FO Model Checking on Posets of Bounded Width

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Abstract

Over the past two decades the main focus of research into first-order (FO) model checking algorithms have been sparse relational structures—culminating in the FPT-algorithm by Grohe, Kreutzer and Siebertz for FO model checking of nowhere dense classes of graphs [STOC'14], with dense structures starting to attract attention only recently. Bova, Ganian and Szeider [CSL–LICS'14] initiated the study of the complexity of FO model checking on partially ordered sets (posets). Bova, Ganian and Szeider showed that model checking *existential* FO logic is fixed-parameter tractable (FPT) on posets of bounded width, where the width of a poset is the size of the largest antichain in the poset. The existence of an FPT algorithm for general FO model checking on posets of bounded width, however, remained open. We resolve this question in the positive by giving an algorithm that takes as its input an n-element poset $\mathcal{P}$ of width w and an FO logic formula φ, and determines whether φ holds on $\mathcal{P}$ in time $f(\varphi, w) \cdot n^2$.

Keywords

first-order logic; partially ordered sets; parameterized complexity; algorithmic meta-theorems

I. Introduction

Algorithmic meta-theorems are general algorithmic results applying to a whole range of problems, rather than just to a single problem alone. Such results are some of the most sought-after in algorithmic research. Many prominent algorithmic meta-theorems are about *model checking*; such theorems state that for certain kinds of logic $\mathcal{L}$, and all classes $\mathcal{C}$ that have a certain structure, there is an algorithm that takes as an input a formula $\varphi \in \mathcal{L}$ and a structure $S \in \mathcal{C}$ and efficiently determines whether $S \models \varphi$. Here $S \models \varphi$ is read as "S models φ" or "φ holds on S". Examples of theorems of this kind include the classic theorem of Courcelle [1], as well as a large body of work on model checking first-order (FO) logic [2], [3], [4], [5], [6], [7], [8], [9], [10].

Most of the research on algorithms for FO model checking has focused on graphs. On general graphs there is a naive brute-force algorithm that takes as an input an n-vertex graph G and a formula φ and determines whether $G \models \varphi$ in time $n^{O(|\varphi|)}$ by enumerating all the possible ways to instantiate the variables of φ. On the other hand, the problem is PSPACE-complete (see e.g. [11]) and encodes the Clique problem, thus it admits no algorithm with running time $f(\varphi)n^{o(|\varphi|)}$ for any function f [12], assuming the Exponential Time Hypothesis [13] (ETH). Thus, assuming the ETH the naive algorithm is the best possible, up to constants in the exponent. Furthermore, FO model checking remains PSPACE-complete on any fixed graph containing at least two vertices (again, see [11]). Hence, it is futile to look for restricted classes of graphs in which FO model checking can be done in polynomial time without restricting φ. Therefore, research has focused on obtaining algorithms with running time $f(\varphi)n^{O(1)}$ on restricted classes of graphs and other structures. Algorithms with such a running time are said to be *fixed*

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parameter tractable (FPT) parameterized by φ. Even though FPT algorithms are not polynomial time algorithms, due to unlimited f, they significantly outperform brute-force.

The parameterized complexity of FO model checking on *sparse* graph classes is now well understood. In 1994 Seese [10] showed an FPT algorithm for FO model checking on graphs of bounded degree. Seese's algorithm was followed by a long line of work [6], [5], [3], [4] giving FPT algorithms for progressively larger classes of sparse graphs, culminating in the FPT algorithm of Grohe, Kreutzer and Siebertz [9] on any nowhere dense graph classes. To complement this, Kreutzer [14] and Dvořák et al. [4] proved that if a class $\mathcal{C}$ closed under taking subgraphs is not nowhere dense, then deciding first-order properties of graphs in $\mathcal{C}$ is not fixed-parameter tractable unless FPT=W[1] (a complexity-theoretic collapse which is considered to be unlikely). Hence, this suggests that the result of Grohe, Kreutzer and Siebertz [9] captures *all* subgraph-closed sparse graph classes on which FO model checking is fixed parameter tractable.

However, for other types of structures, such as dense graphs or algebraic structures, the parameterized complexity of FO model checking is largely uncharted territory. Grohe [15] notes that "*it would also be very interesting to study the complexity of model-checking problems on finite algebraic structures such as groups, rings, fields, lattices, et cetera*". From this perspective it is particularly interesting to investigate model checking problems on *partially ordered sets* (posets), since posets can be seen both as dense graphs and as algebraic structures. Motivated by Grohe's survey [15], Bova, Ganian and Szeider [2], [16] initiated the study of FO model checking on posets. As a preliminary result, they show that FO model checking on posets parameterized by φ is not fixed parameter tractable unless FPT = W[1], motivating the study of FO model checking on restricted classes of posets. Bova, Ganian and Szeider [2] identified posets of bounded width as a particularly interesting class to investigate. Their main technical contribution has been an FPT algorithm for model checking *existential* FO logic on posets of bounded width, and they left the existence of an FPT algorithm for model checking FO logic on posets of bounded width as an open problem. In subsequent work Gajarský et al. [8] gave a simpler and faster algorithm for model checking existential FO logic on posets of bounded width. Nevertheless, the existence of an FPT algorithm for model checking full FO logic remained open.

Our contribution: In this paper we resolve the open problem of Bova, Ganian and Szeider by designing a new algorithm for model checking FO logic on posets. The running time of our algorithm on an n-element poset of width w is $f(\varphi, w) \cdot n^2$. Thus our algorithm is not only FPT when parameterized by φ on posets of bounded width, it is also FPT by the compound parameter $\varphi + w$. We demonstrate the generality and applicability of our main result by showing that a simple FO-interpretation can be used to obtain an FPT-algorithm for another natural dense graph class, namely k-fold proper interval graphs. This generalizes and simplifies the main result of Ganian et al. [7].

Our algorithm is based on a new locality lemma for posets. More concretely, we show that for every poset $\mathcal{P}$ and formula φ one can efficiently iteratively construct a directed graph D such that (a) the vertex set of D are the elements of $\mathcal{P}$, (b) every element of $\mathcal{P}$ has bounded out-degree in D, and (c) it is possible to determine whether $\mathcal{P} \models \varphi$ by checking whether φ holds on sub-posets of $\mathcal{P}$ induced by constant-radius balls in D.

The statement of our lemma sounds very similar to that of Gaifman's locality theorem, the crucial differences being that the digraph D is not the Gaifman graph of $\mathcal{P}$ and that D depends on the quantifier rank of φ. Indeed, constant radius balls in the Gaifman graph of constant width posets typically contain the entire poset. Thus a naive application of Gaifman's theorem would reduce the problem of deciding whether $\mathcal{P}$ is a model of φ to itself. The crucial difficulty we have to overcome is that we have to make the digraph D "dense enough" so that (c) holds, while keeping it "sparse enough" so that the vertices in D still have bounded out-degree. The latter is necessary to ensure that constant radius balls in D have constant size, making it feasible to use the naive model checking algorithm for determining whether φ holds on sub-posets of $\mathcal{P}$ induced by constant-radius balls in D. The construction of the graph D and the proof that it indeed has the desired properties relies on a delicate inductive argument thoroughly exploiting properties of posets of bounded width.

Organization of the paper: In Section II we set up the definitions and the necessary notation. In Section III we define the digraph D used in our poset locality lemma, and prove some useful structural properties of D. In

Section IV we prove the locality lemma for posets and give the FPT algorithm for FO logic model checking. We then proceed to show in Section V how our algorithm can be used to give an FPT algorithm for model checking FO logic on k-fold proper interval graphs. Finally, in Section VI we conclude with a discussion of further research directions.

II. Preliminaries

A. Graphs and Posets

We deal with directed graphs (shortly *digraphs*) whose vertices and arcs bear auxiliary labels, and which may contain parallel arcs. For a directed graph D, a vertex $v \in V(D)$, and an integer r, we denote by $R_r^D(v)$, the set of vertices of D that are reachable from v via a directed path of length at most r. Slightly abusing the notation, we extend the function R_r^D to multiple vertices as follows; $R_r^D(v_1, \dots, v_k) = \bigcup_{i=1}^{k} R_r^D(v_i)$. Moreover, for $v, v' \in V(D)$ we denote by $\mathrm{dist}_D(v, v')$ the length of a shortest directed path from v to v' in D.

A *poset* $\mathcal{P}$ is a pair $(P, \leq^{\mathcal{P}})$ where P is a finite set and $\leq^{\mathcal{P}}$ is a reflexive, anti-symmetric, and transitive binary relation over P. The *size* of a poset $\mathcal{P} = (P, \leq^{\mathcal{P}})$ is $\|\mathcal{P}\| := |P|$. We say that p and p' are *incomparable* (in $\mathcal{P}$), denoted $p \parallel^{\mathcal{P}} p'$, if neither $p \leq^{\mathcal{P}} p'$ nor $p' \leq^{\mathcal{P}} p$ hold. We say that p' is *above* p (and p is *below* p') if $p \leq^{\mathcal{P}} p'$ and $p \neq p'$. A *chain* C of $\mathcal{P}$ is a subset of P such that $x \leq^{\mathcal{P}} y$ or $y \leq^{\mathcal{P}} x$ for every $x, y \in C$. A *chain partition* of $\mathcal{P}$ is a tuple $(C_1, \dots, C_k)$ such that $\{C_1, \dots, C_k\}$ is a partition of P and for every i with $1 \leq i \leq k$ the poset induced by C_i is a chain of $\mathcal{P}$. An *anti-chain* A of $\mathcal{P}$ is a subset of P such that for all $x, y \in P$ it is true that $x \parallel^{\mathcal{P}} y$. The *width* of a poset $\mathcal{P}$, denoted by $\mathrm{width}(\mathcal{P})$ is the maximum cardinality of any anti-chain of $\mathcal{P}$. Note that for a poset of width w there is a chain partition with w chains, which can be efficiently constructed:

Proposition II.1 ([17, Theorem 1.]). *Let $\mathcal{P}$ be a poset. Then in time $\mathcal{O}(\mathrm{width}(\mathcal{P}) \cdot \|\mathcal{P}\|^2)$, it is possible to compute both $\mathrm{width}(\mathcal{P}) = w$ and a corresponding chain partition $(C_1, \dots, C_w)$ of $\mathcal{P}$.*

B. Parameterized Complexity

Here we introduce the most basic concepts of parameterized complexity theory. For more details, we refer to the many existing text books on the topic [18], [19], [20]. An instance of a parameterized problem is a pair $\langle x, k \rangle$ where x is the input and k a parameter. A parameterized problem $\mathcal{P}$ is *fixed-parameter tractable (FPT)* if, for every instance $\langle x, k \rangle$, it can be decided whether $\langle x, k \rangle \in \mathcal{P}$ in time $f(k) \cdot |x|^c$, where f is a computable function, and c is a constant.

C. First-order Logic

In this paper we deal with the, well known, relational first-order (FO) logic. Formulas of this logic are built from (a finite set of) variables, relational symbols, logical connectives ($\wedge, \vee, \neg$) and quantifiers ($\exists, \forall$). A *sentence* is a formula with no free variables. We restrict ourselves to formulas that are in *negation normal form*; a first-order formula is in negation normal form if all negation symbols occur only in front of the atoms. Obviously, any first-order formula can be, in linear time, converted into an equivalent one in negation normal form.

The problem we are interested in is a *model checking problem* for FO formulas on posets, which is formally defined as follows:

Poset FO Model Checking	**Parameter:** $\mathrm{width}(\mathcal{P})$, $|\varphi|$
Input: A first-order sentence φ and a poset $\mathcal{P} = (P, \leq^{\mathcal{P}})$.	
Question: Is it true $\mathcal{P} \models \varphi$, i.e., is $\mathcal{P}$ a model of φ?	

All first-order formulas in this paper are evaluated over posets as follows. The vocabulary consists of the one binary relation $\leq^{\mathcal{P}}$ and a finite set of arbitrary unary relations ("colors" of poset elements). Atoms of these FO formulas can be equalities between variables ($x = y$), applications of the predicate $\leq^{\mathcal{P}}$ (with the natural meaning of $x \leq^{\mathcal{P}} y$ in the poset $\mathcal{P}$), or applications of one of the unary predicates $c(x)$ (with the meaning that x is of color c). For a more detailed treatment of the employed setting, we refer the reader to [2].

While for any fixed sentence φ one can easily decide whether $\mathcal{P} \models \varphi$ in polynomial time, by a brute-force expansion of all the quantifiers, such a solution is not FPT since the exponent depends on φ. Our aim is to provide an FPT solution in the case when we additionally parameterize the input by the width of $\mathcal{P}$.

It is well known that the model checking problem for almost any logic can be formulated as finding a winning strategy in an appropriate model checking game, often called the *Hintikka game* (see e.g. [11]). In our case the game $\mathcal{G}(\mathcal{P}, \varphi)$ for a poset $\mathcal{P}$ and an FO formula $\varphi = \varphi(x_1, \ldots, x_k)$ in negation normal form (where $x_1, \ldots, x_k$ are the free variables of φ) is defined as follows:

The game is played by two players, the existential player (Player $\exists$, Verifier), who tries to prove that $\mathcal{P} \models \varphi$, and the universal player (Player $\forall$, Falsifier), who tries to disprove that claim. The positions of this game $\mathcal{G}(\mathcal{P}, \varphi)$ are pairs (ψ, β), where $\psi \equiv \psi(x_1, \ldots, x_\ell)$ is a subformula of φ, and $\beta : \{x_1, \ldots, x_\ell\} \to P$ assigns free variables of ψ elements of the poset $\mathcal{P}$. We write $\psi(p_1, \ldots, p_\ell)$ for a position (ψ, β), where $\beta(x_i) = p_i$ for the free variables x_i of ψ. The initial position of the game is (φ, β_0), where β_0 is the initial assignment (if φ has free variables, or β_0 is empty).

The game is played as follows: the existential player (Verifier) moves from positions associated with disjunctions and formulas starting with the existential quantifier. From a position $\psi_1 \vee \psi_2$ he moves to either ψ_1 or ψ_2. From a position $\psi(p_1, \ldots, p_i) \equiv \exists y.\, \psi'(p_1, \ldots, p_i, y)$ he moves to any position $\psi'(p_1, \ldots, p_i, p)$, where $p \in P$. The universal player plays similarly from conjunctions and universally quantified formulas. At atoms which, in our case, are the positions $\sigma(p_1)$ of the form $c(p_1)$, or $\sigma(p_1, p_2)$ of the form $p_1 = p_2$, $\neg(p_1 = p_2)$, $p_1 \leq^{\mathcal{P}} p_2$ or $\neg(p_1 \leq^{\mathcal{P}} p_2)$, the existential player wins if $\mathcal{P} \models \sigma(p_1, p_2)$, and otherwise the universal player wins. The equivalence of these games to the standard semantic of FO is given by the following claim:

Proposition II.2. *The existential player has a winning strategy in the Hintikka game $\mathcal{G}(\mathcal{P}, \varphi)$ for a poset $\mathcal{P}$ and a first-order sentence φ if, and only if, $\mathcal{P} \models \varphi$.*

III. Poset Structure and Types

For the rest of the paper we fix a poset $\mathcal{P} = (P, \leq^{\mathcal{P}})$ of width w, a mapping $\lambda : P \to \Lambda$ where Λ is a (fixed) finite set of *colors*, and a chain partition $(C_1, \ldots, C_w)$ of $\mathcal{P}$. To emphasize that $\mathcal{P}$ is associated with an auxiliary coloring λ, we sometimes call $\mathcal{P}$ a *colored poset*. For $p \in C_j$ with $1 \leq j \leq w$, we denote by $C(p)$ the chain C_j. The purpose of this section is to find a description of the structure of a colored poset $\mathcal{P}$ suitable for applying "locality tools" of finite model theory. This turns out to be a delicate job requiring a careful inductive definition.

For an integer $s \geq 0$, we set $r_s := 3 \cdot 4^s - 1$ and inductively define a) a labeling function $\tau_s : P \to \mathbb{N}$, and b) a vertex-labeled and arc-labeled directed graph D_s on the vertex set $V(D_s) := P$ as follows:

Definition III.1. For an integer $s \geq 0$ and an element $p \in P$, we shortly denote by $P_s(p)$ the set $R^{D_s}_{r_s}(p)$ (i.e., the set of vertices reachable in D_s from p at distance $\leq r_s$). We let $\tau_0(p) := \langle \lambda(p), j \rangle$, where j is the index s.t. $C_j = C(p)$. Inductively for every integer $s \geq 0$, we define D_s as the digraph with the vertex set P and vertex labels given by τ_s, containing the following arcs:

- for every $p \in P$ and every $j \in \{1, \ldots, w\}$, D_s contains an arc from p with label *'max'* to the topmost element of C_j;
- for every $p \in P$ and every $j \in \{1, \ldots, w\}$, D_s contains an arc from p with label *'min'* to the bottommost element of C_j;
- for every $p \in P$, every $j \in \{1, \ldots, w\}$, and every $t \in \{\, \tau_s(q) \mid q \in C_j \,\}$, D_s contains an arc from p to p' with label *'up'*, where $p' \neq p$ is the bottommost element of C_j such that $\tau_s(p') = t$ and $p \leq^{\mathcal{P}} p'$ (if such an element p' exists);
- for every $p \in P$, every $j \in \{1, \ldots, w\}$, and every $t \in \{\, \tau_s(q) \mid q \in C_j \,\}$, D_s contains an arc from p to p' with label *'down'*, where $p' \neq p$ is the topmost element of C_j such that $\tau_s(p') = t$ and $p' \leq^{\mathcal{P}} p$ (if such an element p' exists).

Then, having defined D_s as above, we set for every element $p \in P$

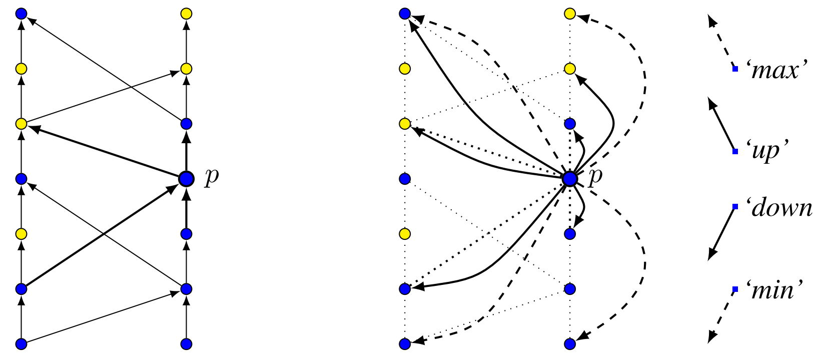

Figure 1. The picture, on the left, shows an upward-directed Hasse diagram of a bicolored poset $\mathcal{P}$ (where $\mathcal{P}$ is the reflexive and transitive closure of it). On the right, the picture shows the arcs of D_0 starting from a selected element $p \in P$, as by Definition III.1.

- $\tau_{s+1}(p)$ to be the isomorphism type of a relational structure $\mathcal{A}_s^{\mathcal{P}}(p)$, where $\mathcal{A}_s^{\mathcal{P}}(p)$ is formed by the vertex- and arc-labeled induced subdigraph $D_s[P_s(p)]$ rooted at p with the additional binary relation $\leq^{\mathcal{P}}$ restricted to $P_s(p)$.

The values $\tau_s(p)$, $p \in P$, will also be called the *types of rank* s (of elements of $\mathcal{P}$), where the rank will often be implicit from the context. It is useful to notice that the considered coloring λ of the poset $\mathcal{P}$ elements is fully determined by their types of rank 0 in D_0 (and so also by their types of any higher rank). Therefore, we may skip an explicit reference to λ in the rest of this section.

The definition of D_0 is illustrated in Figure 1. Informally, the type of an element $p \in P$ captures its "local neighborhood" (which is growing in size with the rank s), and the digraph D_s contains *'up'*-arcs (*'down'*-arcs) from p to the next higher (next lower) elements of $\mathcal{P}$ of each appearing type. Moreover, there are shortcut arcs, labeled *'min'* and *'max'*, from p to the extreme elements of each chain of $\mathcal{P}$. It is important that, since we use a fixed finite number of colors in $\mathcal{P}$ and since $\mathcal{P}$ is of bounded width, the outdegrees in D_s are inductively bounded for every fixed s independently of the size of $\mathcal{P}$.

To start we need the following basic properties of the digraph D_s and the labeling function τ_s, which are easy to prove. The first two of these simple claims establish that the sequence of labeled digraphs $D_0, D_1, D_2, \ldots$ indeed presents an increasingly finer resolution of a "local structure" of the poset $\mathcal{P}$. For all the claims, let $\mathcal{P} = (P, \leq^{\mathcal{P}})$ be a poset and $P_s, D_s, \mathcal{A}_s^{\mathcal{P}}$ and τ_s be as in Definition III.1. We refer to the full paper [21] for their proofs.

Lemma III.2. *For every $p, p' \in P$ and $s \geq 0$, if $\tau_s(p) \neq \tau_s(p')$, then also $\tau_{s+1}(p) \neq \tau_{s+1}(p')$.*

Lemma III.3. *For every $s \geq 0$, if D_s contains an arc from some vertex $p \in P$ to some $p' \in P$, then D_{s+1} also contains an arc from p to p' with the same label. In other words, D_s is a spanning subdigraph of D_{s+1} (neglecting the vertex-labels).*

Another simple property of Definition III.1 is that pairs of arcs of the same vertex- and arc-labels in the digraph D_s never "cross one another", which is formalized as follows:

Lemma III.4. *Let $\mathcal{P} = (P, \leq^{\mathcal{P}})$ be a poset and P_s, D_s and τ_s be as in Definition III.1. Assume that $p, p', q, q' \in P$ are such that $\tau_s(p) = \tau_s(p')$ and $\tau_s(q) = \tau_s(q')$, and that both (p, q) and (p', q') are arcs of the same label in D_s. If $p \leq^{\mathcal{P}} p'$ then $q \leq^{\mathcal{P}} q'$.*

The subsequent claims are more involved and technical. Informally, they together show that for any $p_1, \ldots, p_k \in P$, a property or relation of other element(s) of $\mathcal{P}$ to $p_1, \ldots, p_k$ can also be observed in a given bounded neighborhood of $p_1, \ldots, p_k$ in D_s. Importantly, the richer local property is observed, the higher index s in D_s

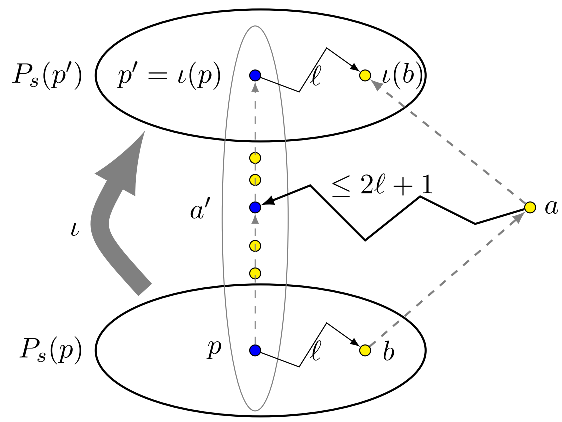

Figure 2. An illustration of the statement of Lemma III.6: the dashed arcs depict the poset relation $\leq^{\mathcal{P}}$ while the solid arcs represent directed paths in the digraph D_s.

is used. The easy base case of $s = 0$ is covered by Lemma III.5 while the general case of s is inductively established by Lemma III.6 and reformulated in Corollary III.7. We refer to Section IV for details on using these claims, and to the full paper [21] for their proofs.

Lemma III.5. *Let $\mathcal{P} = (P, \leq^{\mathcal{P}})$ be a poset and D_0 the digraph defined in Definition III.1. For any $k \geq 1$ and $p, p_1, \ldots, p_k \in P$, there exists an element $p' \in R_2^{D_0}(p_1, \ldots, p_k)$ such that $\tau_0(p') = \tau_0(p)$, and p', p are in the same relation with respect to all of $p_1, \ldots, p_k$ in $\mathcal{P}$: formally, for every $i \in \{1, \ldots, k\}$, it holds that $p' \leq^{\mathcal{P}} p_i$ if and only if $p \leq^{\mathcal{P}} p_i$, and $p_i \leq^{\mathcal{P}} p'$ if and only if $p_i \leq^{\mathcal{P}} p$.*

We use the following shorthand notation. For $p, p', q \in P$ we say that q *discerns p from p'*, with respect to $\leq^{\mathcal{P}}$, if (at least) one of the following four conditions holds true; $p \leq^{\mathcal{P}} q$ and $p' \not\leq^{\mathcal{P}} q$, $p \not\leq^{\mathcal{P}} q$ and $p' \leq^{\mathcal{P}} q$, $q \leq^{\mathcal{P}} p$ and $q \not\leq^{\mathcal{P}} p'$, or $q \not\leq^{\mathcal{P}} p$ and $q \leq^{\mathcal{P}} p'$. For example, the conclusion of Lemma III.5 is equivalent to saying "neither of $p_1, \ldots, p_k$ discerns p from p'". Since the poset $\mathcal{P}$ is fixed for this section, we will often skip an explicit reference to $\leq^{\mathcal{P}}$.

Lemma III.6. *Let $\mathcal{P} = (P, \leq^{\mathcal{P}})$ be a poset, $s \geq 0$ an integer, D_s and τ_s be as in Definition III.1, and $p, p' \in P$. Assume that $p \leq^{\mathcal{P}} p'$ and $\tau_{s+1}(p) = \tau_{s+1}(p')$, where the latter is witnessed by an isomorphism $\iota : P_s(p) \to P_s(p')$ of the structures $\mathcal{A}_s^{\mathcal{P}}(p)$ and $\mathcal{A}_s^{\mathcal{P}}(p')$. If $a \in P$ and $b \in P_s(p)$ are such that a discerns b from $\iota(b)$ (with respect to $\leq^{\mathcal{P}}$), then there exists a directed path from a to some element $a' \in C(p)$ of type $\tau_{s+1}(p)$ with $p \leq^{\mathcal{P}} a' \leq^{\mathcal{P}} p'$, in D_s of length at most $2\ell + 1$ where $\ell = dist_{D_s}(p, b)$.*

The technical statement of Lemma III.6 deserves an informal explanation. For start, if elements $p, p' \in P$ such that $p' \in C(p)$ are discerned in the poset $\mathcal{P}$ by an element $a \in P$ then, clearly, $a = a' \in C(p)$ or the Hasse diagram of $\mathcal{P}$ contains an arc between a and some $a' \in C(p)$ such that a' lies between p and p' on $C(p)$. Lemma III.6 then largely extends this simple observation to the setting of Definition III.1 and for discerned elements in neighborhoods of p and p'. The statement is illustrated in Figure 2.

The way we shall use Lemma III.6 in Section IV can be informally summarized as follows. If p and p' are elements of the same type which are next to each other on their chain in $\mathcal{P}$, then their D_s-neighborhoods appear "the same" with respect to $\leq^{\mathcal{P}}$ to all poset elements which are sufficiently far away from p as measured by D_{s+1}. The precise formulation is next.

Corollary III.7. *Let $\mathcal{P} = (P, \leq^{\mathcal{P}})$ be a poset, $s \geq 0$ an integer, the digraphs D_s and D_{s+1} be as in Definition III.1, $p \neq p' \in P$ be such that $\tau_{s+1}(p) = \tau_{s+1}(p')$, and the isomorphism map $\iota : P_s(p) \to P_s(p')$ be as in Lemma III.6. Assume, moreover, that D_{s+1} contains an arc (p', p) with label 'down'. For any given $p_1, \ldots, p_k \in P$, $k \geq 1$,*

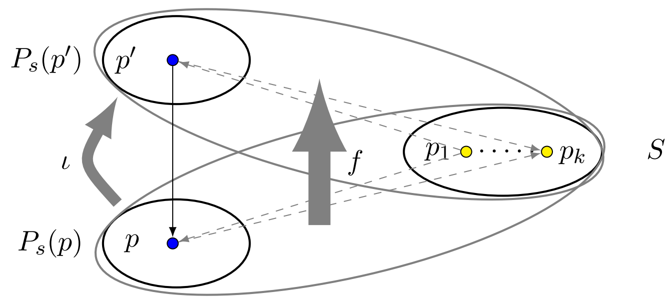

Figure 3. An illustration of the statement of Corollary III.7: f is a color-preserving isomorphism between the induced subposets $\mathcal{P}[S \cup P_s(p)]$ and $\mathcal{P}[S \cup P_s(p')]$.

denote by $S := R^{D_s}_{r_s}(p_1, \ldots, p_k)$ *and define a mapping* $f : S \cup P_s(p) \to S \cup P_s(p')$ *such that* $f(e) = \iota(e)$ *for* $e \in P_s(p)$ *and* $f(e) = e$ *otherwise. If* $p \notin R^{D_{s+1}}_{r_{s+1}-r_s}(p_1, \ldots, p_k)$*, then* f *is a color-preserving isomorphism between the induced subposets* $\mathcal{P}[S \cup P_s(p)]$ *and* $\mathcal{P}[S \cup P_s(p')]$.

The statement of Corollary III.7 is illustrated in Figure 3.

Proof: For the sake of contradiction assume that $p \notin R^{D_{s+1}}_{r_{s+1}-r_s}(p_1, \ldots, p_k)$ and f is not a poset isomorphism. Then either f is not a bijection or there are two elements $a, b \in S \cup P_s(p)$ such that a and b are in a different relation with respect to $\leq^{\mathcal{P}}$ than $f(a)$ and $f(b)$.

Observe that in the latter case, since a and b are in a different relation with respect to $\leq^{\mathcal{P}}$ than $f(a)$ and $f(b)$, and by the definition of f we clearly see that, up to symmetry, $a \in S \setminus P_s(p)$ and $b \in P_s(p)$. Hence, the latter case means that $a = f(a)$ discerns b from $f(b) = \iota(b)$.

In the former case consider the function f' defined analogously to the function f but from the "perspective of p'", i.e., $f' : S \cup P_s(p') \to S \cup P_s(p)$ is defined by setting $f'(e) = \iota^{-1}(e)$ for every $e \in P_s(p')$ and $f'(e) = e$ otherwise. Then f is a bijection if and only if f' is. Moreover, f is a bijection if and only if both f and f' are injective. If f is not injective, then there exists a $b \in P_s(p)$ such that $a = f(b) = \iota(b) \in S \setminus P_s(p)$. Since $a \neq b$, the element a discerns b from $\iota(b) = a$. On the other hand, if f' is not injective, then there exists a $b' \in P_s(p')$ such that $a' = f'(b') = \iota^{-1}(b') \in S \setminus P_s(p')$. Since $a' \neq b'$, the element a' discerns a' from b'. Setting $a := a'$ and $b := a'$, we again see that a discerns $b = a'$ from $\iota(b) = b'$.

In either of the three subcases above, the elements a and b satisfy the conditions of Lemma III.6. Hence there exists a directed path from a to an element $a' \in C(p)$ of type $\tau_{s+1}(p)$ with $p \leq^{\mathcal{P}} a' \leq^{\mathcal{P}} p'$ in D_s of length at most $2r_s + 1$. Because of Lemma III.3 this path exists also in D_{s+1}. Furthermore, since (p', p) is an arc with label *'down'* in D_{s+1}, there are no elements of type $\tau_{s+1}(p)$ on $C(p)$ "between" p and p', and so either $a' = p$ or $a' = p'$. Hence, there is a directed path from a to p in D_{s+1} of length at most $2r_s + 1 + 1$.

Finally, since $a \in S$, there is a directed path from some of the elements $p_1, \ldots, p_k$ to a of length at most r_s. In a summary, there exists a directed path in D_{s+1} from one of $p_1, \ldots, p_k$ to p of length at most (recall $r_s = 3 \cdot 4^s - 1$)

$$2r_s + 2 + r_s = (4r_s + 3) - r_s - 1 = r_{s+1} - r_s - 1$$

which contradicts the assumption $p \notin R^{D_{s+1}}_{r_{s+1}-r_s}(p_1, \ldots, p_k)$.

Color-preservation by f immediately follows from the same property of ι. ■

IV. The Model Checking Algorithm

We would like to use the Hintikka game to solve the poset FO model checking problem $\mathcal{P} \models \varphi$, by Proposition II.2. Though, the number of possible distinct plays in the game $\mathcal{G}(\mathcal{P}, \varphi)$ grows roughly as $\mathcal{O}(\|\mathcal{P}\|^{|\varphi|})$ which is not FPT. To resolve this problem, we are going to show that in fact only a small subset of all plays of the game $\mathcal{G}(\mathcal{P},\varphi)$ is necessary to determine the outcome—only those plays which are suitably locally constrained with the use of Definition III.1.

Let $\mathcal{P} = (P, \leq^{\mathcal{P}})$ be a poset, with an implicitly associated coloring $\lambda : P \to \Lambda$ as in Section III, and φ an FO formula. The *r-local Hintikka game* $\mathcal{G}_r(\mathcal{P},\varphi)$ (where "r" refers to the sequence $r_s = 3\cdot 4^s - 1$, cf. Definition III.1) is played on the same set of positions by the same rules as the ordinary Hintikka game $\mathcal{G}(\mathcal{P},\varphi)$, with the following additional restriction: for each $Q \in \{\exists, \forall\}$, and for any position of the form $\psi(p_1,\dots,p_i) \equiv Qy.\, \psi'(p_1,\dots,p_i,y)$ where $i \geq 1$, Player Q has to move to a position $\psi'(p_1,\dots,p_i,p)$ such that $p \in R^{D_q}_{r_q - r_{q-1}}(p_1,\dots,p_i)$, where $q \geq 1$ is the quantifier rank of ψ'. If $q = 0$, i.e., for quantifier-free ψ', the restriction is $p \in R^{D_0}_{r_0}(p_1,\dots,p_i)$.

Lemma IV.1. *Let $\mathcal{P}$ be a poset, $p_1,\dots,p_j \in P$, and φ be an FO formula of quantifier rank q. In the r-local Hintikka game $\mathcal{G}_r(\mathcal{P},\varphi)$ with an initial position $\varphi(p_1,\dots,p_j)$, $j \geq 1$, every reachable game position $\psi(p_1,\dots,p_k)$, $k > j$, is such that*

$$p_{j+1},\dots,p_k \in R^{D_{q-1}}_{r_{q-1}}(p_1,\dots,p_j).$$

Proof: We proceed by induction on $(k-j)$, starting with the trivial degenerate case $k = j$ and proving a stronger statement

$$p_{j+1},\dots,p_k \in R^{D_{q-1}}_{r_{q-1} - r_{q+j-k-1}}(p_1,\dots,p_j).$$

Let the claim hold for a position $\psi(p_1,\dots,p_{k-1})$ and consider a next position $\psi'(p_1,\dots,p_{k-1},p_k)$, where the quantifier rank of ψ' is $q' = q-(k-j)$. Then, by the definition, p_k is at distance at most $r_{q'} - r_{q'-1}$ in $D_{q'}$ from one of the elements $p_1,\dots,p_{k-1}$, and the same holds also in D_{q-1} by Lemma III.3 as $q' \leq q-1$. Since each of $p_1,\dots,p_{k-1}$ is at distance at most $r_{q-1} - r_{q+j-(k-1)-1} = r_{q-1} - r_{q'}$ from one of $p_1,\dots,p_j$ (and this holds also for $k-1 = j$), we get that p_k is at distance at most $r_{q-1} - r_{q'} + r_{q'} - r_{q'-1} = r_{q-1} - r_{q'-1} = r_{q-1} - r_{q+j-k-1}$ from one of $p_1,\dots,p_j$ in D_{q-1}. ∎

Now we get to the crucial technical claim of this paper. For $Q \in \{\exists, \forall\}$, and with a neglectable abuse of terminology, we say that Player Q *wins* the (ordinary or r-local) Hintikka game *from a position* $\psi(p_1,\dots,p_i)$ if Player Q has a winning strategy in the game $\mathcal{G}(\mathcal{P},\psi)$ (or in $\mathcal{G}_r(\mathcal{P},\psi)$, respectively) with the initial position $\psi(p_1,\dots,p_i)$. Otherwise, Player Q *loses* the game.

Lemma IV.2. *Let $\mathcal{P} = (P, \leq^{\mathcal{P}})$ be a colored poset and φ be an FO formula in negation normal form. For $Q \in \{\exists, \forall\}$ and $i \geq 0$, consider a position $\psi(p_1,\dots,p_i)$ in the Hintikka game $\mathcal{G}(\mathcal{P},\varphi)$. If Player Q wins the game $\mathcal{G}(\mathcal{P},\varphi)$ from the position $\psi(p_1,\dots,p_i)$, then Player Q wins also the r-local Hintikka game $\mathcal{G}_r(\mathcal{P},\psi)$ from an initial position $\psi(p_1,\dots,p_i)$.*

Proof: Let $\bar{Q}$ denote the other player, that is, $\{Q, \bar{Q}\} = \{\exists, \forall\}$. For the sake of contradiction, assume that we have got a counterexample with $\mathcal{P}$, φ, and $\psi(p_1,\dots,p_i)$ of quantifier rank $q+1$; meaning that Player Q wins $\mathcal{G}(\mathcal{P},\varphi)$ from $\psi(p_1,\dots,p_i)$ but Q loses $\mathcal{G}_r(\mathcal{P},\psi)$ from initial $\psi(p_1,\dots,p_i)$. Assume, moreover, that $Q \in \{\exists, \forall\}$ and the counterexample are chosen such that the pair $\langle q, |\psi| \rangle$ is lexicographically minimal.

Since a game move associated with a conjunction or disjunction of formulas, and a (possible) initial move for $i = 0$, are not in any way restricted in the r-local Hintikka game, our minimality setup guarantees that ψ starts with a quantifier, and so $q+1 \geq 1$ and $i \geq 1$. If this leading quantifier of ψ was $\bar{Q}$, then we would again get a contradiction to the minimality of q. Therefore,

$$\psi(p_1,\dots,p_i) \equiv Qx_{i+1}.\, \psi'(p_1,\dots,p_i,x_{i+1}). \tag{1}$$

Note that q is the quantifier rank of ψ'. If $q = 0$, then ψ' is actually quantifier-free and Player Q can make his move x_{i+1} with the element $p' \in R^{D_0}_{r_0}(p_1,\dots,p_i)$, $r_0 = 2$, as in Lemma III.5. Since this is a contradiction to Player Q losing $\mathcal{G}_r(\mathcal{P},\psi)$ from $\psi(p_1,\dots,p_i)$, we may further assume that $q \geq 1$.

Let $p \in P$ be such that Player Q wins the game $\mathcal{G}(\mathcal{P},\varphi)$ from (1) $\psi(p_1,\dots,p_i)$ by moving to the position $\psi'(p_1,\dots,p_i,p)$, and assume that p is chosen maximal with respect to $\leq^{\mathcal{P}}$ with this property. By our minimal choice of q, the statement of Lemma IV.2 holds for the game $\mathcal{G}(\mathcal{P},\varphi)$ from the position $\psi'(p_1,\ldots,p_i,p)$, and so

$$\text{Player } Q \text{ wins the } r\text{-local game } \mathcal{G}_r(\mathcal{P},\psi') \text{ from the initial position } \psi'(p_1,\ldots,p_i,p). \tag{2}$$

Consequently,

$$p \notin R^{D_q}_{r_q - r_{q-1}}(p_1,\ldots,p_i) \tag{3}$$

since, otherwise, Player Q would win also the r-local game $\mathcal{G}_r(\mathcal{P},\psi)$ from $\psi(p_1,\ldots,p_i)$ which is not the case by our assumption.

Let $p' \in C(p)$ be the bottommost element such that $p \lneq^P p'$ and $\tau_q(p') = \tau_q(p)$. Observe that such p' does exist since, otherwise, D_q would contain an arc from the topmost element of $C(p)$ to p with label *'down'*, and hence $p \in R^{D_q}_2(p_1)$ contradicting (3). By our maximal choice of p with respect to $\leq^{\mathcal{P}}$, Player Q loses the game $\mathcal{G}(\mathcal{P},\varphi)$ from $\psi(p_1,\ldots,p_i)$ after moving to the position $\psi'(p_1,\ldots,p_i,p')$. In other words, Player $\bar{Q}$ wins the game $\mathcal{G}(\mathcal{P},\varphi)$ from the position $\psi'(p_1,\ldots,p_i,p')$ and, by our choice of a counterexample with minimum q;

$$\text{Player } \bar{Q} \text{ wins the } r\text{-local game } \mathcal{G}_r(\mathcal{P},\psi') \text{ from the initial position } \psi'(p_1,\ldots,p_i,p'). \tag{4}$$

Now, we employ Corollary III.7 for $s = q-1$, $k = i$ and $S = R^{D_{q-1}}_{r_{q-1}}(p_1,\ldots,p_i)$. By (3), we hence get that there exists a color-preserving isomorphism map f between the induced colored subposets $\mathcal{P}[S \cup P_{q-1}(p)]$ and $\mathcal{P}[S \cup P_{q-1}(p')]$. By Lemma IV.1, all the elements e played in any play of an r-local Hintikka game $\mathcal{G}_r(\mathcal{P},\psi')$ from an initial position $\psi'(p_1,\ldots,p_i,p')$ belong to the set $R^{D_{q-1}}_{r_{q-1}}(p_1,\ldots,p_i,p') = S \cup P_{q-1}(p')$.

The latter finding implies that every play of the r-local Hintikka game $\mathcal{G}_r(\mathcal{P},\psi')$ from $\psi'(p_1,\ldots,p_i,p')$ can be duplicated, with the same outcome, in the same game from $\psi'(p_1,\ldots,p_i,p)$ via the isomorphism map f^{-1}. Therefore, by (4), Player $\bar{Q}$ wins the r-local game $\mathcal{G}_r(\mathcal{P},\psi')$ from the initial position $\psi'(p_1,\ldots,p_i,p)$, too. However, this contradicts (2), and so there cannot be a counterexample to the statement of the lemma. ∎

Remark IV.3. Notice that it is actually not necessary to explicitly use Lemma III.5 in the proof of Lemma IV.2 — with a slightly modified setting this base case comes out "for free". Though, we think that the current proof with Lemma III.5 is easier to read and to understand.

We can now easily formulate and prove the main result:

Theorem IV.4. *Let $\mathcal{P} = (P,\leq^{\mathcal{P}})$ be a poset, associated with $\lambda : P \to \Lambda$ where Λ is a finite set of colors, and let φ be an FO sentence in negation normal form. The existential player has a winning strategy in the r-local Hintikka game $\mathcal{G}_r(\mathcal{P},\varphi)$ if, and only if, $\mathcal{P} \models \varphi$.*

Proof: By Proposition II.2, it is enough to prove the following:

- Player Q, where $Q \in \{\exists,\forall\}$, has a winning strategy in the Hintikka game $\mathcal{G}(\mathcal{P},\varphi)$ if, and only if, Player Q has a winning strategy in the r-local Hintikka game $\mathcal{G}_r(\mathcal{P},\varphi)$.

The "only if" direction follows from Lemma IV.2 for $i = 0$, and so we deal with the "if" direction. Written in a contrapositive way it states that if Q does not have a winning strategy in $\mathcal{G}(\mathcal{P},\varphi)$, then also Q does not have a winning strategy in $\mathcal{G}_r(\mathcal{P},\varphi)$. But this is, by determinacy of Hintikka games, equivalent to saying if $\bar{Q}$ has a winning strategy in $\mathcal{G}(\mathcal{P},\varphi)$, then also $\bar{Q}$ has a winning strategy in $\mathcal{G}_r(\mathcal{P},\varphi)$, which again follows from Lemma IV.2.

∎

The following is then a straightforward observation.

Corollary IV.5. *For a poset $\mathcal{P}$ let $\mathcal{T}_i(\mathcal{P})$ denote the set of types τ_i of rank i occurring in $\mathcal{P}$, i.e. $\mathcal{T}_i(\mathcal{P}) := \{\tau_i(p) : p \in P\}$, where τ_i is as in Definition III.1. Assume $\mathcal{P}^1,\mathcal{P}^2$ are colored posets and φ an FO sentence of quantifier rank q in negation normal form. If $\mathcal{T}_{q-1}(\mathcal{P}^1) = \mathcal{T}_{q-1}(\mathcal{P}^2)$, then it holds; $\mathcal{P}^1 \models \varphi$ if and only if $\mathcal{P}^2 \models \varphi$.*

We finish with the main result:

Theorem IV.6. *Let $\mathcal{P} = (P, \leq^{\mathcal{P}})$ be a poset of width w, with elements colored by $\lambda : P \to \Lambda$ where Λ is a finite set, and let φ be an FO sentence in negation normal form. There is an algorithm which decides whether $\mathcal{P} \models \varphi$ in FPT time $f(w, \varphi) \cdot \|\mathcal{P}\|^2$.*

Proof: According to Corollary IV.5, it is enough to know the set $\mathcal{T}_{q-1}(\mathcal{P})$ in order to decide whether $\mathcal{P} \models \varphi$. We thus proceed the algorithm in two steps:

1) We compute the set $\mathcal{T}_{q-1}(\mathcal{P})$ (of rank-$(q-1)$ types).
2) We decide whether $\mathcal{P} \models \varphi$ using the set from Step 1.

First of all, we show that the set of all possible types of a given rank is finite if the poset width w is bounded. Since each type is a (sub)digraph of bounded radius, it is enough to argue that the out-degrees in D_i are bounded. Indeed, by induction, the outdegree in D_0 is $4w|\Lambda|$. In D_{i+1}, the outdegree is bounded from above by a function of w and the number of all possible types of rank $i+1$, which is finite by the inductive assumption for D_i.

Therefore, Step 2 is a finite problem and we may decide whether $\mathcal{P} \models \varphi$ by a brute-force evaluation of φ on each member of $\mathcal{T}_{q-1}(\mathcal{P})$. This takes time $f'(w, \varphi)$. As for Step 1, we start with computing a chain partition of width w, in time $g(w) \cdot \|\mathcal{P}\|^2$ by Proposition II.1. We then proceed exactly along the iterations of constructive Definition III.1. Since the number of possible types is finite, this computation takes time at most $f''(w, \varphi) \cdot \|\mathcal{P}\|^2$; by traversing, in every iteration D_i for $i = 0, 1, \ldots, q-2$, for each $p \in P$ every chain of $\mathcal{P}$ and finding the appropriate out-neighbors. ■

V. Application to Interval Graphs

Besides the very successful story of FPT FO model checking on sparse graph classes, culminating with the ultimate and outstanding result of Grohe, Kreutzer, and Siebertz [9], only a few such results have been published for dense graph classes (especially, for graph classes which cannot be easily interpreted in nowhere dense classes). One of such notable papers is [7], dealing with FO model checking on interval graphs.

It has been shown [7] that FO model checking of interval graphs is FPT when the intervals are restricted to have lengths from a fixed finite set of reals (Corollary V.2), while the problem is W-hard whenever the intervals are allowed to have lengths from any dense subset of a positive-length interval of reals.

A graph G is an *interval graph* if there exists a set $\mathcal{I}$ of intervals on the real line such that $V(G) = \mathcal{I}$ and $E(G)$ is formed by the intersecting pairs of intervals. For a set L of reals, a set $\mathcal{I}$ of intervals is called an *L-interval representation* if every interval from $\mathcal{I}$ has its length in L. This notion generalizes well-studied *unit interval graphs* (where all interval lengths are 1), which are also known under the name of proper interval graphs. A set $\mathcal{I}$ of intervals is called a *proper interval representation* if there is no pair of intervals $J_1, J_2 \in \mathcal{I}$ such that J_1 is strictly contained in J_2 ($J_1 \subsetneq J_2$). We call $\mathcal{I}$ a *k-fold proper interval representation* if there exists a partition $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_k$ such that each $\mathcal{I}_j$ is a proper interval representation for $j = 1, \ldots, k$.

We can demonstrate the strength and usefulness of our main result by stating the following Theorem V.1 which is a straightforward consequence of Theorem IV.6; see a proof in the full paper [21].

Theorem V.1. *Let φ be a graph FO sentence. Assume G is an interval graph given along with its k-fold proper interval representation $\mathcal{I}$. Then the FO model checking problem $G \models \varphi$, parameterized by k and φ, is FPT.*

We can now match the main result of [7], except the precise runtime:

Corollary V.2 (Ganian et al. [7])**.** *For every finite set L of reals, the FO model checking problem of L-interval graphs (given alongside with an L-interval representation) is FPT when parameterized by the FO sentence φ and $|L|$.*

Proof: Let $\mathcal{I}$ be a given L-interval representation, and set $k := |L|$. We partition $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_k$ such that each $\mathcal{I}_i$ contains intervals of the same length (from L), and then apply Theorem V.1. ■

Although, the formulation of Theorem V.1 is more general than [7]. We can, for instance, in the same way derive fixed-parameter tractability also for FO model checking of well-studied *k-proper interval graphs*, introduced in [22] as those having an interval representation such that no interval is properly contained in more than k other intervals.

VI. Conclusions

Our result can be seen as an initial step towards an understanding of the complexity of FO model checking on non-sparse classes of structures and we hope that the techniques developed here will be useful for future research in this direction, e.g., to investigate the complexity of FO model checking on other algebraic structures such as finite groups and lattices as suggested by Grohe [9], as well as on other dense graph classes, to which the established "locality" tools of finite model theory do not apply.

The result may also be used directly towards establishing fixed-parameter tractability for FO model checking of other graph classes. Given the ease with which it implies the otherwise non-trivial result on interval graphs [7], it is a natural to ask which other (dense) graph classes can be interpreted in posets of bounded width.

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