^{1/3}$. (In fact, it does not touch them at all, because q and s are not "passed" to the induction step.) We still pay a small price for this approach, involving the extra factor $2^{c\sqrt{\log m}}$ in the "leading terms" $m^{2/5}n^{4/5} + m$ (but not in the "lower-dimensional" terms). Nevertheless, when m is "not too close to" $n^{4/3}$, as specified in the theorems, induction is not needed, and a direct analysis yields the sharper bound in (4), without this extra factor.

The idea of using a "small" degree for the partitioning polynomial is not new, and has been applied also in [32, 46]. However, the induction process in [32] results in weaker lower-dimensional terms, which we avoid here with the use of two different partitionings. We note that we have recently applied this approach in the aforementioned "warm-up" study of point-line incidences in three dimensions [37], with a simpler analysis (than that in [11, 13]) and an improved bound than the one in [11].

Incidences on the zero set: Large degree. The main part of the analysis is still in handling incidences within $Z(f)$ in the first partitioning step, for the large value of $D = \deg(f)$ in (5). (Similar issues arise in the second step too, but the bounds there are generally sharper than those obtained in the first step, because the degree is smaller.) This is done as follows.

Flecnode polynomials. We first show that we can ignore the singular points on $Z(f)$, and assume that f is irreducible. (Essentially, we assign each point and each line to some variety (fully) containing it, where each of these varieties is the zero set of some irreducible component of f or of one of its partial derivatives. We then show that the number of incidences between points and lines that are assigned to different components is only $O(nD)$ which is subsumed in the bounds (3), (4).

We define (a four-dimensional variant of) the *flecnode polynomial* $g := \mathsf{FL}_f^4$ of f (see Salmon [29] for the more classical three-dimensional variant, as used in Guth and Katz [13]), which vanishes at those points $p \in Z(f)$ that are incident to a line that *osculates* to $Z(f)$ (agrees with $Z(f)$ sufficiently near p) up to order four (and in particular to lines that are fully contained in $Z(f)$); see Section 2 and the full version for precise definitions and properties. We show that $g = \mathsf{FL}_f^4$ is a polynomial of degree $O(D)$.

When FL_f^4 vanishes identically on $Z(f)$: If $g \equiv 0$ on $Z(f)$ then $Z(f)$ is ruled by lines (as follows from Landsberg's work [22], given as Theorem 23 below, which extends the classical Cayley-Salmon theorem, which is used in [13] for the three-dimensional case). We handle this case by first reducing it to the case where $Z(f)$ is "infinitely ruled" by lines, meaning that most of its points are incident to infinitely many lines that are contained in $Z(f)$ (otherwise, we can show, using Bézout's theorem, that most points are incident to at most 6 lines, for a total of $O(m)$ incidences), and then by using the aforementioned result of Severi [31] from 1901, which shows that in this case $Z(f)$ is ruled by 2-flats (each point on $Z(f)$ is incident to a 2-flat that is fully contained in $Z(f)$), unless $Z(f)$ is a hyperplane or a quadric. This allows us to reduce the problem to several planar incidence problems, each involving at most s lines, which are reasonably easier to handle. (These are in fact handled in much the same way as in the second case, discussed next.)

When FL_f^4 intersects $Z(f)$ in a two-dimensional variety: (i) General weak bounds. The other case is where the common zero set $Z(f, g)$ of f and g is two-dimensional; it is of degree $O(D^2)$, as follows from the generalization of Bézout's theorem in Fulton [10] (see Theorem 6). In this case, we decompose $Z(f, g)$ into its irreducible components, and show that the number of incidences between points of P and lines of L fully contained in irreducible components that are not 2-flats is

$$\min\left\{O(mD^2 + nD),\ O(m + nD^4)\right\}. \tag{6}$$

Both terms are too large for the standard "large" values of D in (5), but their derivation is still rather intricate. They are useful tools for slightly improving the bound (into the one in (4), and for simplifying the analysis considerably when D is not too large—see below). The derivation of these bounds is based on a new study of point-line incidences within two-dimensional varieties that are ruled by lines, provided in a companion paper [36].

(ii) Incidences on 2-flat components of $Z(f, g)$: Linearly flat and flat points. Handling the irreducible components that are 2-flats is involved because their number can be $\deg(Z(f, g)) = O(D^2)$, a number that turns out (as in the non-flat components) to be too large for the purpose of our incidence bound, when a naive analysis (with a large value of D) is used, so some care is needed in this case. The difficult step in this part is when there are many points, each contained in at least three (and in general many) 2-flats fully contained in $Z(f, g)$ (and thus in $Z(f)$). Non-singular points of this kind are called *linearly flat* points of $Z(f)$, naturally generalizing Guth and Katz's notion of linearly flat points in $\mathbb{R}^3$ [13] (see also Kaplan et al. [8]). Linearly flat points are also *flat* points, i.e., points where the *second fundamental form* of $Z(f)$ vanishes (e.g., see Pressley [25]). Flatness of a point p can be expressed, again by a suitable generalization to four dimensions of the techniques in [8, 13], by the vanishing of nine polynomials, each of degree $\le 3D - 4$, at p, which are constructed from f and from its first and second-order derivatives. See Section 2 and the full version for details. The problem is then reduced to the case where all the points and lines are flat (a line is flat, when not all of its points are singular points of $Z(f)$, and all of its non-singular points are flat). With a careful (and somewhat intricate) probing into the geometric properties of flat lines, we can bound the number of incidences with flat lines by reducing the problem into several incidence problems in three dimensions (specifically, within hyperplanes tangent to $Z(f)$ at the flat points), and then using an extension of Guth and Katz's bound (2) for each of these problems, where, in this application, we exploit the fact that each hyperplane is q-restricted, to obtain a better, q-dependent bound.

Preparing for a second partitioning. However, as noted, the terms $O(mD^2)$ (when $n^{6/7} \le m \le n^{4/3}$) and $O(nD^4)$ (when $n^{4/3} \le m \le n^{5/3}$) are too large (for the choices of our "large" values of D in (5)). We retain them for the second partitioning step, when the degree of the partitioning polynomial is smaller, but finesse them, for the large D, by showing that, after pruning away points and lines whose incidences can be estimated directly (within the bound (4), not using the weaker bound of (6)), we are left with subsets for which every hyperplane or quadric is $O(D^2)$-restricted, and each 2-flat contains at most $O(D)$ lines. However, when $m \le n^{6/7}$ or $m \ge n^{5/3}$, there is no need for this part of the analysis, because D is then sufficiently small, and a direct application of the bounds in (6) yields the sharper bound in (4) and simplifies the proof considerably.

The new partitioning and the induction. In the general case, we then go on to our second partitioning step.

We discard f and start afresh with a new partitioning polynomial h of degree $E \ll D$. As already noted, bounding incidences within the partition cells becomes now non-trivial, and we use induction, exploiting the fact that now the parameters q and s are replaced by $O(D^2)$ and $O(D)$, respectively. On the other hand, bounding incidences within $Z(h)$ is now simpler, because E is smaller, and we can use the bounds in (6) (i.e., $\min\{O(mE^2+nE),\ O(m+nE^4)\}$) to establish the bound in (3) for the (now not so) "problematic" incidences.

The reason for using the weaker requirement that each hyperplane and quadric be q-restricted, instead of just requiring that no hyperplane or quadric contain more than q lines of L, is that we do not know how to bound the overall number of lines in a hyperplane or quadric H by $O(D^2)$, because of the potential existence of ruled components of $Z(f,g)$ within H, which can accommodate any number of lines. A major difference between this case and the analysis of ruled components in Guth and Katz's study [13] is that here the overall degree of $Z(f,g)$ is $O(D^2)$, as opposed to the degree of $Z(f)$ being only D in [13]. This precludes the application of the techniques of Guth and Katz to our scenario—they would lead to bounds that are too large—and requires novel, more sophisticated machinery.

We also note that our analysis of incidences within $Z(f)$ is actually carried out in the *complex projective* 4-space, which makes it simpler, and facilitates the application of numerous tools from algebraic geometry that are developed for that setting. The passage from the complex projective setup back to the real affine one is straightforward—the former is a generalization of the latter. The real affine setup is needed only for the construction of a polynomial partitioning, which is meaningless in complex projective spaces. Once we are within the variety $Z(f)$, we can switch to the complex projective setup, with the benefits just noted.

Note that, in spite of these improvements, Theorem 3 still has the peculiar feature, which is not needed in Guth and Katz [13] (for the incidence bound of Theorem 1), that also requires that every *quadric* be q-restricted (or, in the simpler version in Theorem 2, contain at most q lines of L).[3] A recent work of Solomon and Zhang [38] indicates that this requirement cannot be dropped.

2 A brief review of the algebraic machinery

Most of the results mentioned below hold over the complex field $\mathbb{C}$, and in complex projective spaces. As hinted above, transferring them back over the real affine case is usually an easy step.

Lines on varieties. We assume familiarity with the basic concepts in algebraic geometry, such as varieties, projective spaces, tangents and singularities, etc. Many classical textbooks provide a comprehensive treatment of these notions [4, 14, 15]

We begin with several basic notions and results in differential and algebraic geometry that we will need (see, e.g., Ivey and Landsberg [16], and Landsberg [22] for more details). For a vector space V (over $\mathbb{R}$ or $\mathbb{C}$), let $\mathbb{P}V$ denote its projectivization. That is, $\mathbb{P}V = V \setminus \{0\}/\sim$, where $v \sim w$ iff $w = \alpha v$ for some non-zero constant α.

For an (affine) algebraic variety X, and a point $p \in X$, let Σ_p denote the set of the complex lines passing through p and contained in X, and let Ξ_p denote the union of these lines (here X is implicit in these notations). For p fixed, the lines in Σ_p can be represented by their directions, as points in the projectivization of the tangent hyperplane $\mathbb{P}T_pX$. Clearly, $\Xi_p \subseteq T_pX$.

Consider the special case where X is a hypersurface in $\mathbb{C}^4$, i.e., $X = Z(f)$, for a non-linear polynomial $f \in \mathbb{C}[x,y,z,w]$, which we assume to be irreducible, where $Z(f) = \{p \in \mathbb{C}^4 \mid f(p) = 0\}$ is the *zero set* of $Z(f)$. A line $\ell_v = \{p + tv \mid t \in \mathbb{C}\}$ passing through p in direction v is said to *osculate* to $Z(f)$ to order k at p, if the Taylor expansion of f around p in direction v vanishes to order k, i.e., if $f(p) = 0$,

$$\nabla_v f(p) = 0, \qquad \nabla_v^2 f(p) = 0, \qquad \ldots, \qquad \nabla_v^k f(p) = 0, \tag{7}$$

where $\nabla_v f$ (which for uniformity we also denote as $\nabla_v^1 f$), $\nabla_v^2 f, \ldots, \nabla_v^k f$ are, respectively, the first, second, and higher order derivatives of f, up to order k, in direction v (where v is regarded as a vector in projective 3-space, and the derivatives are interpreted in a scale-invariant manner—we only care whether they vanish or not). That is, $\nabla_v f = \nabla f \cdot v$, $\nabla_v^2 f = v^T H_f v$, where H_f is the *Hessian matrix* of f, and $\nabla_v^i f$ is similarly defined, for $i > 2$, albeit with more complicated explicit expressions. For simplicity of notation, put $F_i(p; v) := \nabla_v^i f(p)$, for $i \geq 1$.

[3]This is not quite the case: Guth and Katz also require that no *regulus* contains more than s (actually, $\sqrt{n}$) lines, but this is made to bound the number of "2-rich" points, i.e., points incident to just two lines, and is not needed for the incidence bound in Theorem 1.

In fact, one can extend the definition of osculation of lines to arbitrary varieties in any dimension (see, e.g., Ivey and Landsberg [16]). For a variety X, a point $p \in X$, and an integer $k \geq 1$, let $\Sigma_p^k \subset \mathbb{P}T_pX$ denote the variety of the lines that pass through p and osculate to X to order k at p; as before, we represent the lines in Σ_p^k, for p fixed, by their directions, as points in the corresponding projective space. For each $k \in \mathbb{N}$, there is a natural inclusion $\Sigma_p \subseteq \Sigma_p^k$. In analogy with the previous notation, we denote by Ξ_p^k the union of the lines that pass through p with directions in Σ_p^k. We let $F(X)$ denote the variety of lines (fully) contained in X; this is known as the *Fano variety* of X, and it is a subvariety of the $(2d-2)$-dimensional *Grassmannian manifold* of lines in $\mathbb{P}^d(\mathbb{C})$; see Harris [14, Lecture 6, page 63] for details, and [14, Example 6.19] for an illustration, and for a proof that this is indeed a variety. We will sometimes denote $F(X)$ also as Σ (or $\Sigma(X)$), to conform with the notation involving osculating lines. We also let Σ^k denote the variety of the lines osculating to order k at some point of X, and can be thought of as the union of the Σ_p^k over $p \in X$. When representing lines in Σ or Σ^k we can no longer use the local representation by directions, and instead represent them, in the customary manner, as points within the Grassmanian manifold. Here too Σ^k can be shown to be a variety (within the Grassmannian manifold) and $F(X) \subseteq \Sigma^k$ for each k. We also have, for any $p \in X$, $\Sigma_p \subseteq F(X)$ and $\Sigma_p^k \subseteq \Sigma^k$.

Generalized Bézout's theorem. An affine (resp. projective) variety $X \subset \mathbb{C}^d$ (resp. $X \subset \mathbb{P}^d(\mathbb{C})$) is called *irreducible* if, whenever V is written in the form $V = V_1 \cup V_2$, where V_1 and V_2 are affine (resp., projective) varieties, then either $V_1 = V$ or $V_2 = V$.

Theorem 5 (Cox et al. [4, Theorem 4.6.2, Theorem 8.3.6])**.** *Let V be an affine (resp., projective) variety. Then V can be written as a finite union $V = V_1 \cup \cdots \cup V_m$, where V_i is an irreducible affine (resp., projective) variety, for $i = 1, \ldots, m$.*

We next state a generalized version of Bézout's theorem, as given in Fulton [10]. It will be a major technical tool in our analysis.

Theorem 6 (Fulton [10, Proposition 2.3])**.** *Let $V_1, \ldots, V_s$ be subvarieties of $\mathbb{P}^d$, and let $Z_1, \ldots, Z_r$ be the irreducible components of $\bigcap_{i=1}^s V_i$. Then*

$$\sum_{i=1}^{r} \deg(Z_i) \leq \prod_{j=1}^{s} \deg(V_j).$$

A simple application of Theorem 6 yields the following useful results.

Lemma 7. *(i) A curve $\mathcal{C} \subset \mathbb{P}^4$ of degree D can contain at most D lines.*
(ii) Let f and g be two trivariate polynomials without a common factor. Then $Z(f,g) := Z(f) \cap Z(g)$ contains at most $\deg(f) \cdot \deg(g)$ lines.

Flecnode polynomials and ruled surfaces in four dimensions.

Ruled surfaces in three dimensions. We first review several basic properties of ruled two-dimensional surfaces in $\mathbb{R}^3$ or in $\mathbb{C}^3$. For the sake of completeness we provide proofs of these properties in a companion paper [36].

For a modern approach to ruled surfaces, there are many references; see, e.g., Hartshorne [15, Section V.2], Beauville [2, Chapter III], or the Appendix in Kollar's paper [17]. We say that a real (resp., complex) surface X is *ruled by real* (resp., *complex*) *lines* if every point $p \in X$ in a Zariski-open [4] dense set is incident to a real (resp., complex) line that is fully contained in X; see, e.g., [29] or [7] for further details on ruled surfaces.

In three dimensions, a two-dimensional irreducible ruled surface can be either *singly ruled*, or *doubly ruled* (notions that are elaborated below), or a plane. As the following lemma shows, the only doubly ruled surfaces are *reguli*, where a regulus is the union of all lines that meet three pairwise skew lines. There are only two kinds of reguli, both of which are quadrics—hyperbolic paraboloids and hyperboloids of one sheet; see, e.g., Fuchs and Tabachnikov [9] for more details.

The following (folklore) lemma provides a (somewhat stronger than usual) characterization of doubly ruled surfaces; see [36] for a proof.

[4] The Zariski closure of a set Y is the intersection of all varieties X that contain Y. Y is Zariski closed if it is equal to its closure (and is therefore a variety), and is Zariski open if its complement is Zariski closed. See [15] for further details.

Lemma 8. *Let V be an irreducible ruled surface in $\mathbb{R}^3$ or in $\mathbb{C}^3$ which is not a plane, and let $\mathcal{C} \subset V$ be an algebraic curve, such that every non-singular point $p \in V \setminus \mathcal{C}$ is incident to exactly two lines that are fully contained in V. Then V is a regulus.*

When V is an irreducible ruled surface which is neither a plane nor a regulus, it must be *singly ruled*, in the precise sense spelled out in the following theorem (see also [13]); again, see [36, Theorem 10] for a proof.

Theorem 9. *(a) Let V be an irreducible ruled two-dimensional surface of degree $D > 1$ in $\mathbb{R}^3$ (or in $\mathbb{C}^3$), which is not a regulus. Then, except for at most two exceptional lines, the lines that are fully contained in V are parametrized by an irreducible algebraic curve Σ_0 in the Plücker space $\mathbb{P}^5$, and thus yield a 1-parameter family of generator lines $\ell(t)$, for $t \in \Sigma_0$, that depend continuously on the real or complex parameter t. Moreover, if $t_1 \neq t_2$, and $\ell(t_1) \neq \ell(t_2)$, then there exist sufficiently small and disjoint neighborhoods Δ_1 of t_1 and Δ_2 of t_2, such that all the lines $\ell(t)$, for $t \in \Delta_1 \cup \Delta_2$, are distinct.*

(b) There exists a one-dimensional curve $\mathcal{C} \subset V$, such that any point p in $V \setminus \mathcal{C}$ is incident to exactly one generator line of V.

Following this theorem, we refer to irreducible ruled surfaces that are neither planes nor reguli as *singly ruled*. A line ℓ, fully contained in an irreducible singly ruled surface V, such that every point of ℓ is incident to another line fully contained in V, is called an *exceptional line* of V (these are the lines mentioned in Theorem 9(a)). If there exists a point $p_V \in V$, which is incident to infinitely many lines fully contained in V, then p_V is called an *exceptional point* of V. By Guth and Katz [13], V can contain at most one exceptional point p_V (in which case V is a cone with p_V as its apex), and (as also asserted in the theorem) at most two exceptional lines.

The flecnode polynomial in four dimensions. Let $f \in \mathbb{C}[x, y, z, w]$ be a polynomial of degree $D \geq 4$. A *flecnode* of f is a point $p \in Z(f)$ for which there exists a line that passes through p and osculates to $Z(f)$ to order four at p. Therefore, if the direction of the line is $v = (v_0, v_1, v_2, v_3)$, then it osculates to $Z(f)$ to order four at p if $f(p) = 0$ and

$$F_i(p; v) = 0, \qquad \text{for } i = 1, 2, 3, 4. \tag{8}$$

The four-dimensional *flecnode polynomial* of f, denoted FL_f^4, is the polynomial obtained by eliminating v from the four equations in the system (8). (See Salmon [29], and the relevant applications thereof in [8, 13], for details concerning flecnode polynomials in three dimensions; see also Ivey and Landsberg [16] for a more modern generalization of this concept.) Note that these four polynomials are homogeneous in v (of respective degrees 1, 2, 3, and 4). We thus have a system of four equations in eight variables, which is homogeneous in the four variables v_0, v_1, v_2, v_3. Eliminating those variables results in a single polynomial equation in $p = (x, y, z, w)$. Using standard techniques, as in Cox et al. [5], the resulting polynomial FL_f^4 is the *multipolynomial resultant* $Res_4(F_1, F_2, F_3, F_4)$ of F_1, F_2, F_3, F_4, regarding these as polynomials in v (where the coefficients are polynomials in p). By definition, FL_f^4 vanishes at all the flecnodes of f. The following results are immediate consequences of the theory of multipolynomial resultants, presented in Cox et al. [5].

Lemma 10. *Given a polynomial $f \in \mathbb{C}[x, y, z, w]$ of degree $D \geq 4$, its flecnode polynomial FL_f^4 has degree $O(D)$.*

Lemma 11. *Given a polynomial $f \in \mathbb{C}[x, y, z, w]$ of degree $D \geq 4$, every line that is fully contained in $Z(f)$ is also fully contained in $Z(\mathsf{FL}_f^4)$.*

Ruled Surfaces in four dimensions. Flecnode polynomials are a major tool for characterizing ruled surfaces. This is manifested in the following theorem of Landsberg [22], which is a crucial tool for our analysis. It is established in [22] as a considerably more general result, but we formulate here a special instance that suffices for our needs.

Theorem 12 (Landsberg [22])**.** *Let $f \in \mathbb{C}[x, y, z, w]$ be a polynomial of degree $D \geq 4$. Then $Z(f)$ is ruled by (complex) lines if and only if $Z(f) \subseteq Z(\mathsf{FL}_f^4)$.*

We note that Theorem 12 extends the classical Cayley–Salmon theorem in three dimensions (see Salmon [29]). A quick review of this result is given below. We also note that we will use a refined version of this theorem, also due to Landsberg, given as Theorem 23 in Section 3.

When f is of degree ≤ 3, we have the following simpler situation, whose proof is omitted here.

Lemma 13. *For every polynomial $f \in \mathbb{C}[x, y, z, w]$ of degree ≤ 3, $Z(f)$ is ruled by (possibly complex) lines.*

Back to three dimensions. In three dimensions the analysis is somewhat simpler, and goes back to the 19th century, in Salmon's work [29] and others. The flecnode polynomial $\mathsf{F}L_f$ of f, defined in an analogous (and simpler) manner, is of degree $11\deg(f) - 24$ [29]. Theorem 12 is replaced by the Cayley–Salmon theorem [29], with the analogous assertion that $Z(f)$ is ruled by lines if and only if $Z(f) \subseteq Z(\mathsf{F}L_f)$. A simple proof of the Cayley–Salmon theorem can be found in Terry Tao's blog [44].

We will be using the following result, established by Guth and Katz [12]; see also [8]. It is an immediate consequence of Lemma 7(ii).

Proposition 14. *Let f be a trivariate irreducible polynomial of degree D. If $Z(f)$ fully contains more than $11D^2 - 24D$ lines then $Z(f)$ is ruled by (possibly complex) lines.*

Flat points and the second fundamental form. We continue with the four-dimensional setup. Extending the notation in Guth and Katz [12] (see also [8], and also Pressley [25] and Ivey and Landsberg [16] for more basic references), we call a non-singular point p of $Z(f)$ *linearly flat*, if it is incident to at least three distinct 2-flats that are fully contained in $Z(f)$ (and thus also in the tangent hyperplane $T_pZ(f)$). (The original definition, in [8, 13], for the three-dimensional case, is that a non-singular point $p \in Z(f)$ is linearly flat if it is incident to three distinct lines that are fully contained in $Z(f)$). The condition for a point p to be linearly flat can be worked out as follows, suitably extending the technique used in three dimensions in [8, 13].

Let p be a non-singular point of $Z(f)$, and let $f^{(2)}$ denote the second-order Taylor expansion of f at p. That is, we have, for any direction vector v and $t \in \mathbb{C}$,

$$f^{(2)}(p + tv) = t\nabla f(p) \cdot v + \tfrac{1}{2}t^2 v^T H_f(p) v = tF_1(p; v) + \tfrac{1}{2}t^2 F_2(p; v). \tag{9}$$

If p is linearly flat, there exist three 2-flats π_1, π_2, π_3, contained in the tangent hyperplane $T_pZ(f)$, such that $v^T H_f(p)v = 0$, for all $v \in \pi_1, \pi_2, \pi_3$ (clearly, the first term $\nabla f(p) \cdot v$ also vanishes for any such v). A simple argument, based on the fact that $f^{(2)}$ is quadratic in v, shows that $f^{(2)}$ must be identically zero on $T_pZ(f)$, and we then say that p is a *flat* point of $Z(f)$. Therefore, every linearly flat point of $Z(f)$ is also a flat point of $Z(f)$ (albeit not necessarily vice versa). The same definition applies in three dimensions too. Another characterization of flat points is that the *second fundamental form* of f vanishes at every flat point; see the full version and [8, 13] for details.

We next show that the set of linearly flat points of $Z(f)$ is the zero set of a certain collection of nine polynomials, each of degree at most $3D - 4$, constructed from f and its first- and second-order partial derivatives. See the full version for details. We say that a line $\ell \subset Z(f)$ is a *singular* line of $Z(f)$, if all of its points are singular. We say that a line $\ell \subset Z(f)$ is a *flat* line of $Z(f)$ if it is not a singular line of $Z(f)$, and all of its non-singular points are flat. An easy observation is that a flat line can contain at most $D - 1$ singular points of $Z(f)$ (these are the points on ℓ where all four first-order partial derivatives of f vanish). Similarly, a non-singular line is flat if (and only if) it is incident to at least $3D - 3$ flat points.

A curious property of flat lines, over the complex field, is the following; see the full version for a proof.

Lemma 15. *If a line $\ell \subset Z(f)$ is flat, then the tangent space $T_pZ(f)$ is fixed for all the non-singular points $p \in \ell$.*

Finitely and infinitely ruled surfaces in four dimensions, and u-resultants. Recall again the definition of Ξ_p, for a polynomial $f \in \mathbb{C}[x, y, z, w]$ and a point $p \in Z(f)$, which is the union of all (complex) lines passing through p and fully contained in $Z(f)$, and that of Σ_p, as the set of directions (considered as points in $\mathbb{P}T_pZ(f)$) of these lines.

Fix a line $\ell \in \Xi_p$, and let $v = (v_0, v_1, v_2, v_3) \in \mathbb{P}^3$ represent its direction. Since $\ell \subset Z(f)$, the four terms $F_i(p; v) = \nabla_v^i f(p)$, for $i = 1, 2, 3, 4$, must vanish at p. These terms, which we denote shortly as $F_i(v)$ at the fixed p, are homogeneous polynomials of respective degrees $1, 2, 3$, and 4 in $v = (v_0, v_1, v_2, v_3)$. (Note that when $D \leq 3$, some of these polynomials are identically zero.)

We provide a (partial) *algebraic* characterization of points $p \in Z(f)$ for which $|\Sigma_p|$ is infinite; that is, points that are incident to infinitely many lines that are fully contained in $Z(f)$. We refer to this situation by saying that $Z(f)$ is *infinitely ruled* at p. To be precise, here we only characterize points that are incident to infinitely many lines that osculate to $Z(f)$ to order three. The passage from this to the full characterization will be done during the analysis in the next section.

u-resultants. The algebraic tool that we use for this purpose are *u-resultants*. Specifically, following and specializing Cox et al. [5, Chapter 3.5, page 116], define, for a vector $u = (u_0, u_1, u_2, u_3) \in \mathbb{P}^3$,

$$U(p; u_0, u_1, u_2, u_3) = Res_4\Big(F_1(p;v), F_2(p;v), F_3(p;v), u_0v_0 + u_1v_1 + u_2v_2 + u_3v_3\Big),$$

where $Res_4(\cdot)$ denotes, as earlier, the multipolynomial resultant of the four respective (homogeneous) polynomials, with respect to the variables v_0, v_1, v_2, v_3. For fixed p, this is the so-called *u-resultant* of $F_1(v), F_2(v), F_3(v)$. The following properties of u-resultants, proved in the full version, are important for our analysis.

Theorem 16. *The function $U(p; u_0, u_1, u_2, u_3)$ is a homogeneous polynomial of degree six in the variables u_0, u_1, u_2, u_3, and is a polynomial of degree $O(D)$ in $p = (x, y, z, w)$. For fixed $p \in Z(f)$, $U(p; u_0, u_1, u_2, u_3)$ is identically zero as a polynomial in u_0, u_1, u_2, u_3, if and only if there are infinitely many (complex) directions $v = (v_0, v_1, v_2, v_3)$, such that the corresponding lines $\{p + tv \mid t \in \mathbb{C}\}$ osculate to $Z(f)$ to order three at p.*

Corollary 17. *Fix $p \in Z(f)$. The polynomial $U(p; u_0, u_1, u_2, u_3)$ is identically zero, as a polynomial in u_0, u_1, u_2, u_3, if and only if there are more than six (complex) lines osculating to $Z(f)$ to order 3 at p.*

3 A tour of the proof of Theorem 3

Let P, L, m, n, q, and s be as in the theorem. The proof proceeds by induction on m, where the base cases of the induction are the ranges $m \le \sqrt{n}$ and $m \le M_0$, for a sufficiently large constant M_0. In both cases we have $I(P, L) \le A(m + n)$, for a suitable choice of A. Assume then that the bound holds for all $m' < m$, and consider an instance involving sets P, L, with $|P| = m > \sqrt{|L|} = \sqrt{n}$.

First partitioning scheme. We apply the polynomial partitioning theorem of Guth and Katz (see [13] and [19, Theorem 2.6]), to obtain a 4-variate (real) partitioning polynomial f of degree D, which is (recall 5) $O(m^{2/5}/n^{1/5})$ if $m = O(n^{4/3})$, and $O(n/m^{1/2})$ if $m = \Omega(n^{4/3})$. That is, each of the $O(D^4)$ connected components of $\mathbb{R}^4 \setminus Z(f)$ contains at most $O(m/D^4)$ points of P, where, as above, $Z(f)$ denotes the zero set of f.

Set $P_0 := P \cap Z(f)$ and $P' := P \setminus P_0$. Each line $\ell \in L$ is either fully contained in $Z(f)$ or intersects it in at most D points. Let L_0 denote the subset of lines of L that are fully contained in $Z(f)$ and put $L' = L \setminus L_0$. Then $I(P, L) = I(P_0, L_0) + I(P_0, L') + I(P', L')$.

Since each line in L' intersects $Z(f)$ in at most D points, it follows that $I(P_0, L') \le |L'| \cdot D \le nD$. To estimate $I(P', L')$, we proceed exactly as in the three-dimensional case. For example, when $m = O(n^{4/3})$, we have $k = O(D^4)$ cells, $\tau_1, \ldots, \tau_k$, and we define $P_i = P \cap \tau_i$, and L_i to be the set of lines of L that cross τ_i, for $i = 1, \ldots, k$. Put $m_i = |P_i| = O(m/D^4)$, and $n_i = |L_i|$, for each i; note that $\sum_i n_i \le n(D + 1) = O(nD)$, because each line crosses the cell boundaries in at most D points. This immediately yields $I(P_0, L') \le nD$. Using the trivial bounds $I(P_i, L_i) = O(m_i^2 + n_i)$ for $i = 1, \ldots, k$, and summing these bounds, we get

$$I(P_0, L') + I(P', L') = I(P_0, L') + \textstyle\sum_i I(P_i, L_i) =$$

$$O(nD) + \textstyle\sum_i O(m_i^2 + n_i) = O(D^4 \cdot (m/D^4)^2 + nD) = O(m^2/D^4 + nD) = O(m^{2/5}n^{4/5}),$$

for our choice of D. The same reasoning yields $I(P_0, L') + I(P', L') = O(m)$ for $m = \Omega(n^{4/3})$.

Estimating $I(P_0, L_0)$. We next bound the number of incidences between points and lines that are contained in $Z(f)$. Assume, to simplify the notation, that $P_0 = P$ and $L_0 = L$, so $|P_0| = m$ and $|L_0| = n$. By the nature of its construction, f is in general reducible (see [13]). However, certain steps of the analysis require that f be irreducible, and that the points of P be non-singular points of $Z(f)$. As we show in the full version, this can be done, by *partitioning* the points and lines among the irreducible components of $Z(f)$, and some other varieties, all of degree $\le D$, in such a way that it suffices to bound the number of incidences in every single component (i.e., between the points and lines assigned to the same component), and sum up the bounds; the number of "cross-component" incidences, where a point is assigned to one component, and an incident line to another component, is shown to be only $O(nD)$.

The flecnode polynomial FL_f^4 of f (see Section 2) vanishes identically on every line of L, and thus also on P, and is of degree $O(D)$. If FL_f^4 does not vanish identically on $Z(f)$, then $Z(f, \mathsf{FL}_f^4) := Z(f) \cap Z(\mathsf{FL}_f^4)$ is a two-dimensional variety that contains P, and all the lines of L, and is of degree $\deg(f) \cdot \deg(\mathsf{FL}_f^4) = O(D^2)$. The other possibility is that FL_f^4 vanishes identically on $Z(f)$, and then Theorem 23 (see also the full version) implies that $Z(f)$ is ruled by (possibly complex) lines.

First case: $Z(f, \mathsf{FL}_f^4)$ **is two-dimensional.** Put $g = \mathsf{FL}_f^4$. Our analysis only uses the facts that $\deg(g) = O(D)$, and that $Z(f,g)$ is two-dimensional, so the analysis applies for any such g.

We have a set P of m points and a set L of n lines in $\mathbb{C}^4$, so that P, and every line of L is contained in the two-dimensional algebraic variety $Z(f,g) \subset \mathbb{C}^4$. By pruning away all the lines containing at most $\max(D, \deg(g))$ points of P, we lose $O(nD)$ incidences, and all the surviving lines are contained in $Z(f,g)$, as is easily checked. For simplicity of notation, we continue to denote by L the set of surviving lines.

Let $Z(f,g) = \bigcup_{i=1}^{s} V_i$ be the decomposition of $Z(f,g)$ into its irreducible components, as described in Section 2. By Theorem 6, we have $\sum_{i=1}^{s} \deg(V_i) \le \deg(f)\deg(g) = O(D^2)$.

Lemma 18 (Sharir and Solomon [36, Theorem 15])**.** *Let V be a possibly reducible two-dimensional algebraic surface of degree $D > 1$ in $\mathbb{R}^3$ or in $\mathbb{C}^3$, with no linear components. Let P be a set of m distinct points on V and let L be a set of n distinct lines fully contained in V. Then there exists a subset $L_0 \subseteq L$ of at most $O(D^2)$ lines, such that the number of incidences between P and $L \setminus L_0$ satisfies*

$$I(P, L \setminus L_0) = O\left(m^{1/2} n^{1/2} D^{1/2} + m + n\right). \tag{10}$$

We can now proceed, by deriving two upper bounds for certain types of incidences between P and L. The first bound is relevant for the range $m = O(n^{4/3})$, and the second bound is relevant for the range $m = \Omega(n^{4/3})$. Nevertheless, both bounds apply to the entire range of m and n.

Proposition 19. *The number of incidences involving non-singular points of $Z(f)$ that are contained in components of $Z(f,g)$ that are not 2-flats is*

$$\min\left\{O(mD^2 + nD),\ O(m + nD^4)\right\}. \tag{11}$$

Sketch of Proof. We only establish here the bound $O(mD^2 + nD)$. Let $p \in Z(f)$ be a non-singular point. The irreducible decomposition of $S_p := Z(f,g) \cap T_p Z(f)$ is the union of one- and two-dimensional components. Clearly, S_p contains all the lines that are incident to p and are fully contained in $Z(f,g)$; it is a variety, embedded in 3-space (namely, in $T_p Z(f)$), of degree $O(D^2)$. The union of the one-dimensional components is a curve of degree $O(D^2)$, so, by Lemma 7(i), it can contain at most $O(D^2)$ lines; when summing over all $p \in P$, the total number of incidences with those lines is $O(mD^2)$.

It remains to bound incidences involving the two-dimensional components of S_p that are not 2-flats. By Sharir and Solomon [37, Lemma 5], the number of lines incident to p inside these two-dimensional components of S_p is at most $O(D^2)$, except possibly for lines that lie in a component that is a cone and has p as its apex. Summing over all $p \in P$, we get a total of $O(mD^2)$ incidences for this case too, ignoring lines that lie only in conic (or flat) components.

Note that each two-dimensional component of S_p is necessarily also a two-dimensional irreducible component of $Z(f,g)$. Hence the analysis performed so far takes care of all incidences except for those that occur on conic two-dimensional components of $Z(f,g)$ (and on 2-flats, which we totally ignore in this proposition). We omit this part, merely noting that we only need to consider points that are apices of such conic components, and we show (in the full version) that the number of incidences with these points is $O(mD + nD)$. See the full version for more details and for the proof of the second bound. ◻

Restrictedness of hyperplanes and quadrics, and lines on 2-flats. The bounds in Proposition 19 are too large, for the current choices of D, because of the respective terms $O(mD^2)$ and $O(nD^4)$. For the large D we therefore abandon Proposition 19, and instead we show the following, in which L_1 is obtained from L by pruning away the lines contained in planar components of $Z(f,g)$.

Lemma 20. *(a) Each hyperplane or quadric is $O(D^2)$-restricted with respect to L_1.*
(b) Each 2-flat contains at most $O(D)$ lines of L_1.

(Recall that, for a hyperplane or quadric H, being $O(D^2)$-restricted with respect to a set L_1 of lines means that, except for $O(D^2)$ lines, all the lines of L_1 within H lie in ruled components of some polynomial of degree $O(D)$, that are not 2-flats.)

Proof. (a) Fix a hyperplane or quadric H. Recall that all the lines in the current set L_1 are contained in $Z(f,g)$. Let V be an irreducible component of $Z(f,g)$, which is not a 2-flat. If $V \cap H$ is a curve, then (recalling Theorem 6) its degree is at most $\deg(V)$ (when H is a hyperplane) or $2\deg(V)$ (when H is a quadric), and can therefore contain at most $2\deg(V)$ lines, by Lemma 7(i). Therefore, the union of all the irreducible components V of $Z(f,g)$ which intersect H in a curve, contains at most $2\sum_V \deg(V) = O(D^2)$ lines. Assume then that $V \cap H$ is two-dimensional. Since V is irreducible, we must have $V \cap H = V$, so V is fully contained in H. Moreover, V is an irreducible two-dimensional surface contained in $Z(f) \cap H$, and therefore must be an irreducible component of $Z(f) \cap H$, which is a two-dimensional surface of degree $\le D$. By Theorem 6, $\sum_{V \subset H} \deg(V) \le \deg(Z(f) \cap H) \le \deg(f) \le D$. If V is not ruled by lines (and, by assumption, is not a 2-flat), then by Proposition 14, it contains at most $11\deg(V)^2$ lines, and summing over all such components V within H, we get a total of at most $\sum_V 11\deg(V)^2 = O(D^2)$ lines.

The remaining irreducible (two-dimensional) components V of $Z(f,g)$ that meet H (if such components exist) are fully contained in H, and are ruled by lines. As already observed, these components are also irreducible components of $Z(f) \cap H$, and so, with the exception of $O(D^2)$ lines (those contained in the components already analyzed), all the lines of L_1 that lie in H are contained in components of $Z(f) \cap H$ that are ruled by lines. Since f restricted to H is a polynomial of degree $\le D$, and since we are interested in lines of L_1 that are not contained in planar components of $Z(f) \cap H$ we conclude that H is $O(D^2)$-restricted, with respect to the subset of L_1 mentioned in the lemma.

(b) We argue that it suffices to consider 2-flats π that are fully contained in $Z(f,g)$. The intersection $Z(f) \cap \pi$ is either π itself, or a curve of degree $\le D$. The latter case implies (using Lemma 7) that π contains at most D lines that are fully contained in $Z(f)$. In the former case $\pi \subset Z(f)$. By assumption, π is not contained in $Z(f,g)$, implying that g intersects π in a curve of degree $O(D)$ (since $\pi \cap Z(f,g) = \pi \cap Z(g)$), and can therefore contain at most $O(D)$ lines that are fully contained in $Z(f)$. □

Incidences within hyperplanes and quadrics. To analyze incidences within 2-flat components of $Z(f,g)$ (involving the lines that we have just discarded), and also to handle the case where f is linear or quadratic, we establish the following proposition.

Proposition 21. *Let $H_1, \ldots, H_t$ be a finite collection of hyperplanes and quadrics. Assume that the points of P and the lines of L are partitioned among $H_1, \ldots, H_t$, so that each point $p \in P$ (resp., each line $\ell \in L$) is assigned to a unique hyperplane or quadric that contains p (resp., fully contains ℓ), and assume further that each H_i is q-restricted with respect to L, and that each 2-flat contains at most s lines of L. Then the overall number of incidences between points and lines that are assigned to the same surface is*

$$O\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + n\right). \tag{12}$$

We omit the proof. It is a straightforward application of the bound of Guth and Katz (see Theorem 1 and ([13]), applied separately within each H_i (when H_i is a hyperplane), or within a generic projection of H_i to some 3-flat (when it is a quadric). In each such application we exploit the properties that each H_i contains at most q lines, and each 2-flat contains at most s lines. We then sum up the bounds and use Hölder's inequality to conclude that this sum satisfies (12).

In particular, this bound takes care of the cases where f is linear or quadratic.

Incidences within 2-flats fully contained in $Z(f,g)$. The strategy here is to distribute the points of P and the lines of L among the 2-flats that contain them (lines not contained in any 2-flat are contained in some other component of $Z(f,g)$ and are dealt with separately—see below. Points that belong to at most two such 2-flats get duplicated at most twice, and we then apply the planar bound (1) to each 2-flat separately, and sum up the bounds, to get $O(m^{2/3}n^{1/3}s^{1/3} + m + n)$ (see details in the full version). Points that belong to at least three 2-flats contained in $Z(f,g)$ are *linearly flat* and thus *flat* (see Section 2). In this case, using the characterization of flat points presented in Section 2, we partition the points of P among their tangent hyperplanes (to $Z(f)$). Lines that are incident to at most $3D-4$ flat points contribute only $O(nD)$ incidences, and the "heavier" lines are flat. By Lemma 15, all the (flat) points that lie on a flat line have the *same* tangent hyperplane. We use this property to obtain a partition of the points and lines into distinct (tangent) hyperplanes, so that it suffices to bound the number of incidences within each hyperplane. We then use Proposition 21 and obtain again, with some care (see the full version for details) the bound in (12).

In summary, the overall outcome of the analysis for the first case is summarized in the following proposition.

Proposition 22. *Let g be any polynomial of degree $O(D)$ such that $Z(f,g)$ is two-dimensional, let P be a set of m*

points contained in $Z(f,g)$, and let L be a set of n lines contained in $Z(f,g)$. Then

$$I(P,L) = I(P^*,L^*) + O\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + nD\right), \tag{13}$$

where P^* and L^* are subsets of P and L, respectively, so that each hyperplane or quadric is $O(D^2)$-restricted with respect to L^*, and each 2-flat contains at most $O(D)$ lines of L^*. We also have the explicit estimate

$$I(P^*,L^*) = \min\{O\left(mD^2 + nD\right),\ O\left(m + nD^4\right)\}. \tag{14}$$

Second case: $Z(f)$ is ruled by (complex) lines.

We next consider the case where the four-dimensional flecnode polynomial FL_f^4 vanishes identically on $Z(f)$. By Theorem 12, this implies that $Z(f)$ is ruled by (possibly complex) lines.

In what follows we assume that $D \geq 3$ (the cases $D = 1, 2$ have already been treated earlier, using Proposition 21). We prune away points $p \in P$, with $|\Sigma_p| \leq 6$ (the number of incidences involving these points is at most $6m = O(m)$). For simplicity of notation, we still denote the set of surviving points by P. Thus we now have $|\Sigma_p| > 6$, for every $p \in P$.

Recalling the properties of the u-resultant of f (that is, the u-resultant associated with $F_1(p;v)$, $F_2(p;v)$, $F_3(p;v)$), as reviewed in Section 2, we have, by Corollary 17, that $U(p;u_0,u_1,u_2,u_3) \equiv 0$ (as a polynomial in $u_0,\ldots,u_3$) for every $p \in P$.

We will use the following theorem of Landsberg, which generalizes Theorem 12. It is stated here in a specialized and slightly revised form that suffices for our purposes. Recall that Σ^3 is the union of Σ_p^3 over all $p \in X$, namely, it is the set of all lines that osculate to $Z(f)$ to order three at some point on $Z(f)$.

Theorem 23 (Landsberg [16, Theorem 3.8.7])**.** *Consider $Z(f)$ as a variety in $\mathbb{P}^4(\mathbb{C})$, and assume that there is an irreducible component $\Sigma_0^3 \subset \Sigma^3$ satisfying, for every point p in a Zariski open set $\mathcal{O} \subset Z(f)$, $\dim \Sigma_{0,p}^3 \geq 1$, where $\Sigma_{0,p}^3$ is the set of lines in Σ_0^3 incident to p. Then, for each point $p \in \mathcal{O}$, all lines in $\Sigma_{0,p}^3$ are contained in $Z(f)$.*

If $U(p;u_0,u_1,u_2,u_3)$ does not vanish identically (as a polynomial in u_0,u_1,u_2,u_3) at every point $p \in Z(f)$, then at least one of its coefficients, call it c_U, does not vanish identically on $Z(f)$. In this case, as U vanishes identically at every point of P (as a polynomial in u_0,u_1,u_2,u_3), it follows that P is contained in the two-dimensional variety $Z(f,c_U)$. Since c_U has degree $O(D)$ in x,y,z,w (by Theorem 16), we can proceed exactly as we did in the case where $Z(f,\mathsf{FL}_f^4)$ was 2-dimensional. That is, we obtain the bound (13) in Proposition 22, namely,

$$I(P,L) = I(P^*,L^*) + O\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + nD\right), \tag{15}$$

where P^* and L^* are subsets of P and L, respectively, so that each hyperplane or quadric is $O(D^2)$-restricted with respect to L^*, and each 2-flat contains at most $O(D)$ lines of L^*. We also have the explicit estimate

$$I(P^*,L^*) = \min\{O\left(mD^2 + nD\right),\ O\left(m + nD^4\right)\}.$$

Therefore, since this case does not require any further analysis, it suffices to consider the complementary situation, where we assume that $U(p;u_0,u_1,u_2,u_3) \equiv 0$ at every point $p \in Z(f)$ (as a polynomial in u_0,u_1,u_2,u_3). By Theorem 16, Σ_p^3 is infinite, so its dimension is positive, for each such p.

Informally, the analysis proceeds as follows. Since Σ_p^3 is (at least) one-dimensional for every point $p \in Z(f)$, the set Σ^3, which is the union of Σ_p^3, over all $p \in Z(f)$, has (at least) three degrees of freedom—three for specifying p, at least one for specifying the line in Σ_p^3, and one removed because the same line may arise at each of its points (if it is fully contained in $Z(f)$). The following major technical result, whose proof is omitted in this version, facilitates the application of Theorem 23 in our context.

Theorem 24. *There exists an irreducible component Σ_0^3 of Σ^3 of dimension at least three, such that for each non-singular $p \in Z(f)$, the variety $\Sigma_{0,p}^3$ is at least one-dimensional.*

Theorem 23, with Σ_0^3 as specified by Theorem 24, then implies that $Z(f)$ is *infinitely ruled* by lines, in the sense defined in Section 2; that is, each point $p \in Z(f)$ is incident to infinitely many lines that are fully contained in $Z(f)$, and, moreover, $\Sigma_{0,p}^3 = \Sigma_{0,p}$ (which is the set of lines in Σ_0 incident to p). That is, we have shown that $\Sigma_0^3 = \Sigma_0$. In other words, for each $p \in Z(f)$, $\Sigma_{0,p}$ is of dimension at least 1, or, equivalently, the cone $\Xi_{0,p}$ (which is the union of the lines in $\Sigma_{0,p}$) is at least two-dimensional. If, for some non-singular $p \in Z(f)$, the cone $\Xi_{0,p}$ were three-dimensional, then, one can show that $Z(f)$ must be a hyperplane, contrary to assumption. Thus, for each non-singular point $p \in Z(f)$, the cone $\Xi_{0,p}$ is two-dimensional, and $\Sigma_{0,p}$ is one-dimensional. We also have $\dim(\Sigma_0) = \dim(\Sigma_0^3) \geq 3$. We thus have

Corollary 25. *The union of lines in $\Sigma_0^3 = \Sigma_0$ is equal to $Z(f)$, and $\dim(\Sigma_0) = \dim(\Sigma_0^3) \geq 3$.*

Severi's theorem. The following theorem is a major ingredient in the present part of our analysis. It has been obtained by Severi [31] in 1901. A variant of this result has also been obtained by Segre [30]; see also the more recent works [23, 27, 28]. As a small service to the community, we sketch in Appendix A of the full version a proof of this theorem (or rather a special case of the theorem that arises in our context), suggested to us by A. J. de Jong.

We state here a special case of the theorem that we need; see the full version for the general formulation.

Theorem 26 (Severi's Theorem [31])**.** *Let $X \subset \mathbb{P}^4(\mathbb{C})$ be a three-dimensional irreducible variety, and let Σ_0 be a component of maximal dimension of $\Sigma = \Sigma(X)$, such that the lines of Σ_0 cover X. Then the following holds. (i) If $\dim(\Sigma_0) = 4$, then X is a hyperplane. (ii) If $\dim(\Sigma_0) = 3$, then either X is a quadric, or X is ruled by 2-flats.*

(Informally, $\dim(\Sigma_0) = 3$ corresponds to the case where X is infinitely ruled by lines of Σ_0: There are four degrees of freedom to specify a line in Σ_0—three to specify $p \in Z(f)$, and one to specify the line in $\Sigma_{0,p}$, but one degree of freedom has to be removed, to account for the fact that the same line arises at each of its points. Severi's theorem asserts that in this case the infinite family of lines of $\Sigma_{0,p}$ must form a 2-flat, unless X is a quadric or a hyperplane.)

Applying the second case in Severi's theorem to $Z(f)$, which is justified by the preceding arguments, we conclude that either $Z(f)$ is a quadric or it is ruled by 2-flats. The case where $Z(f)$ is a quadric (or a hyperplane) has already been dealt with by Proposition 21, so we consider only the case where $Z(f)$ is ruled by (complex) 2-flats.

The case where $Z(f)$ is ruled by 2-flats. In the remaining case, every point[5] $p \in Z(f)$ is incident to at least one 2-flat $\tau_p \subset Z(f)$. Let D_p denote the set of 2-flats that pass through p and are contained in $Z(f)$.

For a non-singular point $p \in Z(f)$, if $|D_p| > 2$, then p is a (linearly flat and thus) flat point of $Z(f)$. Repeating the analysis of incidences with flat points, as in the treatment of the first case, we conclude that the number of such incidences is

$$O\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + n\right).$$

In what follows we therefore assume that all points of P are non-singular and non-flat, and therefore $|D_p| = 1$ or 2, for each such p. The analysis of this case is very similar to the analysis of the same setup in the first case (with some subtle differences, discussed in the full version). It yields the same bound $O(m^{2/3}n^{1/3}s^{1/3} + m + n)$ on the number of these incidences.

In summary, combining the bounds that we have obtained for the various subcases of the second case, we get the following proposition.

Proposition 27. *Let P be a set of m points contained in $Z(f)$, and let L be a set of n lines contained in $Z(f)$, and assume that $Z(f)$ is ruled by lines and that f is of degree ≥ 3. Then*

$$I(P, L) = I(P^*, L^*) + O\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + nD\right), \tag{16}$$

where P^* and L^* are subsets of P and L, respectively, so that each hyperplane or quadric is $O(D^2)$-restricted with respect to L^*, and each 2-flat contains at most $O(D)$ lines of L^*. We also have the explicit estimate

$$I(P^*, L^*) = \min\{O\left(mD^2 + nD\right),\ O\left(m + nD^4\right)\}. \tag{17}$$

The induction. Although the actual inductive process is somewhat more involved, as it has to consider the various irreducible components of $Z(f)$ and of its various derivatives, and the corresponding partition of P and L among

[5]Similar to the definition in Section 2 for the case of lines, it suffices to require this property for every point in some Zariski-open subset of X. Here too one can show that the two definitions are equivalent. See also the companion paper [36, Lemma 11].

these varieties, we describe it focusing on a single set P of non-singular points and a single set L of lines within some irreducible variety $Z(f)$ of degree at most D. We recall that a direct attempt to bound $I(P, L)$, via Propositions 22 or 27 fails because of the weak terms $O(mD^2)$ and $O(nD^4)$, which are too large when D is large. However, as in the propositions, "most" of the incidences can be handled directly (as in (13) or (16)), and we are left with the subsets P^*, L^* (which we denote here as P and L, for simplicity) for which the analysis fails, but now every hyperplane and quadric is $O(D^2)$-restricted with respect to $L = L^*$, and every 2-flat contains at most $O(D)$ lines of L.

We now discard f, forget the fact that the points of P and the lines of L lie in $Z(f)$, and start afresh with a new partitioning polynomial h, of degree $E \ll D$. We obtain $O(E^4)$ cells, each containing at most $O(m/E^4)$ points of P, and each line of L either crosses at most $E + 1$ cells, or is fully contained in $Z(h)$.

Set $P_0 := P \cap Z(h)$ and $P' := P \setminus P_0$. Similarly, denote by L_0 the set of lines of L that are fully contained in $Z(h)$, and put $L' := L \setminus L_0$. We repeat the whole analysis done so far, but with h and its degree E instead of f and D, for the points of P and the lines of L. Arguing as above, $I(P_0, L') = O(nE)$. $I(P_0, L_0)$ is now bounded by Propositions 22 or 27 with E replacing D, and with plugging the explicit bound $\min\{O\left(mE^2 + nE\right),\ O\left(m + nE^4\right)\}$ for the resulting "problematic" subsets P^*, L^*. As we show (see the full version), since $E \ll D$, this still keeps us within the bound in (3).

The problem, as already noted, is in handling $I(P', L')$, namely incidences within the partition cells (the "opposite" problem to those that arise when D is large). Here we finally use the induction hypothesis within each cell, and the fact that every hyperplane and quadric is $O(D^2)$-restricted with respect to L, and every 2-flat contains at most $O(D)$ lines of L. Since each cell involves, roughly, m/E^4 points and (on average) n/E^3 lines, an easy calculation shows that the induction yields the overall bound

$$I(P', L') \le b \cdot 2^{c\sqrt{\log(m/E^4)}} m^{2/5} n^{4/5} + bA\left(m^{1/2} n^{1/2} D^{1/2} E^{1/2} + m^{2/3} n^{1/3} D^{1/3} E^{1/3} + m + nE\right).$$

As we show, a suitable choice of E makes this bound subsumed in (3). This establishes the induction step and thus, finally, completes the proof of the upper bound (3) in the theorem. The sharper bound (4), when applicable, can be obtained without using induction, as noted above, and the lower bound is presented in the full version. Altogether, Theorem 3, and thus also Theorem 2, follow. □

4 Discussion

The results of this paper (almost) settle the problem of point-line incidences in four dimensions, but they raise several interesting and challenging open problems. Among them are:

(a) Get rid of the factor $2^{c\sqrt{\log m}}$ in the bound. We have achieved this improvement when m is not too close to $n^{4/3}$, so to speak, allowing us to use the weak but non-inductive bounds and complete the analysis in one step. We believe that the ranges of m where this can be done can be enlarged, e.g., by improving these weak bounds. It would also be interesting to improve the bound using the strategy in [32, 37], which generates a sequence of ranges of m, converging to $m = \Theta(n^{4/3})$, where in each range the improved bound (4) holds, with a different constant of proportionality A.

(b) Extend (and sharpen) the bound of Corollary 4 for any value of k. In particular, is it true that the number of intersection points of the lines (this is the case $k = 2$; the intersection points are also known as 2-*rich points*) is $O(n^{4/3} + nq^{1/2} + ns)$? We conjecture that this is indeed the case. A deeper question, extending a similar open problem in three dimensions that has been posed by Guth and others (see, e.g., Katz's expository note [21]), is whether the above conjectured bound can be improved when $q = o(n^{2/3})$ and $s = o(n^{1/3})$, that is, when the second and third terms in the conjectured bound become much smaller than $n^{4/3}$. We also note that if we could establish such a bound for the number of k-rich points, for any constant k (when q and s are not too large), then the case of large m (that is, $m = \Omega(n^{4/3})$) would become vacuous, as only $O(n^{4/3})$ points could be incident to more than k lines.

(c) Extend the study to five and higher dimensions. In a preliminary study, joint with Adam Sheffer [33], we do it using a constant-degree partitioning polynomial, with the disadvantages discussed above (slightly weaker bounds, significantly more restrictive assumptions, and inferior "lower-dimensional" terms). The leading terms in the resulting bounds, for points and curves in $\mathbb{R}^d$, are $O(m^{2/(d+1)+\varepsilon} n^{d/(d+1)} + m^{1+\varepsilon})$, for any $\varepsilon > 0$. Obtaining sharper results, like the ones obtained in this paper, is quite challenging algebraically, although some of the tools developed in this work seem promising for higher dimensions too.

(d) If we are given in advance that the points and lines lie in some algebraic surface of a given (low) degree $D > 2$, can we improve the bound and/or simplify the analysis? In our companion work [36] we achieve these goals for two-dimensional surfaces (embedded in any dimension), improving in particular the bound of Guth and Katz [13] in this special case.

(e) Elaborating on item (a) above, we note that the "culprit" Proposition 19, which produces the weak bounds that force us to go into the induction, is only used in the case where $Z(f, g)$ is two-dimensional, and the difficulty there lies in bounding the number of incidences within a two-dimensional ruled surface (be it either one irreducible ruled surface of large degree, or the union of many irreducible ruled surfaces of small degree). The analysis of the three-dimensional analogous situation (addressed in Guth and Katz [13]), cannot be applied here, since the degree of the underlying surface is $O(D^2)$ instead of D in [13]. In a recent study of Szemerédi-Trotter type theorems in three dimensions [17], Kollár uses the *arithmetic genus* of curves to prove effective bounds on the number of point-line incidences in three dimensions. In four dimensions, the situation is more involved, but we hope that the arithmetic genus of the surface $Z(f, g)$ may yield effective bounds for the number of incidences within this surface.

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