

## The Submodular Secretary Problem Goes Linear

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### Abstract

During the last decade, the matroid secretary problem (MSP) became one of the most prominent classes of online selection problems. The interest in MSP is twofold: on the one hand, there are many interesting applications of MSP; and on the other hand, there is strong hope that MSP admits  $O(1)$ -competitive algorithms, which is the claim of the well-known matroid secretary conjecture. Partially linked to its numerous applications in mechanism design, substantial interest arose also in the study of nonlinear versions of MSP, with a focus on the submodular matroid secretary problem (SMSP). The fact that submodularity captures the property of diminishing returns, a very natural property for valuation functions, is a key reason for the interest in SMSP. So far,  $O(1)$ -competitive algorithms have been obtained for SMSP over some basic matroid classes. This created some hope that, analogously to the matroid secretary conjecture, one may even obtain  $O(1)$ -competitive algorithms for SMSP over any matroid. However, up to now, most questions related to SMSP remained open, including whether SMSP may be substantially more difficult than MSP; and more generally, to what extent MSP and SMSP are related.

Our goal is to address these points by presenting general black-box reductions from SMSP to MSP. In particular, we show that any  $O(1)$ -competitive algorithm for MSP, even restricted to a particular matroid class, can be transformed in a black-box way to an  $O(1)$ -competitive algorithm for SMSP over the same matroid class. This implies that the matroid secretary conjecture is equivalent to the same conjecture for SMSP. Hence, in this sense SMSP is not harder than MSP. Also, to find  $O(1)$ -competitive algorithms for SMSP over a particular matroid class, it suffices to consider MSP over the same matroid class. Using our reductions we obtain many first and improved  $O(1)$ -competitive algorithms for SMSP over various matroid classes by leveraging known algorithms for MSP. Moreover, our reductions imply an  $O(\log \log(\text{rank}))$ -competitive algorithm for SMSP, thus, matching the currently best asymptotic algorithm for MSP, and substantially improving on the previously best  $O(\log(\text{rank}))$ -competitive algorithm for SMSP.

### Keywords

matroids; online algorithms; secretary problem; submodular functions

### I. INTRODUCTION

Secretary problems are a very natural class of online selection problems with many interesting applications. The origin of the secretary problem is hard to track and dates back to at least the '60s [12], [17], [19], [20], [29]. In its original form, also called the *classical secretary problem*, the task is to hire the best secretary out of a set  $E$  of candidates of known size  $n = |E|$ . Secretaries get interviewed (or *appear*) one by one in a random order. All secretaries that appeared so far can be compared against each other according to an underlying linear ordering. Whenever a secretary got interviewed, one has to decide immediately and irrevocably whether to hire (or *select*) him. The task is to hire the best secretary with as high a probability as possible. Dynkin [12] provided an asymptotically optimal algorithm for this problem, which hires the best secretary with probability at least  $1/e$ .

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The classical secretary problem is naturally interpreted as a stopping time problem and, not surprisingly, was mostly studied by probabilists.

During the last decade, interest in generalized versions of the classical secretary problem surged. One reason for this is a variety of applications in mechanism design (see [1], [3], [4], [25] and the references therein). These generalizations allow hiring of more than one secretary, subject to a given set of (down-closed) constraints. Each secretary reveals a non-negative weight at appearance, and the task is to hire a maximum weight set of secretaries. The arguably most canonical generalization is the problem of hiring  $k$  out of  $n$  secretaries instead of a single one (see [25]). However, more general constraints are required for many interesting applications.

A considerably more general setting, known as the *matroid secretary problem* (MSP for short) and introduced in [4], allows for selecting a subset of  $E$  that is independent in a given matroid  $\mathcal{M} = (E, \mathcal{I})$ .<sup>1</sup> Similar to the classical secretary problem, the number  $n = |E|$  of candidates, or *elements*, is known upfront, elements appear in random order, and no assumption is made on their weights. Access to the matroid  $\mathcal{M}$  is provided by an independence oracle that can be called on appeared elements, *i.e.*, for any subset  $S \subseteq E$  of elements that appeared so far, one can check whether  $S \in \mathcal{I}$  or not.

MSP attracted considerable interest recently. It is very appealing due to the fact that it captures a wide set of interesting selection problems in a single framework. Furthermore, matroids are highly structured constraints, which gives reasonable hope that strong online algorithms exist. Indeed, there is a famous conjecture, which we simply call the *matroid secretary conjecture*, claiming the existence of an  $O(1)$ -competitive algorithm for the matroid secretary problem [4]. We recall that an algorithm is  $\alpha$ -competitive if the expected weight collected by the algorithm is at least  $\frac{1}{\alpha} \cdot w(\text{OPT})$ , where  $w(\text{OPT})$  is the maximum weight of any feasible set, *i.e.*, the offline optimum. Whereas this conjecture remains open,  $O(1)$ -competitive algorithms have been obtained for various special cases of matroids (see [4], [9], [11], [23], [24], [26], [31], [35]). The currently strongest asymptotic competitive ratio obtained for general MSP—without any restriction on the underlying matroid—is  $O(\log \log(\text{rank}))$  [27], [16], where “rank” is the rank of  $\mathcal{M}$ , *i.e.*, the cardinality of a maximum cardinality independent set in  $\mathcal{M}$ .

Recently, increased interest arose in nonlinear versions of the secretary problem, with a focus on the maximization of a non-negative submodular function<sup>2</sup> [5], [6], [14], [22], [31], leading to the *submodular secretary problem*. Submodular functions have widespread use as valuation functions because they reflect the property of diminishing returns, *i.e.*, the marginal value of an element is the bigger the fewer elements have been selected so far. This makes them natural candidates for the matroid secretary setting.

Additionally, submodular weight functions capture further generalizations of the secretary problem. For example, Gilbert and Mosteller [21] and Freeman [18] considered a variation of the classical secretary problem where one can select  $k$  elements with the goal of maximizing the value of the highest-valued element. This problem can be phrased as a submodular secretary problem with the submodular function  $f : 2^E \rightarrow \mathbb{R}^+$  defined by  $f(S) = \max\{w(e) \mid e \in S\}$  for  $S \subseteq E$ , where  $w(e)$  is the weight revealed by element  $e$ .

The currently best asymptotic competitive ratio for the submodular matroid secretary problem (SMSP) is  $O(\log(\text{rank}))$  [22]. Furthermore,  $O(1)$ -competitive algorithms have been obtained for special classes of matroids, including uniform matroids [6], [15], [22], partition matroids [15], [22], and laminar and transversal matroids [31] (both only for monotone<sup>3</sup> submodular functions).

In general, our understanding of secretary problems is much more limited when dealing with submodular weights instead of linear ones, leading to many open questions. In particular, is there hope to get an  $O(1)$ -competitive algorithm for SMSP? Notice that this corresponds to the matroid secretary conjecture translated to

<sup>1</sup> A matroid  $\mathcal{M} = (E, \mathcal{I})$  consists of a finite set  $E$  and a non-empty family  $\mathcal{I} \subseteq 2^E$  of subsets of  $E$ , called *independent sets* that satisfy: (i) if  $I \in \mathcal{I}$  and  $J \subseteq I$  then  $J \in \mathcal{I}$ , and (ii) if  $I, J \in \mathcal{I}$  with  $|I| > |J|$  then there is an element  $e \in I \setminus J$  such that  $J \cup \{e\} \in \mathcal{I}$ . For further basic matroidal concepts, such as rank and span, we refer to [34, Volume B].

<sup>2</sup>A non-negative submodular function  $f$  on a ground set  $E$  is a function  $f : 2^E \rightarrow \mathbb{R}^+$  giving a non-negative weight to every subset of  $E$  and satisfying the following *diminishing returns* property:  $f(A \cup \{e\}) - f(A) \geq f(B \cup \{e\}) - f(B)$  for  $A \subseteq B \subseteq E$  and  $e \in E \setminus B$ .

<sup>3</sup>A submodular function is monotone if  $S \subseteq T \subseteq E$  implies  $f(S) \leq f(T)$ .

the submodular case. Or may the submodular case be substantially harder than the linear one? Do monotone SMSP admit considerably better competitive ratios than nonmonotone ones? Can we leverage strong algorithms for MSP to obtain results for SMSP?

The goal of this paper is to address these questions and get a deeper understanding of SMSP and its relation to MSP, independently of the structure of particular classes of underlying matroids.

### A. Our Results

Our main result below shows an intimate relation between the (linear) matroid secretary problem and the submodular version. More precisely, we show that one can use any algorithm for MSP as a black box to obtain an algorithm for SMSP with a slightly weaker competitive ratio.

**Theorem I.1.** *Given an arbitrary algorithm  $Linear$  for MSP and a value  $\alpha \geq 1$ , there exists an algorithm for SMSP whose competitive ratio is at most  $24\alpha(3\alpha + 1) = O(\alpha^2)$  for every matroid  $\mathcal{M}$  on which  $Linear$  is guaranteed to be at least  $\alpha$ -competitive.*

Theorem I.1 has several interesting implications. In particular, if there is an  $O(1)$ -competitive algorithm for the linear case, then there is an  $O(1)$ -competitive algorithm for the submodular case. Hence, the matroid secretary conjecture is equivalent to the same statement for the submodular version. This provides strong hope that constant-competitive algorithms exist for SMSP.

Furthermore, Theorem I.1 implies many new results for SMSP, both for the general version without any restriction on the matroid as well as for many special classes of matroids, by leveraging algorithms for MSP. In particular, the known  $O(\log \log(\text{rank}))$ -competitive algorithms for MSP [27], [16] imply  $O((\log \log(\text{rank}))^2)$ -competitive algorithms for SMSP, which already considerably improves on the previously best  $O(\log(\text{rank}))$ -competitive algorithm for SMSP [22]. We later strengthen this result to match the asymptotically best algorithm for MSP.

The only matroid classes for which  $O(1)$ -competitive algorithms for SMSP have been explicitly given, without assuming monotonicity of the submodular weight function, are uniform matroids [6], [22], unitary partition matroids [22] and matroids for which a reduction to unitary partition matroids is known. Such reductions are known for graphic matroids [2], [22], cographic matroids [35], and for max-flow min-cut matroids [11]. These reductions have originally been used to obtain  $O(1)$ -competitive algorithms for MSP over these matroids, but they lead also to algorithms for SMSP over the same matroid classes when combined with an algorithm for SMSP over unitary partition matroids. For other classes of matroids, Theorem I.1 implies the first  $O(1)$ -competitive algorithm for SMSP, by leveraging known  $O(1)$ -competitive algorithms for MSP, such as the ones known for transversal matroids [26] and laminar matroids [23]. Furthermore, we also improve the competitive ratios for most matroid classes for which  $O(1)$ -competitive algorithms have already been known.

The analysis of the algorithm that we use to prove Theorem I.1 can easily be improved in many cases, if the algorithm  $Linear$  obeys some natural properties. Theorem I.2 below gives a first strengthening of Theorem I.1 and allows us to derive an  $O(\log \log(\text{rank}))$ -algorithm for SMSP, thus, matching the currently best algorithm for MSP up to a constant factor, and improving on the previously best  $O(\log(\text{rank}))$  competitive algorithm for SMSP.

We highlight that Theorem I.1 can be obtained from Theorem I.2 by setting  $k = 1$ .

**Theorem I.2.** *Given an algorithm  $Linear$  for MSP and value  $\alpha \geq 1$ , there exists an algorithm for SMSP whose competitive ratio is at most  $24k\alpha(3\alpha + 1) = O(k \cdot \alpha^2)$  for every matroid  $\mathcal{M}$  on which the behavior of  $Linear$  can be characterized as follows.*

- For every MSP instance over the matroid  $\mathcal{M}$ , there exists a set of  $k$  (correlated) random sets  $\{P_i\}_{i=1}^k$  such that each set  $P_i$  is always independent in  $\mathcal{M}$  and  $\mathbb{E}[w(\bigcup_{i=1}^k P_i)] \geq w(\text{OPT}_w)/\alpha$ , where  $w$  is the weight function of the MSP instance and  $\text{OPT}_w$  is the maximum weight independent set given  $w$ .
- $Linear$  outputs a uniformly random set from  $\{P_i\}_{i=1}^k$ .

Observe that such an algorithm *Linear* is  $(k \cdot \alpha)$ -competitive for  $\mathcal{M}$ .

**Corollary I.3.** *There exists an  $O(\log \log(\text{rank}))$ -competitive algorithm for SMSP.*

*Proof:* Feldman et al. [16] describe an  $O(\log \log(\text{rank}))$ -competitive algorithm for MSP. Their analysis of the algorithm shows that in fact it obeys the requirements of Theorem I.2 for *Linear* with  $\alpha = O(1)$  and  $k = O(\log \log(\text{rank}))$ . Notice that the algorithm guaranteed by Theorem I.2 does not depend on  $k$ . Hence, there is no need to know the rank ahead of time to use this algorithm. ■

In what follows we present further useful strengthenings of Theorem I.1 that are readily obtained through our derivation of Theorem I.1. These results allow for obtaining stronger competitive ratios if *Linear* fulfills some very typical properties, or if the submodular valuation function is monotone. When dealing with monotone submodular functions, we talk about the *monotone submodular secretary problem* (MSMSP). We first state the improved reductions and later present their implications in terms of competitive ratios in the form of a table.

**Theorem I.4.** *Given an algorithm *Linear* for MSP and values  $\alpha \geq 1$  and  $q \in (0, 1]$ , there exists an algorithm for SMSP whose competitive ratio is at least  $24\alpha(3q\alpha + 1) = O(q \cdot \alpha^2)$  for every matroid  $\mathcal{M}$  on which *Linear* is guaranteed to:*

- be at least  $\alpha$ -competitive.
- select every element with probability at most  $q$ .

The results proved above can be somewhat improved for monotone objective functions. The full version of the paper provides details of how the proofs of Theorems I.1, I.2 and I.4 need to be changed in order to get the following theorem.

**Theorem I.5.** *When SMSP is replaced with MSMSP in Theorems I.1, I.2 and I.4 their guarantees can be improved to  $8\alpha(\alpha + 1)$ ,  $8k\alpha(\alpha + 1)$ , and  $8\alpha(\alpha q + 1)$ , respectively.*

For monotone functions there is one additional natural property of *Linear* that can be used to get a stronger result. Intuitively this property is that *Linear* selects every element of the optimal solution with probability at least  $\alpha^{-1}$ , and thus, is  $\alpha$ -competitive. Many algorithms have this property when items have disjoint weights, and thus, there is a single optimal solution. Such algorithms are often extended to general inputs by introducing a random tie breaking rule. The following theorem is designed to deal with algorithms obtained this way; its proof can be found in the full version of the paper.

**Theorem I.6.** *Given an arbitrary algorithm *Linear* for MSP and a value  $\alpha \geq 1$ , there exists an algorithm for MSMSP whose competitive ratio is at most  $16\alpha = O(\alpha)$  for every matroid  $\mathcal{M}$  on which *Linear* has the following property: For every MSP instance over the matroid  $\mathcal{M} = (\mathcal{N}, \mathcal{I})$ , there exists a random set  $S \subseteq E$  obeying:*

- $S$  is always an optimal solution of the MSP instance.
- For every element  $u \in E$ ,  $\Pr[u \text{ is selected by } \textit{Linear}] \geq \Pr[u \in S]/\alpha$ .

The following table summarizes the competitive ratios we obtain, for SMSP and MSMSP over particular matroid classes, by leveraging the above-presented reductions.<sup>4</sup> A straightforward application of these reductions leads to improvements or even first results for all matroid classes listed in the table. However, using some additional observations we can sometimes get further improvements. Further details on how the stated results are implied by our reductions can be found in the full version of the paper. Our improvement for unitary partition matroids also implies improvements by the same factor for all matroid classes for which a reduction to unitary partition matroids is known.

<sup>4</sup>For readability, the competitive ratios in the table have been rounded to the closest integer.

Matroid Type	Known Competitive Ratio	Our Competitive Ratio
Unitary partition matroids	1297 <sup>(1)</sup> [22]	261
Transversal matroids	—	2496
$k$ -sparse linear matroids <sup>(2)</sup>	—	$24ke(3k + 1)$
$k$ -sparse linear matroids (MSMSP)	—	$8ke(k + 1)$
Laminar matroids	—	585
Laminar matroids (MSMSP)	211 [31]	144

<sup>(1)</sup> Gupta et al. [22] do not explicitly calculate the competitive ratio of their algorithm, however, its competitive ratio is no better than  $48000/37 \approx 1297$ .

<sup>(2)</sup> A  $k$ -sparse linear matroid is a linear matroid with a matrix representation using at most  $k$  non-zeros per column.

We highlight that apart from the unitary partition matroid case, all other results in the above table for particular matroid classes assume prior knowledge of the matroid, in addition to the size of its ground set. This is due to the fact that the corresponding MSP algorithms that we put into our framework to get results for SMSP make this assumption.

**Remark:** The proofs of all the above theorems use `Linear` in a black-box manner. Hence, these theorems apply also to many models allowing the algorithm more information, such as the model in which the algorithm has full knowledge about the matroid from the beginning (but not about the objective function). Also, we emphasize that all the above results are general reductions from MSP to SMSP, not assuming any particular structure about the underlying matroid. This is in stark contrast to almost all results on SMSP so far.

### B. Further Related Work

Progress has been made on the matroid secretary conjecture for variants of MSP which modify the assumptions on the order in which elements arrive and the way weights are assigned to elements. One simpler variant of MSP is obtained by assuming *random weight assignment*. Here, an adversary can only choose  $n = |E|$  not necessarily distinct weights, and the weights are assigned to the elements uniformly at random. In this model, a 5.7187-competitive algorithm can be obtained for any matroid [35]. Additionally, a  $16(1 - 1/e)$ -competitive procedure can still be obtained in the random weight assignment model even if elements are assumed to arrive in an adversarial order [33], [35]. Hence, this variant is, in a sense, the opposite of the classical MSP, where weights are adversarial and the arrival order is random. Furthermore, a 4-competitive algorithm can be obtained in the so-called *free order model*. Here, the weight assignment is adversarial; however, the algorithm can choose the order in which elements arrive [1], [24]. Among the above-discussed variants, this is the only variant with adversarial weight assignments for which an  $O(1)$ -competitive algorithm is known. For more information on recent advances on MSP and its variants we refer to the survey [10].

We also highlight that SMSP is an online version of submodular function maximization (SFM) over a matroid constraint. Interestingly, even in the offline setting,  $O(1)$ -approximations for SFM over a matroid constraint have only been discovered very recently, starting with a  $(4 + \epsilon)$ -approximation presented in [28]. Considerable progress has been made in the meantime [33], [15]. The currently strongest approximation algorithm has an approximation ratio of about  $e \approx 2.718$  [15]. We refer the interested reader to [8], [15] for more information on constrained SFM.

### C. Organization of the paper

We start by formally introducing our problem and some basic notation and results in Section II. Section III presents our main algorithm that we use to prove our results. Finally, Section IV provides details on the analysis of our algorithm.

## II. PRELIMINARIES

In this section we formally define our problem and state some notation and known results that are used later in the paper.

### A. Problems and Standard Notation

An instance of the Submodular Matroid Secretary Problem (SMSP) consists of a ground set  $E$ , a non-negative submodular objective  $f: E \rightarrow \mathbb{R}^+$  and a matroid constraint  $\mathcal{M} = (E, \mathcal{I})$ . An algorithm for this problem faces the elements of  $E$  in a uniformly random order, and must accept or reject each element immediately upon arrival. The algorithm has access to  $n = |E|$  and two oracles:

- A value oracle that, given a subset  $S \subseteq E$  of elements that *already arrived*, returns  $f(S)$ .
- An independence oracle that, given a subset  $S \subseteq E$  of elements that *already arrived*, determines whether  $S \in \mathcal{I}$ .

The objective of the algorithm is to accept an independent set of elements maximizing  $f$ .

The Matroid Secretary Problem (MSP) can be viewed as a restriction of SMSP to linear objective functions. More formally, an instance of MSP is an instance of SMSP in which the function  $f(S)$  is defined by  $f(S) = \sum_{u \in S} w(u)$  for some set of non-negative weights  $\{w(u) \mid u \in E\}$ . Similarly, the Monotone Submodular Matroid Secretary Problem (MSMSP) is a restriction of SMSP to non-negative monotone submodular objective functions.

The following notation comes handy in our proofs. Given a set  $S$  and an element  $u$ , we use  $S + u$  and  $S - u$  to denote the sets  $S \cup \{u\}$  and  $S \setminus \{u\}$ , respectively. Additionally, given a weight function  $w: E \rightarrow \mathbb{R}^+$ , we use  $w(S)$  as a shorthand for  $\sum_{u \in S} w(u)$ . Finally, given a set function  $f: 2^E \rightarrow \mathbb{R}$ , we denote by  $f(u \mid S)$  the marginal contribution of adding  $u$  to  $S$ . More formally,  $f(u \mid S) := f(S + u) - f(S)$ .

### B. The Function $f_w$

Given a set function  $f: 2^E \rightarrow \mathbb{R}$  and a weight vector  $w \in \mathbb{R}^E$ , let  $f_w: 2^E \rightarrow \mathbb{R}$  be the function defined as:

$$f_w(S) = \min_{A \subseteq S} \{f(A) + w(S \setminus A)\} \quad \forall S \subseteq E .$$

This construction of  $f_w$  out of  $f$  and  $w$  is well known in the field of submodular function optimization, and is sometimes called *convolution* (see, for example, [30]). We state some basic properties of  $f_w$ . Proofs of these results can, for example, be found in [32, Chapter 10].

**Property II.1.** *For every set  $S \subseteq E$ ,  $f(S) \geq f_w(S)$ , and for  $S = \emptyset$  the inequality holds with equality.*

**Property II.2.** *If  $f$  is a non-negative submodular function and  $w$  is non-negative, then  $f_w$  is also non-negative and submodular. Moreover, if  $f$  is also monotone, then so is  $f_w$ .*

### C. Known Lemmata

We need the following known lemmata. The lemmata have been rephrased a bit to make the difference between them clearer.

**Lemma II.3** (Lemma 2.2 of [13]). *Let  $g: 2^E \rightarrow \mathbb{R}^+$  be a submodular function. Denote by  $A(p)$  a random subset of  $A$  where each element appears with probability  $p$  (not necessarily independently). Then,*

$$\mathbb{E}[g(A(p))] \geq (1 - p)g(\emptyset) + p \cdot g(A) .$$

**Lemma II.4** (Lemma 2.2 of [7]). *Let  $g: 2^E \rightarrow \mathbb{R}^+$  be a non-negative submodular function. Denote by  $A(p)$  a random subset of  $A$  where each element appears with probability at most  $p$  (not necessarily independently). Then,*

$$\mathbb{E}[g(A(p))] \geq (1 - p)g(\emptyset) .$$

### III. ALGORITHM

The algorithm used to prove our results for SMSP is given as Algorithm 1. Observe that the algorithm has a single probability parameter  $p \in (0, 1)$ . Additionally, the algorithm uses an arbitrary procedure **Linear** for MSP whose existence is assumed by Theorem I.1 and our other results for SMSP. Finally, the algorithm also uses as a subroutine the standard greedy algorithm for maximizing a submodular function subject to matroid constraints (denoted by **Greedy**), which can be found, for completeness, in the full version of the paper.<sup>5</sup> While reading the description of Algorithm 1, the only important thing one has to know about **Greedy** is that it creates its solution by starting with the empty set and adding elements to it one by one.

A key challenge in trying to leverage **Greedy** in an algorithm for SMSP, is that **Greedy** is not a constant-factor approximation algorithm for submodular function maximization over a matroid (or even in the unconstrained setting). In our analysis we show that Algorithm 1 manages to circumvent this issue, and the set  $M$ , produced within Algorithm 1 using **Greedy**, does provide a constant approximation for the optimal solution.

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**Algorithm 1:** `Online( $p$ )`

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// Learning Phase
1 Choose  $X$  from the binomial distribution  $B(n, 1/2)$ .
2 Observe (and reject) the first  $X$  elements of the input. Let  $L$  be the set of these elements.

// Selection Phase
3 Let  $M$  be the output of Greedy on the set  $L$ .
4 Let  $N \leftarrow \emptyset$ .
5 for each arriving element  $u \in E \setminus L$  do
6     Let  $w(u) \leftarrow 0$ .
7     if  $u$  is accepted by Greedy when applied to  $M + u$  then
8         with probability  $p$  do
9             Add  $u$  to  $N$ .
10            Let  $M_u \subseteq M$  be the solution of Greedy immediately before it adds  $u$  to it.
11            Update  $w(u) \leftarrow f(u \mid M_u)$ .
12     Pass  $u$  to Linear with weight  $w(u)$ .
13 return  $Q \cap N$ , where  $Q$  is the output of Linear.

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Observe that Algorithm 1 can be implemented online because whenever an element  $u$  is fed to **Linear**, the algorithm already knows whether  $u \in N$ , and thus, can determine the membership of  $u$  in  $Q \cap N$  immediately after **Linear** determines the membership of  $u$  in  $Q$ . Additionally, note that Algorithm 1 applies **Linear** to the restriction<sup>6</sup> of its input matroid  $\mathcal{M}$  to the set  $E \setminus L$ . We assume in the analysis of Algorithm 1 that **Linear** is  $\alpha$ -competitive for that restriction of  $\mathcal{M}$  whenever it is  $\alpha$ -competitive for  $\mathcal{M}$ . Many algorithms for MSP obey this property without any modifications, but for some we need the following proposition which proves that this assumption can be justified in general.

**Definition III.1.** *Partial-MSP is a variant of MSP where an instance consists also of a set  $L \subseteq E$  known to the algorithm. The elements of  $E \setminus L$  arrive at a uniformly random order as usual. The elements of  $L$  never arrive, and thus, cannot be added to the solution. However, the oracles can answer queries about them as if they arrived before the first element of  $E \setminus L$ .*

<sup>5</sup>For non-monotone functions there are two versions of **Greedy**: one that stops picking elements once they all have negative marginal contributions, and one that continues as long as possible. In this paper we use the first version.

<sup>6</sup>The *restriction* of a matroid  $\mathcal{M} = (E, \mathcal{I})$  to a subset  $E' \subseteq E$  of its ground set is the matroid  $\mathcal{M}' = (E', \mathcal{I} \cap 2^{E'})$ .

**Proposition III.2.** *Given an algorithm `Linear` for MSP, there exists an algorithm for Partial-MSP whose competitive ratio for every matroid  $\mathcal{M}$  is as good as the competitive ratio of `Linear` for this matroid.*

The proof of Proposition III.2 can be found in Appendix A. Notice that `Linear` indeed faces an instance of Partial-MSP since the set  $L$  is fully known before `Linear` is invoked for the first time, and the weights assigned to elements depend solely on  $L$ . The following simple observation is well known. A proof of it can be found, e.g., as Lemma A.1. of [16].

**Observation III.3.** *The set  $L$  constructed by Algorithm 1 contains every element of  $E$  with probability  $1/2$ , independently.*

As is, it is difficult to prove some claims about Algorithm 1. For that purpose, we present Algorithm 2, which is an offline algorithm sharing the same joint distribution (as Algorithm 1) of the set  $M$  and the output set (a justification of this claim can be found in the full version of the paper). Clearly, the competitive ratio of Algorithm 1 is equal to the approximation ratio of Algorithm 2.

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**Algorithm 2: Simulated( $p$ )**

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// Initialization
1 Let  $M, N, N_0 \leftarrow \emptyset$ .
2 Let  $E' \leftarrow E$ .

// Main Loop
3 while  $E' \neq \emptyset$  do
4   Let  $u$  be the element of  $E'$  maximizing  $f(u \mid M)$ , and remove  $u$  from  $E'$ .
5   Let  $w(u) \leftarrow f(u \mid M)$ .
6   if  $M + u$  is independent in  $\mathcal{M}$  and  $w(u) \geq 0$  then
7     with probability  $1/2$  do Add  $u$  to  $M$ .
8     otherwise with probability  $p$  do Add  $u$  to  $N$ .
9     otherwise
10      Update  $w(u) \leftarrow 0$ .
11      Add  $u$  to  $N_0$ .
12   else
13     Let  $w(u) \leftarrow 0$ .
14     Add  $u$  to  $N_0$  with probability  $1/2$ .
15 Run Linear with  $N \cup N_0$  as the input (in a uniformly random order) and the weights defined by  $w$ .
16 return  $Q \cap N$ , where  $Q$  is the output of Linear.

```

---

Notice that Algorithm 2 also makes `Linear` face an instance of Partial-MSP (with  $L = E \setminus (N \cup N_0)$ ). Hence, by Proposition III.2, we can assume the competitive ratio of `Linear` for  $\mathcal{M}$  extends to the instance it faces in Algorithm 2.

Algorithms 1 and 2 can be viewed as generalizations of algorithms presented by [31] for MSMSP over laminar matroids. In particular, the idea of defining a surrogate weight function  $w$  based on a subset of the elements sampled at the beginning of the algorithm was already used in [31]. Although our analysis of Algorithms 1 and 2 is quite different and much more general than the analysis of [31], it does borrow some ideas from [31]. The concept of an offline algorithm simulating a more difficult to analyze online algorithm has been previously used, even specifically for offline simulations of an online greedy algorithm [9], [26], [31]. A key novel contribution of our analysis, compared to [31], is that we manage to relate the expected  $w$ -weight of a maximum  $w$ -weight independent set in  $N$  to  $\mathbb{E}[w(M)]$  (see Lemma IV.8). Furthermore, we overcome several technical hurdles by



first comparing values of constructed sets with respect to the convoluted submodular function  $f_w$  instead of the original function  $f$ . Finally, our use (and analysis) of a modified greedy algorithm allows us to deal with non-monotone submodular functions.

#### IV. ANALYSIS OF ALGORITHMS 1 AND 2

Throughout this section (except in Section IV-D) we fix an arbitrary matroid  $\mathcal{M} = (E, \mathcal{I})$  for which Linear is  $\alpha$ -competitive, and analyze the approximation ratio of Algorithm 2 for this matroid. Since Algorithms 1 and 2 share their output distribution, the approximation ratio we prove for Algorithm 2 implies an identical competitive ratio for Algorithm 1. The analysis of the approximation ratio consists of three main stages. In each stage we study one of the sets  $M$ ,  $N$  and  $N \cap Q$ . Specifically, we show bounds on the expected values assigned to each one of these sets by  $w$  and  $f_w$ . Notice that once we have a bound on  $\mathbb{E}[f_w(N \cap Q)]$ , we also get a bound on the expected value of the solution produced by Algorithm 2 since  $f(N \cap Q) \geq f_w(N \cap Q)$  by Property II.1.<sup>7</sup>

Following is some notation that is useful in many of the proofs below. For every element  $u \in E$ , let  $E_u$  be the set of elements processed by Algorithm 2 before  $u$ . Then, we define:

$$N_u = N \cap E_u \quad \text{and} \quad M_u = M \cap E_u .$$

Similarly, we also define  $E'_u = E_u + u$  and:

$$N'_u = N \cap E'_u \quad \text{and} \quad M'_u = M \cap E'_u .$$

##### A. First Stage

We now begin with the first stage of the analysis, *i.e.*, bounding  $\mathbb{E}[w(M)]$  and  $\mathbb{E}[f_w(M)]$ . The following lemma shows that both values are in fact strongly related to  $\mathbb{E}[f(M)]$ .

**Lemma IV.1.**  $w(M) + f(\emptyset) = f(M) = f_w(M)$ .

*Proof:* Observe that, by construction,  $w(u) = f(u \mid M_u)$  for every  $u \in M$ . Hence, the equality  $w(M) + f(\emptyset) = f(M)$  holds since:

$$w(M) = \sum_{u \in M} f(u \mid M_u) = f(M) - f(\emptyset) .$$

Let us now prove the equality  $f_w(M) = f(M)$ . By Property II.1,  $f_w(M) \leq f(M)$ . Thus, we only need to prove the reverse inequality. For every set  $A \subseteq M$ :

$$\begin{aligned} f(A) + w(M \setminus A) &= f(\emptyset) + \sum_{u \in A} f(u \mid A \cap M_u) + \sum_{u \in M \setminus A} f(u \mid M_u) \\ &\geq f(\emptyset) + \sum_{u \in A} f(u \mid M_u) + \sum_{u \in M \setminus A} f(u \mid M_u) = f(M) , \end{aligned}$$

where the inequality follows from submodularity. Hence, by the definition of  $f_w(M)$ :

$$f_w(M) = \min_{A \subseteq M} \{f(A) + w(M \setminus A)\} \geq \min_{A \subseteq M} f(M) = f(M) .$$

The bounds given by the above lemma are in terms of  $f(M)$ . To make these bounds useful, we need to bound also  $\mathbb{E}[f(M)]$ . This is not trivial since Greedy is not a constant-factor approximation algorithm for submodular function maximization over a matroid. Different approaches are known to adapt or extend Greedy such that it provides an  $O(1)$ -approximation in the offline setting. However, these are not well-suited for the way we simulate greedy online. The next two lemmata show a very simple way to transform Greedy into an  $O(1)$ -approximation

<sup>7</sup>Observe that the weights  $w$  chosen by Algorithm 2 are always non-negative, thus, we can use all the properties of  $f_w$  stated in Section II.

algorithm for submodular function maximization; and most importantly, this adjustment of **Greedy** is trivial to simulate online. We are not aware of any previously known variation of **Greedy** that provides an  $O(1)$ -approximation for submodular function maximization over a matroid constraint, and that can easily be simulated in the online setting.

The next lemma is similar to Lemma 3 of [22], but does not assume that  $f$  is normalized (i.e.,  $f(\emptyset) = 0$ ). The proof of the lemma is deferred to Appendix A since it goes along the same lines as the proof of [22].

**Lemma IV.2.** *For a matroid and a non-negative submodular function  $f$ , if  $S$  is the independent set returned by **Greedy**, then for any independent set  $C$ ,  $f(S) \geq f(C \cup S)/2$ .*

The following lemma gives the promised variant of **Greedy**.

**Lemma IV.3.** *Let  $S$  be a random set containing every element of  $E$  with probability  $1/2$ , independently, then  $\mathbb{E}[f(\text{Greedy}(S))] \geq f(\text{OPT})/8$ .*

*Proof:* Let  $T$  be the output of **Greedy**. By Lemma IV.2, the following inequality always holds:

$$f(T) \geq \frac{f(T \cup (\text{OPT} \cap S))}{2}.$$

Hence,

$$\begin{aligned} \mathbb{E}[f(T)] &= \sum_{A \subseteq \text{OPT}} \Pr[\text{OPT} \cap S = A] \cdot \mathbb{E}[f(T) \mid \text{OPT} \cap S = A] \\ &\geq \frac{1}{2} \cdot \sum_{A \subseteq \text{OPT}} \Pr[\text{OPT} \cap S = A] \cdot \mathbb{E}[f(T \cup A) \mid \text{OPT} \cap S = A]. \end{aligned}$$

Let  $g_A(T) = f(T \cup A)$ . It is easy to check that  $g_A$  is a non-negative submodular function for every choice of set  $A$ . Additionally,  $T \subseteq S$ , and thus, contains every element of  $E \setminus A$  with probability at most  $1/2$  even conditioned on  $\text{OPT} \cap S = A$ . Hence, by Lemma II.4:

$$\mathbb{E}[f(T \cup A) \mid \text{OPT} \cap S = A] = \mathbb{E}[g_A(T \setminus A) \mid \text{OPT} \cap S = A] \geq \frac{g_A(\emptyset)}{2} = \frac{f(A)}{2}.$$

Combining the two last inequalities gives:

$$\mathbb{E}[f(T)] \geq \frac{1}{2} \cdot \sum_{A \subseteq \text{OPT}} \left( \Pr[\text{OPT} \cap S = A] \cdot \frac{f(A)}{2} \right) = \frac{\mathbb{E}[f(\text{OPT} \cap S)]}{4} \geq \frac{f(\text{OPT})}{8},$$

where the last inequality follows by Lemma II.3. ■

**Corollary IV.4.**  $\mathbb{E}[f(M)] \geq f(\text{OPT})/8$ .

*Proof:* We prove the corollary for the set  $M$  of Algorithm 1, which is fine since it has the same distribution as the set  $M$  of Algorithm 2. Observe that the set  $L$  of Algorithm 1 contains every element with probability  $1/2$ , independently, and  $M$  is the result of applying **Greedy** to this set. Thus, the corollary holds by Lemma IV.3. ■

### B. Second Stage

In the second stage we use the bounds proved in the first stage to get bounds also on  $\mathbb{E}[w(N)]$  and  $\mathbb{E}[f_w(N)]$ .

**Lemma IV.5.**  $\mathbb{E}[w(N)] = p \cdot \mathbb{E}[w(M)]$ .

*Proof:* Consider an arbitrary element  $u \in E$  processed by Algorithm 2, and let us fix all history up to the point before  $u$  is processed. If  $M + u \notin \mathcal{I}$  or  $f(u \mid M) \leq 0$ , then there is a zero expected increase in both  $w(M)$  and  $w(N)$  during the processing of  $u$ . Otherwise, the expected increase in  $w(M)$  is  $w(u)/2$ , while the expected

increase in  $w(N)$  is  $(p/2) \cdot w(u)$ . Hence, if we denote by  $\Delta_M$  the expected increase in  $w(M)$  and by  $\Delta_N$  the expected increase in  $w(N)$ , then:

$$\Delta_N = \frac{(p/2) \cdot w(u)}{w(u)/2} \cdot \Delta_M = p \cdot \Delta_M .$$

The lemma now follows from the linearity of expectation. ■

Getting a bound on  $f_w(N)$  is somewhat more involved.

**Lemma IV.6.**  $\mathbb{E}[f_w(N)] \geq \frac{f(\emptyset)}{1+p} + \frac{p(1-p)}{1+p} \cdot \mathbb{E}[f_w(M)]$ .

*Proof:* Consider the auxiliary function  $\Phi(M, N) = f_w(N) - \frac{p}{1+p} \cdot f_w(M \cup N)$ . Observe that  $\Phi$  is in fact a random function since it depends on the random vector  $w$ . Our first objective in this proof is to show that

$$\mathbb{E}[\Phi(M, N)] \geq (1+p)^{-1} \cdot f(\emptyset) . \quad (1)$$

For that purpose, we define  $\Delta_u$  as the change in  $\Phi(M, N)$  when  $u$  is processed by Algorithm 2. More formally,  $\Delta_u = \Phi(M'_u, N'_u) - \Phi(M_u, N_u)$ . Additionally, let  $R_u$  be an event encoding all the random decisions of Algorithm 2 up to the moment before it processes  $u$ , and let  $\mathcal{R}_u$  be the set of all such events. Then, since, by Property II.1,  $\Phi(\emptyset, \emptyset) = (1+p)^{-1} \cdot f_w(\emptyset) = (1+p)^{-1} \cdot f(\emptyset)$ :

$$\mathbb{E}[\Phi(M, N)] - (1+p)^{-1} \cdot f(\emptyset) = \sum_{u \in E} \mathbb{E}[\Delta_u] = \sum_{u \in E} \sum_{R_u \in \mathcal{R}_u} (\Pr[R_u] \cdot \mathbb{E}[\Delta_u \mid R_u]) .$$

Thus, to prove Inequality (1) it is enough to show  $\mathbb{E}[\Delta_u \mid R_u] \geq 0$  for an arbitrary element  $u \in E$  and event  $R_u \in \mathcal{R}_u$ . Notice that conditioned on  $R_u$ , the sets  $M_u$  and  $N_u$  and the part of the vector  $w$  corresponding to  $E'_u$  are all deterministic. If  $M_u + u \notin \mathcal{I}$  or  $f(u \mid M_u) < 0$ , then we are done since  $M_u = M'_u$  and  $N_u = N'_u$ . Thus, we only need to consider the case  $M_u + u \in \mathcal{I}$  and  $f(u \mid M_u) \geq 0$ . In this case  $u$  is added to  $M$  with probability  $1/2$  and to  $N$  with probability  $p/2$ . Thus,

$$\begin{aligned} \mathbb{E}[f_w(N'_u) - f_w(N_u) \mid R_u] &= \Pr[u \in N \mid R_u] \cdot \mathbb{E}[f_w(u \mid N_u) \mid R_u] \\ &\geq \Pr[u \in N \mid R_u] \cdot \mathbb{E}[f_w(u \mid M_u \cup N_u) \mid R_u] \\ &= \frac{p/2}{1/2 + p/2} \cdot \Pr[u \in M \cup N \mid R_u] \cdot \mathbb{E}[f_w(u \mid M_u \cup N_u) \mid R_u] \\ &= \frac{p}{1+p} \cdot \mathbb{E}[f_w(M'_u \cup N'_u) - f_w(M_u \cup N_u) \mid R_u] , \end{aligned}$$

where the inequality holds since  $f_w$  is submodular by Property II.2. Rearranging the last inequality yields  $\mathbb{E}[\Delta_u \mid R_u] \geq 0$ , which completes the proof of Inequality (1).

Next, observe that for an element to enter  $N$ , three things have to happen: first it must hold that  $M_u + u \in \mathcal{I}$  and  $f(u \mid M_u) \geq 0$ , then the algorithm must randomly decide not to add the element to  $M$  and finally the algorithm must randomly decide (with probability  $p$ ) to add the element to  $N$ . The last decision does not affect the future development of  $M$  and  $w$ , and thus, even conditioned on  $M$  and  $w$ , every element belongs to  $N$  with probability at most  $p$ . To use the last observation, let  $g_{w,M}(S) = f_w(M \cup S)$ . One can observe that  $g_{w,M}$  is non-negative and submodular. Thus, by Lemma II.4:

$$\mathbb{E}[f_w(M \cup N) \mid M, w] = \mathbb{E}[g_{w,M}(N) \mid M, w] \geq (1-p) \cdot g_{w,M}(\emptyset) = (1-p) \cdot f_w(M) .$$

By the law of total expectation, the above inequality implies:

$$\mathbb{E}[f_w(M \cup N)] \geq (1-p) \cdot \mathbb{E}[f_w(M)] ,$$

which implies the lemma when combined with Inequality (1). ■

### C. Third Stage

In the third stage we use the bounds proved in the first and second stages to get bounds on  $\mathbb{E}[w(Q \cap N)]$  and  $\mathbb{E}[f_w(Q \cap N)]$ . For that purpose, let us define  $\text{OPT}_w(N)$  as the maximum weight independent set in  $N$  with respect to the weight function  $w$ . The fact that **Linear** is  $\alpha$ -competitive for  $\mathcal{M}$  implies the following observation.

**Observation IV.7.**  $\mathbb{E}[w(Q \cap N)] = \mathbb{E}[w(Q)] \geq \frac{1}{\alpha} \cdot \mathbb{E}[w(\text{OPT}_w(N \cup N_0))] = \frac{1}{\alpha} \cdot \mathbb{E}[w(\text{OPT}_w(N))]$ .

Thus, to get a lower bound on  $\mathbb{E}[w(Q \cap N)]$  it suffices to get a lower bound on  $\mathbb{E}[w(\text{OPT}_w(N))]$ .

**Lemma IV.8.**  $\mathbb{E}[w(\text{OPT}_w(N))] \geq \frac{p}{1+p} \cdot \mathbb{E}[w(M)]$ .

*Proof:* For this proof we need Algorithm 2 to maintain two additional sets  $N'$  and  $H$ . The set  $N'$  is the set of elements of  $N$  that are not spanned by previous  $N$ -elements when added to  $N$ . More formally, the set  $N'$  is originally empty. Whenever an element  $u$  is added to  $N$ , it is also added to  $N'$  if it is not spanned by previous elements of  $N$ . Clearly, the set  $N' \subseteq N$  at the end of the procedure is an independent set, and we can, thus, use  $\mathbb{E}[w(N')]$  as a lower bound on  $\mathbb{E}[w(\text{OPT}_w(N))]$ .<sup>8</sup> Hence, to prove the lemma it is enough to show  $\mathbb{E}[w(N')] \geq \frac{p}{1+p} \mathbb{E}[w(M)]$ .

The set  $H$  is maintained by the following rules:

- Originally  $H$  is empty.
- Whenever an element  $u$  is added to  $N'$ , it is also added to  $H$  if  $H + u \in \mathcal{I}$ .
- Whenever an element  $u$  is added to  $M$ , it is also added to  $H$ . If that addition makes  $H$  non-independent, then an arbitrary element  $\phi(u) \in H \cap N'$  such that  $H - \phi(u) \in \mathcal{I}$  (such an element exists since  $H \setminus N' = M$  is independent) is removed from  $H$ .

Consider now an arbitrary element  $u \in E$  processed by Algorithm 2, and let us fix all history up to the point before  $u$  is processed. Notice that at this point  $w(u)$  is no longer a random variable. We are interested in the expected increase of  $w(M)$  and  $w(N')$  when  $u$  is processed. If  $M + u \notin \mathcal{I}$  or  $f(u | M) < 0$ , then Algorithm 2 does not add  $u$  to either  $M$  or  $N$ , and thus, there is a zero increase in both  $w(M)$  and  $w(N')$ . Otherwise, if  $N' + u \in \mathcal{I}$ , then the expected increase in  $w(M)$  is  $w(u)/2$ , while the expected increase in  $w(N')$  is  $(p/2) \cdot w(u)$ . Finally, we need to consider the case that  $M + u \in \mathcal{I}$  and  $f(u | M) \geq 0$  but  $N' + u \notin \mathcal{I}$ . In this case the expected increase in  $w(M)$  is still  $w(u)/2$ , but the expected increase in  $w(N')$  is 0. To fix that, we charge  $(p/2) \cdot w(u)$  to the element of  $H$  that becomes  $\phi(u)$  if  $u$  is added to  $M$  (regardless of whether  $u$  is really added to  $M$  or not).

Let  $c(u)$  be the amount charged to an element  $u$ . By the above discussion we clearly have:

$$\frac{\mathbb{E}[w(M)]}{\mathbb{E}[w(N')] + \sum_{u \in E} \mathbb{E}[c(u)]} = \frac{1/2}{p/2} = p^{-1} . \quad (2)$$

To complete the proof we upper bound the charge to every element  $u \in E$ . Let us fix all history up to the point after  $u$  is processed. If  $u \notin H \cap N'$  at this point, then, by definition,  $c(u) = 0$ . Otherwise, from this point on, till  $u$  leaves  $H \cap N'$ , every arriving element  $u'$  can have one of two behaviors:

- $u'$  does not cause a charge to be added to  $u$ .
- $u'$  causes  $u$  to be charged by  $(p/2) \cdot w(u') \leq (p/2) \cdot w(u)$  (the inequality holds because Algorithm 2 considers elements in a non-increasing weights order). In this case, with probability  $1/2$ ,  $u'$  is added to  $M$  and  $u$  is removed from  $H \cap N'$  (and therefore, will not be charged again in the future).

From the above two options we learn that when  $u$  gets into  $H \cap N'$ ,  $c(u)$  is upper bounded by  $(p/2) \cdot w(u) \cdot X$ , where  $X$  is distributed according to the geometric distribution  $G(1/2)$ . Thus, in this case:

$$\mathbb{E}[c(u)] \leq (p/2) \cdot w(u) \cdot \mathbb{E}[X] = p \cdot w(u) .$$

<sup>8</sup>In fact,  $N'$  is a maximum weight independent set in  $N$  since the process creating it is equivalent to running greedy on  $N$  with the objective function  $w$ .

Combining both cases, we get that the following inequality always holds:

$$\mathbb{E}[c(u)] \leq p \cdot \mathbb{E}[w(N' \cap \{u\})] .$$

Plugging the last inequality into (2) gives:

$$\frac{\mathbb{E}[w(M)]}{\mathbb{E}[w(N')] + p \cdot \mathbb{E}[w(N')]} \leq \frac{1}{p} \Rightarrow \frac{p}{1+p} \cdot \mathbb{E}[w(M)] \leq \mathbb{E}[w(N')] .$$

■

**Corollary IV.9.**  $\mathbb{E}[w(Q \cap N)] \geq \frac{p}{\alpha(1+p)} \cdot \mathbb{E}[w(M)]$ .

*Proof:* Follows immediately from Observation IV.7 and Lemma IV.8. ■

The bound on  $\mathbb{E}[f_w(Q \cap N)]$  is obtained by showing that  $\mathbb{E}[w(Q \cap N) - f_w(Q \cap N)] \leq q \cdot \mathbb{E}[w(N) - f_w(N)]$  for some value  $q \geq \frac{1}{\alpha}$ . We later show that this inequality always holds for  $q = 1$ , which already allows us to prove Theorem I.1. However, it turns out that by exploiting some basic properties of many algorithms for (linear) MSP, the same inequality can be shown for smaller values of  $q$ , which leads to stronger competitive ratios.

**Proposition IV.10.** *If  $\mathbb{E}[w(Q \cap N) - f_w(Q \cap N)] \leq q \cdot \mathbb{E}[w(N) - f_w(N)]$  for some value  $q \geq \frac{1}{\alpha}$ , then  $\mathbb{E}[f(Q \cap N)] \geq \mathbb{E}[f_w(Q \cap N)] \geq \frac{p(1-2pq\alpha)}{8\alpha(1+p)} \cdot f(\text{OPT})$ .*

*Proof:* Observe that:

$$\begin{aligned} \mathbb{E}[f_w(Q \cap N)] &\geq \mathbb{E}[w(Q \cap N)] - q \cdot \mathbb{E}[w(N) - f_w(N)] & (3) \\ &\geq \frac{p}{\alpha(1+p)} \cdot \mathbb{E}[w(M)] - q \cdot \mathbb{E}[w(N)] + q \cdot \mathbb{E}[f_w(N)] \\ &\geq \frac{p}{\alpha(1+p)} \cdot \mathbb{E}[w(M)] - qp \cdot \mathbb{E}[w(M)] + \frac{q \cdot f(\emptyset)}{1+p} + \frac{qp(1-p)}{1+p} \cdot \mathbb{E}[f_w(M)] , \end{aligned}$$

where the first inequality holds by the assumption of the proposition, the second by Corollary IV.9 and the last by Lemmata IV.5 and IV.6.

Lemma IV.1 implies the following inequalities:

$$\begin{aligned} \frac{p}{\alpha(1+p)} \cdot \mathbb{E}[w(M)] &= \frac{p}{\alpha(1+p)} \cdot (\mathbb{E}[f(M)] - f(\emptyset)) \geq \frac{p}{\alpha(1+p)} \cdot \mathbb{E}[f(M)] - \frac{q \cdot f(\emptyset)}{1+p} , \\ -\mathbb{E}[w(M)] &= -(\mathbb{E}[f(M)] - f(\emptyset)) \geq -\mathbb{E}[f(M)] \quad \text{and} \quad \mathbb{E}[f_w(M)] = \mathbb{E}[f(M)] . \end{aligned}$$

Plugging these inequalities into Inequality (3) yields:

$$\begin{aligned} \mathbb{E}[f_w(Q \cap N)] &\geq \frac{p}{\alpha(1+p)} \cdot \mathbb{E}[f(M)] - qp \cdot \mathbb{E}[f(M)] + \frac{qp(1-p)}{1+p} \cdot \mathbb{E}[f(M)] \\ &= \frac{p(1 - \alpha q(1+p) + \alpha q(1-p))}{\alpha(1+p)} \cdot \mathbb{E}[f(M)] = \frac{p(1 - 2qp\alpha)}{\alpha(1+p)} \cdot \mathbb{E}[f(M)] . \end{aligned}$$

The proposition now follows by combining the last inequality with Corollary IV.4. ■

The following observation proves that one can use  $q = 1$  for every algorithm **Linear**.

**Observation IV.11.**  $w(S) - f_w(S)$  is a monotone function of  $S$ , hence,  $w(Q \cap N) - f_w(Q \cap N) \leq w(N) - f_w(N)$ .

*Proof:* Let  $S$  be an arbitrary subset of  $E$  and let  $u \in E \setminus S$ . We need to show that  $w(S+u) - f_w(S+u) \geq w(S) - f_w(S)$ , or equivalently,  $w(u) \geq f_w(S+u) - f_w(S)$ . By definition, there exists a set  $B \subseteq S$  such that:

$$f_w(S) = f(B) + w(S \setminus B) .$$

Hence,

$$\begin{aligned} f_w(S+u) &= \min_{A \subseteq S+u} \{f(A) + w((S+u) \setminus A)\} \leq f(B) + w((S+u) \setminus B) \\ &= f(B) + w(S \setminus B) + w(u) = f_w(S) + w(u) . \end{aligned}$$

The observation now follows by rearranging the above inequality.  $\blacksquare$

We can now prove Theorem I.1.

*Proof of Theorem I.1:* By Observation IV.11, we can plug  $q = 1$  into Proposition IV.10. Choosing  $p = (3\alpha)^{-1} < 1$ , the proposition implies:

$$\mathbb{E}[f(Q \cap N)] \geq \frac{(3\alpha)^{-1}(1-2/3)}{8\alpha(1+(3\alpha)^{-1})} \cdot f(\text{OPT}) = \frac{1}{24\alpha(3\alpha+1)} \cdot f(\text{OPT}) .$$

Hence, Algorithm 2 with  $p = (3\alpha)^{-1}$  is  $24\alpha(3\alpha+1)$ -competitive for the matroid  $\mathcal{M}$ .  $\blacksquare$

Note that  $p = (3\alpha)^{-1}$  used in the above proof is not the minimizer of the expression  $p(1-2p\alpha)/(8\alpha(1+p))$  obtained from Proposition IV.10 by setting  $q = 1$ . We use  $p = (3\alpha)^{-1}$  for simplicity since it leads to a clean expression that is very close to the one obtained by the minimizing  $p$ .

In some cases it is possible to use a smaller value of  $q$  in Proposition IV.10. The following two claims prove one such case. A set function  $g: 2^E \rightarrow \mathbb{R}$  is called *normalized* if  $g(\emptyset) = 0$  and *supermodular* if  $g(A) + g(B) \leq g(A \cup B) + g(A \cap B)$ .

**Lemma IV.12.** *Let  $g: 2^E \rightarrow \mathbb{R}^+$  be a normalized, monotone and supermodular function. Denote by  $A(q)$  a random subset of  $A$  where each element appears with probability at most  $q$  (not necessarily independently). Then,  $\mathbb{E}[g(A(q))] \leq q \cdot g(A)$ .*

*Proof:* Let  $A = \{u_1, \dots, u_{|A|}\}$  be an arbitrary numbering of the elements in  $A$ , and for  $i \in \{0, \dots, |A|\}$  we define  $A_i = \{u_1, \dots, u_i\}$ , where  $A_0 = \emptyset$ . Denote by  $X_i$  an indicator for the event that  $u_i \in A(q)$ , and let  $q_i = \Pr[u_i \in A(q)] = \mathbb{E}[X_i] \leq q$ . Then:

$$\begin{aligned} \mathbb{E}[g(A(q))] &= \mathbb{E} \left[ g(\emptyset) + \sum_{i=1}^{|A|} X_i \cdot g(u_i \mid A_{i-1} \cap A(q)) \right] \leq \mathbb{E} \left[ g(\emptyset) + \sum_{i=1}^{|A|} X_i \cdot g(u_i \mid A_{i-1}) \right] \\ &= g(\emptyset) + \sum_{i=1}^{|A|} q_i \cdot g(u_i \mid A_{i-1}) \leq g(\emptyset) + q \cdot \sum_{i=1}^{|A|} g(u_i \mid A_i) \\ &= (1-q) \cdot g(\emptyset) + q \cdot g(A) = q \cdot g(A) , \end{aligned}$$

where the first inequality follows from the supermodularity of  $g$ , the second one from monotonicity and the fact that  $q_i \leq q$  for  $1 \leq i \leq |A|$ , and the last equality follows by the fact that  $g$  is normalized.  $\blacksquare$

**Corollary IV.13.** *If Linear is guaranteed to pick every element of its input with probability at most  $q$ , then  $\mathbb{E}[w(Q \cap N) - f_w(Q \cap N)] \leq q \cdot \mathbb{E}[w(N) - f_w(N)]$ .*

*Proof:* Let us define the function  $g(S) = w(S) + f_w(\emptyset) - f_w(S)$ . Since  $w(S)$  is linear,  $f_w(\emptyset)$  is a constant and  $f_w(S)$  is submodular,  $g(S)$  is a supermodular function. Additionally,  $g(S)$  is normalized by construction and monotone by Observation IV.11. Hence, for every fixed  $N$ , by Lemma IV.12 and the assumption that no element of  $N$  belongs to  $Q$  with probability larger than  $q$ :

$$\mathbb{E}[g(Q \cap N)] \leq q \cdot g(N) .$$

Since the last inequality holds for every fixed  $N$ , it holds also, in expectation, without fixing  $N$ . Thus:

$$\begin{aligned} \mathbb{E}[w(Q \cap N) - f_w(Q \cap N)] &= \mathbb{E}[g(Q \cap N)] - f_w(\emptyset) \\ &\leq q \cdot \mathbb{E}[g(N)] - f_w(\emptyset) \leq q \cdot \mathbb{E}[w(N) - f_w(N)] , \end{aligned}$$

where the last inequality holds since  $f_w$  is non-negative by Property II.2. ■

We can now prove Theorem I.4.

*Proof of Theorem I.4:* Observe that  $q$  must be at least  $1/\alpha$  since any  $\alpha$ -competitive algorithm for MSP must be able to select an element with probability at least  $1/\alpha$  when this element is the only element having a non-zero weight. Hence, by Corollary IV.13, we can plug  $q$  into Proposition IV.10. Letting  $\beta = \alpha q \geq 1$  and choosing  $p = (3\beta)^{-1} < 1$ , the proposition implies:

$$w(Q \cap N) \geq \frac{\frac{1}{3\beta}(1 - 2q\alpha\frac{1}{3\beta})}{8\alpha(1 + \frac{1}{3\beta})} = \frac{\frac{1}{3\beta}(1 - 2/3)}{8\frac{\beta}{q}(1 + \frac{1}{3\beta})} \cdot f(\text{OPT}) = \frac{q}{24\beta(3\beta + 1)} \cdot f(\text{OPT}) .$$

Hence, Algorithm 2 with  $p = (3\beta)^{-1} = (3\alpha q)^{-1}$  has a competitive ratio of at most:

$$\frac{24\beta(3\beta + 1)}{q} = \frac{24\alpha q(3\alpha q + 1)}{q} = O(q \cdot \alpha^2)$$

for the matroid  $\mathcal{M}$ . ■

#### D. Proof of Theorem I.2

In this section we assume  $\mathcal{M}$  is a matroid for which Theorem I.2 is meaningful. More specifically, the behavior of **Linear** on  $\mathcal{M}$  can be characterized as follows.

- For every MSP instance over the matroid  $\mathcal{M}$ , there exists a set of  $k$  (correlated) random sets  $\{P_i\}_{i=1}^k$  such that each set  $P_i$  is always independent in  $\mathcal{M}$  and  $\mathbb{E}[w(\bigcup_{i=1}^k P_i)] \geq w(\text{OPT}_w)/\alpha$ , where  $w$  is the weight function of the MSP instance and  $\text{OPT}_w$  is the maximum weight independent set given  $w$ .
- **Linear** outputs a uniformly random set from  $\{P_i\}_{i=1}^k$ .

For every  $1 \leq i \leq k$ , let  $Q_i$  be the set  $P_i$  corresponding to the execution of **Linear** within Algorithm 2 with  $p = (3\alpha)^{-1}$ .

**Lemma IV.14.**  $\mathbb{E}[f(\bigcup_{i=1}^k Q_i \cap N)] \geq \frac{1}{24\alpha(3\alpha+1)} \cdot f(\text{OPT})$ .

*Proof:* One can verify that the fact that  $Q$  is produced by **Linear** was used in the above analysis of the competitive ratio of Algorithm 2 only to justify the inequality:  $\mathbb{E}[w(Q)] \geq \frac{1}{\alpha} \cdot \mathbb{E}[w(\text{OPT}_w(N \cup N_0))]$ . Thus, the proof of Theorem I.1 can be viewed as showing that after executing Algorithm 2 with  $p = (3\alpha)^{-1}$  every random subset  $S$  of  $N \cup N_0$  obeying this inequality must obey also:

$$\mathbb{E}[f(S \cap N)] \geq \frac{1}{24\alpha(3\alpha + 1)} \cdot f(\text{OPT}) .$$

The lemma now follows since  $\bigcup_{i=1}^k Q_i$  is a random subset of  $N \cup N_0$  obeying the required inequality. ■

The next lemma is necessary to transform the bound given by the last lemma into a bound on the values of the separate sets  $Q_i$ .

**Lemma IV.15.** Let  $g : 2^E \rightarrow \mathbb{R}^+$  be a non-negative submodular function and let  $S_1, \dots, S_k \subseteq E$ . Then,

$$\sum_{i=1}^k f(S_i) \geq f\left(\bigcup_{i=1}^k S_i\right) .$$

*Proof:* We prove the result by induction on  $k$ . It clearly holds for  $k = 1$ . For  $k > 1$ :

$$\begin{aligned} \sum_{i=1}^k f(S_i) &= f(S_1) + f(S_2) + \sum_{i=3}^k f(S_i) \geq f(S_1 \cup S_2) + f(S_1 \cap S_2) + \sum_{i=3}^k f(S_i) \\ &\geq f(S_1 \cup S_2) + \sum_{i=3}^k f(S_i) \geq f\left(\bigcup_{i=1}^k S_i\right) , \end{aligned}$$

where the first inequality holds by the submodularity of  $f$ , the second by the non-negativity of  $f$  and the last by the induction hypothesis. ■

We are now ready to prove Theorem I.2.

*Proof of Theorem I.2:* Combining Lemmata IV.14 and IV.15, we get:

$$\sum_{i=1}^k \mathbb{E}[f(Q_i \cap N)] \geq \mathbb{E} \left[ f \left( \bigcup_{i=1}^k Q_i \cap N \right) \right] \geq \frac{1}{24\alpha(3\alpha+1)} \cdot f(\text{OPT}) .$$

Since the output set  $Q \cap N$  of Algorithm 2 is a uniformly random subset from  $\{Q_i \cap N\}_{i=1}^k$ , we get:

$$\mathbb{E}[f(Q \cap N)] = \frac{\sum_{i=1}^k \mathbb{E}[f(Q_i \cap N)]}{k} \geq \frac{1}{24k\alpha(3\alpha+1)} \cdot f(\text{OPT}) .$$

Hence, Algorithm 2 with  $p = (3\alpha)^{-1}$  is  $24k\alpha(3\alpha+1)$ -competitive for the matroid  $\mathcal{M}$ . ■

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APPENDIX A.  
MISSING PROOFS

In this appendix we provide the missing proofs for Proposition III.2 and Lemma IV.2.

*A.1 Proof of Proposition III.2*

**Proposition III.2.** *Given an algorithm `Linear` for MSP, there exists an algorithm for Partial-MSP whose competitive ratio for every matroid  $\mathcal{M}$  is as good as the competitive ratio of `Linear` for this matroid.*

The algorithm we use to prove Proposition III.2 is Algorithm 3.

---

**Algorithm 3:** Algorithm for Partial-MSP

---

```

// Initialization
1 Let  $L' \leftarrow L$ .
2 Let  $r \leftarrow n - |L|$ .

// Main Loop
3 while  $|L'| + r > 0$  do
4   with probability  $r/(|L'|+r)$  do
5     Update  $r \leftarrow r - 1$ .
6     Pass the next arriving element to Linear with its original weight  $w(u)$ .
7   otherwise
8     Let  $u$  be a uniformly random element of  $L'$ .
9     Update  $L' \leftarrow L' - u$ .
10    Pass  $u$  to Linear with a weight of 0.
11 Return the output of Linear without the elements of  $L$ .
```

---

**Observation A.1.** *The input passed to `Linear` by Algorithm 3 is a uniformly random permutation of  $E$ , where the weight of an element  $u \in E$  is defined by:*

$$w'(u) = \begin{cases} 0 & \text{if } u \in L, \\ w(u) & \text{otherwise.} \end{cases}$$

*Proof:* The only part of the observation that requires a proof is the claim that the elements of  $E$  are passed to `Linear` in a uniformly random order. To see why this is true, notice that  $r$  is maintained as the number of input elements that have not been passed yet to `Linear` and  $L'$  is the set of elements of  $L$  that have not been

passed yet to **Linear**. Hence, at each time point the set of elements that have not been passed yet to **Linear** is of size  $r + |L'|$ , and the algorithm passes a uniformly random element from this set. ■

The following lemma implies Proposition III.2.

**Lemma A.2.** *Algorithm 3 is an algorithm for Partial-MSP whose competitive ratio for every matroid  $\mathcal{M}$  is as good as the competitive ratio of **Linear** for this matroid.*

*Proof:* Assume **Linear** is  $\alpha$ -competitive for an arbitrary matroid  $\mathcal{M} = (E, \mathcal{I})$ . By Observation A.1, whenever Algorithm 3 faces an instance of Partial-MSP over the matroid  $\mathcal{M}$  with a weight function  $w$ , it makes **Linear** face an instance of MSP over the same matroid with weights given by  $w'$ . Hence, if we denote by  $S$  the output of **Linear**, then, by the guarantee of **Linear**:

$$\mathbb{E}[w'(S)] \geq \alpha \cdot \max_{T \in \mathcal{I}} w'(T) .$$

Recall that  $w'$  is equal to  $w$  for elements of  $E \setminus L$  and is equal to 0 for all other elements. Hence,

$$\mathbb{E}[w(S \setminus L)] = \mathbb{E}[w'(S)] \geq \alpha \cdot \max_{T \in \mathcal{I}} w'(T) = \alpha \cdot \max_{T \in \mathcal{I} \cap 2^{E \setminus L}} w'(T) = \alpha \cdot \max_{T \in \mathcal{I} \cap 2^{E \setminus L}} w(T) .$$

This completes the proof of the lemma since  $S \setminus L$  is the output of Algorithm 3 and  $\max_{T \in \mathcal{I} \cap 2^{E \setminus L}} w(T)$  is the value of the optimal solution for the instance of Partial-MSP faced by Algorithm 3. ■

## A.2 Proof of Lemma IV.2

**Lemma IV.2.** *For a matroid and a non-negative submodular function  $f$ , if  $S$  is the independent set returned by **Greedy**, then for any independent set  $C$ ,  $f(S) \geq f(C \cup S)/2$ .*

First, we need some definitions. Let  $k = |S|$ , and for every  $0 \leq i \leq k$  let  $S_i$  be the solution of **Greedy** after  $i$  elements are added to it. Additionally, for every  $1 \leq i \leq k$ , let  $u_i$  be the single element in  $S_i \setminus S_{i-1}$ . We also need to define the sets  $\{C_i\}_{i=0}^k$  recursively as follows:

- The set  $C_0$  is simply  $C$ . Observe that  $C_0 \cup S_0 = C$  is independent.
- For every  $1 \leq i \leq k$ , the set  $C_i$  is a maximal independent subset of  $C_{i-1} + u_i$  that contains  $S_i$ . It is also useful to define  $C'_i = C_{i-1} \setminus C_i$ , the set of elements that appear in  $C_{i-1}$  but not in  $C_i$ . By matroid properties, since  $S_i - u_i = S_{i-1} \subseteq C_{i-1}$  and  $C_{i-1}$  is independent, the size of  $C'_i$  is at most 1.

Let us now make a few observations regarding the above definitions.

**Observation A.3.**  $f(S_k) \geq f(C_k)$ .

*Proof:* Assume towards a contradiction that  $f(S_k) < f(C_k)$ . Then, by submodularity since  $S_k \subseteq C_k$ , there must exist an element  $u \in C_k \setminus S_k$  obeying  $f(u | S_k) > 0$ . On the other hand, since  $C_k$  is independent we must also have that  $S_k + u$  is independent. However, the existence of an element with these properties contradicts the fact that  $S_k$  is the output of **Greedy**. ■

**Lemma A.4.** *For every  $1 \leq i \leq k$ ,  $f(u_i | S_{i-1}) \geq f(C'_i \cup C_k) - f(C_k)$ .*

*Proof:* If  $C'_i = \emptyset$ , then the right hand side of the inequality we want to prove is 0 while its left hand side is non-negative (since **Greedy** chose to add  $u_i$  to  $S_{i-1}$ ). Thus, we concentrate from this point on the case  $|C'_i| = 1$ .

Let  $u'_i$  be the single element of  $C'_i$ . Since  $C_i$  contains the set  $S_{i-1}$  and is independent, we know that  $u'_i$  is not in  $S_{i-1}$  and could be added to  $S_{i-1}$  by **Greedy** without violating independence. On the other hand, since **Greedy** chose to add  $u_i$  to  $S_{i-1}$ , we must have:

$$f(u_i | S_{i-1}) \geq f(u'_i | S_{i-1}) \geq f(u'_i | C_k) = f(C'_i \cup C_k) - f(C_k) ,$$

where the second inequality follows by submodularity since  $S_{i-1} \subseteq S_k \subseteq C_k$ . ■

We are now ready to prove Lemma IV.2.

$$\begin{aligned}
 f(S) &= \frac{f(S_k) + f(\emptyset) + \sum_{i=1}^k f(u_i | S_{i-1})}{2} \geq \frac{f(C_k) + f(\emptyset) + \sum_{i=1}^k [f(C'_i \cup C_k) - f(C_k)]}{2} \\
 &\geq \frac{f(C_k) + f(\emptyset) + [f(\bigcup_{i=1}^k C'_i \cup C_k) - f(C_k)]}{2} = \frac{f(\emptyset) + f(S \cup C)}{2} \geq \frac{f(S \cup C)}{2},
 \end{aligned}$$

where the first inequality holds by Observation A.3 and Lemma A.4, the second inequality holds by submodularity of  $f$  and the fact that the  $C'_i$  are disjoint, and the last by the non-negativity of  $f$ .