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The Complexity of Counting Edge Colorings and a Dichotomy for Some Higher Domain Holant Problems (Extended Abstract)

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***Abstract*—We show that an effective version of Siegel's Theorem on finiteness of integer solutions for a specific algebraic curve and an application of elementary Galois theory are key ingredients in a complexity classification of some Holant problems. These Holant problems, denoted by $\mathrm{Holant}(f)$, are defined by a symmetric ternary function f that is invariant under any permutation of the $\kappa \geq 3$ domain elements. We prove that $\mathrm{Holant}(f)$ exhibits a complexity dichotomy. The hardness, and thus the dichotomy, holds even when restricted to planar graphs. A special case of this result is that counting edge κ-colorings is #P-hard over planar 3-regular multigraphs for all $\kappa \geq 3$. In fact, we prove that counting edge κ-colorings is #P-hard over planar r-regular multigraphs for all $\kappa \geq r \geq 3$. The problem is polynomial-time computable in all other parameter settings. The proof of the dichotomy theorem for $\mathrm{Holant}(f)$ depends on the fact that a specific polynomial $p(x, y)$ has an explicitly listed finite set of integer solutions, and the determination of the Galois groups of some specific polynomials. In the process, we also encounter the Tutte polynomial, medial graphs, Eulerian partitions, Puiseux series, and a certain lattice condition on the (logarithm of) the roots of polynomials.**

***Keywords*-counting problems; dichotomy theorem; Holant problems; edge coloring;**

I. Introduction

What do Siegel's Theorem and Galois theory have to do with complexity theory? In this paper, we show that an effective version of Siegel's Theorem on finiteness of integer solutions for a specific algebraic curve and an application of elementary Galois theory are key ingredients in a chain of steps that lead to a complexity classification of some counting problems. More specifically, we consider a certain class of counting problems that are expressible as Holant problems with an arbitrary domain of size κ over 3-regular multigraphs (i.e. self-loops and parallel edges are allowed), and prove a dichotomy theorem for this class of problems. The hardness, and thus the dichotomy, holds even when restricted to planar multigraphs. Among other things, the proof of the dichotomy theorem depends on the following: (A) the specific polynomial

$$p(x, y) = x^5 - 2x^3y - x^2y^2 - x^3 + xy^2 + y^3 - 2x^2 - xy$$

has only the integer solutions

$$(x, y) = (-1, 1), (0, 0), (1, -1), (1, 2), (3, 3),$$

and (B) the determination of the Galois groups of some specific polynomials. In the process, we also encounter the Tutte polynomial, medial graphs, Eulerian partitions, Puiseux series, and a certain lattice condition on the (logarithm of) the roots of polynomials such as $p(x, y)$.

A special case of this dichotomy theorem is the problem of counting edge colorings over planar 3-regular multigraphs using κ colors. In this case, the corresponding constraint function is the $\text{All-Distinct}_{3,\kappa}$ function, which takes value 1 when all three inputs from $[\kappa]$ are distinct and 0 otherwise. We further prove that the problem using κ colors over r-regular multigraphs is #P-hard for all $\kappa \geq r \geq 3$, even when restricted to planar multigraphs. The problem is polynomial-time computable in all other parameter settings. This solves a long-standing open problem.

We give a brief description of the framework of Holant problems [20], [18], [15], [17]. The problem $\mathrm{Holant}(\mathcal{F})$, defined by a set of functions $\mathcal{F}$, takes as input a *signature grid* $\Omega = (G, \pi)$, where $G = (V, E)$ is a multigraph, π assigns each $v \in V$ a function $f_v \in \mathcal{F}$, and f_v maps $[\kappa]^{\deg(v)}$ to $\mathbb{C}$ for some integer $\kappa \geq 2$. An edge κ-labeling $\sigma : E \to [\kappa]$ gives an evaluation $\prod_{v \in V} f_v(\sigma \mid_{E(v)})$, where $E(v)$ denotes the incident edges of v and $\sigma \mid_{E(v)}$ denotes the restriction of σ to $E(v)$. The counting problem on the instance Ω is to compute

$$\mathrm{Holant}(\Omega, \mathcal{F}) = \sum_{\sigma : E \to [\kappa]} \prod_{v \in V} f_v\left(\sigma \mid_{E(v)}\right).$$

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Counting edge κ-colorings over r-regular multigraphs amounts to setting $f_v = \text{ALL-DISTINCT}_{r,\kappa}$ for all v. We also use $\text{Pl-Holant}(\mathcal{F})$ to denote the restriction of $\text{Holant}(\mathcal{F})$ to planar multigraphs.

Holant problems appear in many areas under a variety of different names. They are equivalent to counting Constraint Satisfaction Problems (#CSP) [5], [7] with the restriction that all variables are read twice,[1] to the contraction of a tensor network [21], [31], and to the partition function of graphical models in Forney normal form [32], [35] from artificial intelligence, coding theory, and signal processing. Special cases of Holant problems include simulating quantum circuits [42], [36], counting graph homomorphisms [1], [23], [3], [28], [9], and evaluating the partition function of the edge-coloring model [1, Section 3.6].

An edge κ-coloring of a graph G is an edge κ-labeling of G such that any two incident edges have different colors. A fundamental problem in graph theory is to determine how many colors are required to edge color G. The obvious lower bound is $\Delta(G)$, the maximum degree of the graph. By Vizing's Theorem [44], an edge coloring using just $\Delta(G) + 1$ colors always exists for simple graphs (i.e. graphs without self-loops or parallel edges). Whether $\Delta(G)$ colors suffice depends on the graph G.

Consider the edge coloring problem over 3-regular graphs. It follows from the parity condition (Lemma IV.4) that any graph containing a bridge does not have an edge 3-coloring. For bridgeless planar simple graphs, Tait [41] showed that the existence of an edge 3-coloring is equivalent to the Four-Color Theorem. Thus, the answer for the decision problem over planar 3-regular simple graphs is that there is an edge 3-coloring iff the graph is bridgeless.

Without the planarity restriction, determining if a 3-regular (simple) graph has an edge 3-coloring is NP-complete [30]. This hardness extends to finding an edge κ-coloring over κ-regular (simple) graphs for all $\kappa \geq 3$ [33]. However, these reductions are not parsimonious, and, in fact, it is claimed that no parsimonious reduction exists unless $\text{P} = \text{NP}$ [46, p. 118]. The counting complexity of this problem has remained open.

We prove that counting edge colorings over planar regular multigraphs is $\#\text{P}$-hard.[2]

[1]Without this restriction, #CSPs are a special case of Holant problems.

[2]Vizing's Theorem is for simple graphs. In Holant problems as well as counting complexity such as graph homomorphism or #CSP, one typically considers multigraphs (i.e. self-loops and parallel edges are allowed). However, our hardness result for counting edge 3-colorings over planar 3-regular multigraphs also holds for simple graphs. See Theorem 4.9 in [11].

Theorem I.1. *$\#\kappa$-EDGECOLORING is $\#\text{P}$-hard over planar r-regular multigraphs for all $\kappa \geq r \geq 3$.*

See Theorem IV.8 for the proof when $\kappa = r$. Theorem 4.20 in [11] considers $\kappa > r$.

The techniques we develop to prove Theorem I.1 naturally extend to a class of Holant problems with domain size $\kappa \geq 3$ over planar 3-regular multigraphs. Functions such as $\text{ALL-DISTINCT}_{3,\kappa}$ are symmetric, which means that they are invariant under any permutation of its three inputs. But $\text{ALL-DISTINCT}_{3,\kappa}$ has another invariance—it is invariant under any permutation of the κ domain elements. We call the second property *domain invariance*.

A ternary function that is both symmetric and domain invariant is specified by three values, which we denote by $\langle a, b, c\rangle$. The output is a when all inputs are the same, c when all inputs are distinct, and b when two inputs are the same but the third input is different.

We prove a dichotomy theorem for such functions with complex weights.

Theorem I.2. *Suppose $\kappa \geq 3$ is the domain size and $a, b, c \in \mathbb{C}$. Then either $\text{Holant}(\langle a, b, c\rangle)$ is computable in polynomial time or $\text{Pl-Holant}(\langle a, b, c\rangle)$ is $\#\text{P}$-hard. Furthermore, given a, b, c, there is a polynomial-time algorithm that decides which is the case.*

See Theorem 10.1 in [11] for an explicit listing of the tractable cases. Note that counting edge κ-colorings over 3-regular multigraphs is the special case when $\langle a, b, c\rangle = \langle 0, 0, 1\rangle$.

There is only one previous dichotomy theorem for higher domain Holant problems [19]. The important difference is that the present work is for general domain size $\kappa \geq 3$ while the previous result is for domain size $\kappa = 3$. When restricted to domain size 3, the result in [19] assumes that all unary functions are available, while this dichotomy does not assume that; however it does assume domain invariance. Dichotomy theorems for an arbitrary domain size are generally difficult to prove. The Feder-Vardi Conjecture for decision Constraint Satisfaction Problems (CSP) is still open [27]. It was a major achievement to prove this conjecture for domain size 3 [4]. The #CSP dichotomy was proved after a long series of work [6], [5], [3], [22], [2], [15], [8], [12], [24], [29], [13], [7].

Our proof of Theorem I.2 has many components, and a number of new ideas are introduced in this proof. We discuss some of these ideas and give an outline of our proof in Section II.

II. Proof Outline and Techniques

As usual, the difficult part of a dichotomy theorem is to carve out *exactly* the tractable problems in the class, and prove all the rest #P-hard. A dichotomy theorem for Holant problems has the additional difficulty that some tractable problems are only shown to be tractable under a holographic transformation, which can make the appearance of the problem rather unexpected. For example, we show [11] that $\operatorname{Holant}(\langle -3-4i, 1, -1+2i \rangle)$ on domain size 4 is tractable. Despite its appearance, this problem is intimately connected with a tractable graph homomorphism problem defined by the Hadamard matrix $\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$. In order to understand all problems in a Holant problem class, we must deal with such problems. Dichotomy theorems for graph homomorphisms and for #CSP do not have to deal with as varied a class of such problems, since they implicitly assume all EQUALITY functions are available and must be preserved. This restricts the possible transformations.

After isolating a set of tractable problems, our #P-hardness results in both Theorem I.1 and Theorem I.2 are obtained by reducing from evaluations of the Tutte polynomial over planar graphs. A dichotomy is known for such problems (Theorem IV.1).

The chromatic polynomial, a specialization of the Tutte polynomial, is concerned with vertex colorings. On domain size κ, one starting point of our hardness proofs is the chromatic polynomial, which contains the problem of counting vertex colorings using at most κ colors. By the planar dichotomy for the Tutte polynomial, this problem is #P-hard for all $\kappa \geq 3$.

Another starting point for our hardness reductions is the evaluation of the Tutte polynomial at an integer diagonal point (x, x), which is #P-hard for all $x \geq 3$ by the same planar Tutte dichotomy. These are new starting places for reductions involving Holant problems. These problems were known to have a so-called state-sum expression (Lemma IV.3), which is a sum over weighted Eulerian partitions. This sum is not over the original planar graph but over its directed medial graph, which is always a planar 4-regular graph (Figure 1). We show that this state-sum expression is naturally expressed as a Holant problem with a particular quaternary constraint function (Lemma IV.6).

To reduce from these two problems, we execute the following strategy. First, we attempt to construct the unary constraint function $\langle 1 \rangle$, which takes value 1 on all κ inputs. Second, we attempt to interpolate all succinct binary signatures assuming that we have $\langle 1 \rangle$. (See Section III for the definition of a succinct signature.) Lastly, we attempt to construct a ternary signature with a special property assuming that all these binary signatures are available. At each step, there are some problems specified by certain signatures $\langle a, b, c \rangle$ for which our attempts fail. In such cases, we directly obtain a dichotomy without the help of additional signatures.

Below we highlight some of our proof techniques.

Interpolation within an orthogonal subspace: We develop the ability to interpolate when faced with some nontrivial null spaces inherently present in interpolation constructions. In any construction involving an initial signature and a recurrence matrix, it is possible that the initial signature is orthogonal to some row eigenvectors of the recurrence matrix. Previous interpolation results always attempt to find a construction that avoids this. In the present work, this avoidance seems impossible. We prove an interpolation result that can succeed in this situation to the greatest extent possible. We prove that one can interpolate any signature provided that it is orthogonal to the same set of row eigenvectors, and the relevant eigenvalues satisfy a lattice condition.

Satisfy lattice condition via Galois theory: A key requirement for this interpolation to succeed is the lattice condition (Definition V.1), which involves the roots of the characteristic polynomial of the recurrence matrix. We use Galois theory to prove that our constructions satisfy this condition. If a polynomial has a large Galois group, such as S_n or A_n, and its roots do not all have the same complex norm, then we show that its roots satisfy the lattice condition.

Effective Siegel's Theorem via Puiseux series: We need to determine the Galois groups for an infinite family of polynomials, one for each domain size. If these polynomials are irreducible, then we can show they all have the full symmetric group as their Galois group, and hence fulfill the lattice condition. We suspect that these polynomials are all irreducible but are unable to prove it.

A necessary condition for irreducibility is the absence of any linear factor. This infinite family of polynomials, as a single bivariate polynomial in (x, κ), defines an algebraic curve, which has genus 3. By a well-known theorem of Siegel [39], there are only a finite number of integer values of κ for which the corresponding polynomial has a linear factor. However this theorem and others like it are not *effective* in general. There are some effective versions of Siegel's Theorem that can be applied to the algebraic curve, but the best general effective bound is over $10^{20,000}$ [45] and hence cannot be checked in practice. Instead, we use Puiseux series in

Section V to show that this algebraic curve has exactly five explicitly listed integer solutions.

Eigenvalue Shifted Triples: For a pair of eigenvalues, the lattice condition is equivalent to the statement that the ratio of these eigenvalues is not a root of unity. A sufficient condition is that the eigenvalues have distinct complex norms. We prove three results, each of which is a different way to satisfy this sufficient condition. Chief among them is the technique we call an *Eigenvalue Shifted Triple* (EST). In an EST, we have three recurrence matrices, each of which differs from the other two by a nonzero additive multiple of the identity matrix. Provided these two multiples are linearly independent over $\mathbb{R}$, we show at least one of these matrices has eigenvalues with distinct complex norms. (However determining which one succeeds is a difficult task; but we need not know that).

E Pluribus Unum: When the ratio of a pair of eigenvalues is a root of unity, it is a challenge to effectively use this failure condition. Direct application of this cyclotomic condition is often of limited use. We introduce an approach that uses this cyclotomic condition effectively. A direct recursive construction involving these two eigenvalues only creates a finite number of different signatures. We reuse all of these signatures in a multitude of new interpolation constructions, one of which we hope will succeed. If the eigenvalues in all of these constructions also satisfy a cyclotomic condition, then we obtain a more useful condition than any of the previous cyclotomic conditions. This idea generalizes the anti-gadget technique [14], which only reuses the "last" of these signatures.

Local holographic transformation: One reason to obtain all succinct binary signatures is for use in the gadget construction known as a local holographic transformation. This construction mimics the effect of a holographic transformation applied on a single signature. In particular, using this construction, we attempt to obtain a succinct ternary signature of the form $\langle a, b, b\rangle$, where $a \neq b$. This signature turns out to have some magical properties in the Bobby Fischer gadget, which we discuss next.

Bobby Fischer gadget: Typically, any combinatorial construction for higher domain Holant problems produces very intimidating looking expressions that are nearly impossible to analyze. In our case, it seems necessary to consider a construction that has to satisfy multiple requirements involving at least nine polynomials. However, we are able to combine the signature $\langle a, b, b\rangle$, where $a \neq b$, with a succinct binary signature of our choice in a special construction that we call the *Bobby Fischer gadget*. This gadget is able to satisfy seven conditions using just one degree of freedom. This ability to satisfy a multitude of constraints simultaneously in one magic stroke reminds us of some unfathomably brilliant moves by Bobby Fischer, the chess genius extraordinaire.

III. PRELIMINARIES

In this paper, we investigate some complex-weighted Holant problems on domain size $\kappa \geq 3$. A constraint function, or *signature*, of arity n, maps from $[\kappa]^n \to \mathbb{C}$. For consideration of models of computation, functions take complex algebraic numbers.

Graphs (called multigraphs in Section I) may have self-loops and parallel edges. A graph without self-loops or parallel edges is a *simple* graph. A *signature grid* $\Omega = (G, \pi)$ of $\operatorname{Holant}(\mathcal{F})$ consists of a graph $G = (V, E)$, where π assigns each vertex $v \in V$ and its incident edges with some $f_v \in \mathcal{F}$ and its input variables. We say Ω is a *planar signature grid* if G is planar, where the variables of f_v are ordered counterclockwise. The Holant problem on instance Ω is to evaluate

$$\operatorname{Holant}(\Omega; \mathcal{F}) = \sum_{\sigma} \prod_{v \in V} f_v(\sigma \mid_{E(v)}),$$

a sum over all edge labelings $\sigma : E \to [\kappa]$, where $E(v)$ denotes the incident edges of v and $\sigma \mid_{E(v)}$ denotes the restriction of σ to $E(v)$.

A function f_v can be represented by listing its values in lexicographical order as in a truth table, which is a vector in $\mathbb{C}^{\kappa^{\deg(v)}}$, or as a tensor in $(\mathbb{C}^{\kappa})^{\otimes \deg(v)}$. In this paper, we consider symmetric signatures. An example of a symmetric signature is the EQUALITY signature $=_r$ of arity r. A Holant problem is parametrized by a set of signatures.

Definition III.1. *Given a set of signatures $\mathcal{F}$, we define the counting problem* $\operatorname{Holant}(\mathcal{F})$ *as:*
Input: A signature grid $\Omega = (G, \pi)$*;*
Output: $\operatorname{Holant}(\Omega; \mathcal{F})$.

The problem $\operatorname{Pl-Holant}(\mathcal{F})$ is defined similarly using a planar signature grid. Replacing a signature $f \in \mathcal{F}$ by a constant multiple cf, where $c \neq 0$, does not change the complexity of $\operatorname{Holant}(\mathcal{F})$. It introduces a global nonzero factor to $\operatorname{Holant}(\Omega; \mathcal{F})$. We follow the usual conventions about polynomial time Turing reduction $\leq_T$.

We say a signature f is *realizable* or *constructible* from a signature set $\mathcal{F}$ if there is a gadget with some dangling edges such that each vertex is assigned a signature from $\mathcal{F}$, and the resulting graph, when viewed as a black-box signature with inputs on the dangling edges, is exactly f. If f is realizable from a set $\mathcal{F}$, then we can freely add f into $\mathcal{F}$ while preserving the complexity.

Formally, such a notion is defined by an $\mathcal{F}$-gate [15], [16]. An $\mathcal{F}$-gate is similar to a signature grid (G, π) for $\operatorname{Holant}(\mathcal{F})$ except that $G = (V, E, D)$ is a graph with some dangling edges D. The dangling edges define external variables for the $\mathcal{F}$-gate. (See Figure 3 for an example.) We denote the regular edges in E by $1, 2, \ldots, m$ and the dangling edges in D by $m+1, \ldots, m+n$. Then we can define a function Γ for this $\mathcal{F}$-gate as

$$\Gamma(y_1, y_2, \ldots, y_n) = \sum_{x_1, x_2, \ldots, x_m \in [\kappa]} H(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n),$$

where $(y_1, \ldots, y_n) \in [\kappa]^n$ denotes a labeling on the dangling edges and $H(x_1, \ldots, x_m, y_1, \ldots, y_n)$ denotes the value of the signature grid on a labeling of all edges in G, which is the product of evaluations at all vertices. We also call this function Γ the signature of the $\mathcal{F}$-gate. An $\mathcal{F}$-gate is planar if the underlying graph G is a planar graph, and the dangling edges, ordered counterclockwise corresponding to the order of the input variables, are in the outer face in a planar embedding. A planar $\mathcal{F}$-gate can be used in a planar signature grid as if it is just a single vertex with the particular signature.

Using the idea of planar $\mathcal{F}$-gates, we can reduce one planar Holant problem to another. Suppose g is the signature of some planar $\mathcal{F}$-gate. Then we obtain Pl-$\operatorname{Holant}(\mathcal{F} \cup \{g\}) \leq_T$ Pl-$\operatorname{Holant}(\mathcal{F})$, by replacing every appearance of g by the $\mathcal{F}$-gate. Since the signature of the $\mathcal{F}$-gate is g, the Holant values for these two signature grids are identical.

Our main results are about symmetric signatures (i.e. signatures that are invariant under any permutation of inputs). However, we also need some asymmetric signatures in our proofs. When a gadget has an asymmetric signature, we place a diamond on the edge corresponding to the first input. The remaining inputs are ordered counterclockwise around the vertex. (See Figure 3 for an example.)

An arity r signature on domain size κ is fully specified by κ^r values. However, some special cases can be defined using far fewer values. Consider the signature ALL-DISTINCT$_{r,\kappa}$ of arity r on domain size κ that outputs 1 when all inputs are distinct and 0 otherwise. We also denote this signature by $\mathrm{AD}_{r,\kappa}$. In addition to being symmetric, it is also invariant under any permutation of the κ domain elements. We call the second property *domain invariance*. The signature of an $\mathcal{F}$-gate in which all signatures in $\mathcal{F}$ are domain invariant is itself domain invariant.

Definition III.2 (Succinct signature). *Let $\tau = (P_1, P_2, \ldots, P_\ell)$ be a partition of $[\kappa]^r$ listed in some order. We say that f is a* succinct signature *of type τ if f is constant on each P_i. A set $\mathcal{F}$ of signatures is of type τ if every $f \in \mathcal{F}$ has type τ. We denote a succinct signature f of type τ by $\langle f(P_1), \ldots, f(P_\ell)\rangle$, where $f(P) = f(x)$ for any $x \in P$.*

Furthermore, we may omit 0 entries. If f is a succinct signature of type τ, we also say f is a succinct signature *of type τ' with length ℓ', where τ' lists ℓ' parts of the partition τ and we write f as $\langle f_1, f_2, \ldots, f_{\ell'}\rangle$, provided all nonzero values $f(P_i)$ are listed. When using this notation, we will make it clear which zero entries have been omitted.*

For example, a symmetric signature in the Boolean domain (i.e. $\kappa = 2$) has been denoted in previous work [10] by $[f_0, f_1, \ldots, f_r]$, where f_w is the output on inputs of Hamming weight w. This corresponds to the succinct signature type $(P_0, P_1, \ldots, P_r)$, where P_w is the set of inputs of Hamming weight w.

We prove a dichotomy theorem for Pl-$\operatorname{Holant}(f)$ when f is a succinct ternary signature of type τ_3 on domain size $\kappa \geq 3$. For $\kappa \geq 3$, the succinct signature of type $\tau_3 = (P_1, P_2, P_3)$ is a partition of $[\kappa]^3$ with $P_i = \{(x, y, z) \in [\kappa]^3 : |\{x, y, z\}| = i\}$ for $1 \leq i \leq 3$. The notation $\{x, y, z\}$ denotes a multiset and $|\{x, y, z\}|$ denotes the number of distinct elements in it. Succinct signatures of type τ_3 are exactly the symmetric and domain invariant ternary signatures. In particular, the succinct ternary signature for $\mathrm{AD}_{3,\kappa}$ is $\langle 0, 0, 1\rangle$.

IV. Complexity of Counting Edge Colorings

Here we prove that counting edge κ-colorings over planar r-regular graphs is #P-hard provided $\kappa = r \geq 3$. For the proof when $\kappa > r \geq 3$, see Theorem 4.20 in [11]. We reduce from evaluating the Tutte polynomial of a planar graph at the positive integer points on the diagonal $x = y$. For $x \geq 3$, evaluating the Tutte polynomial of a planar graph at (x, x) is #P-hard.

Theorem IV.1 (Theorem 5.1 in [43]). *For $x, y \in \mathbb{C}$, evaluating the Tutte polynomial at (x, y) is #P-hard over planar graphs unless $(x-1)(y-1) \in \{1, 2\}$ or $(x, y) \in \{(1, 1), (-1, -1), (\omega, \omega^2), (\omega^2, \omega)\}$, where $\omega = e^{2\pi i/3}$. In each exceptional case, the computation can be done in polynomial time.*

To state the connection with the diagonal of the Tutte polynomial, we need to consider Eulerian subgraphs in directed medial graphs. We say a graph is an Eulerian

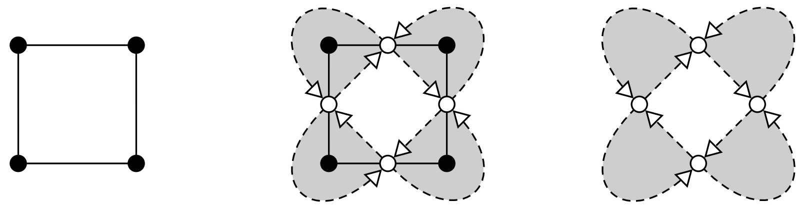

Figure 1: A plane graph ((a)), its directed medial graph ((c)), and both superimposed ((b)).

(di)graph if every vertex has even degree (resp. in-degree equal to out-degree), but connectedness is not required. Now recall the definition of a medial graph and its directed variant.

Definition IV.2 (cf. Section 4 in [25])**.** *For a connected plane graph G (i.e. a planar embedding of a connected planar graph), its* medial graph *G_m has a vertex on each edge of G and two vertices in G_m are joined by an edge for each face of G in which their corresponding edges occur consecutively.*

The directed medial graph *$\vec{G}_m$ of G colors the faces of G_m black or white depending on whether they contain or do not contain, respectively, a vertex of G. Then the edges of the medial graph are directed so that the black face is on the left.*

See Figure 1 for an example. Notice that the directed medial graph is always a planar 4-regular graph. Now we can give the connection with the diagonal of the Tutte polynomial. A monochromatic vertex is a vertex with all its incident edges having the same color.

Lemma IV.3 (Equation (17) in [25])**.** *Suppose G is a connected plane graph and $\vec{G}_m$ is its directed medial graph. For $\kappa \in \mathbb{N}$, let $\mathcal{C}(\vec{G}_m)$ be the set of all edge κ-labelings of $\vec{G}_m$ so that each (possibly empty) set of monochromatic edges forms an Eulerian digraph. Then*

$$\kappa\,\mathrm{T}(G; \kappa+1, \kappa+1) = \sum_{c \in \mathcal{C}(\vec{G}_m)} 2^{m(c)}, \qquad (1)$$

where $m(c)$ is the number of monochromatic vertices in the coloring c.

The Eulerian partitions in $\mathcal{C}(\vec{G}_m)$ have the property that the subgraphs induced by each partition do not intersect (or crossover) each other due to the orientation of the edges in the medial graph. We call the counting problem defined by the sum on the right-hand side of (1) as counting weighted Eulerian partitions over planar 4-regular graphs. This problem also has an expression as a Holant problem using a succinct signature. To define this succinct signature, it helps to know the following basic result about edge colorings.

When the number of available colors coincides with the regularity parameter of the graph, the cuts in any coloring satisfy a well-known parity condition. The parity condition we state here follows from a more general parity argument (see (1.2) and the Parity Argument on page 95 in [40]).

Lemma IV.4 (Parity Condition)**.** *Let G be a κ-regular graph and consider a cut C in G. For any edge κ-coloring of G, $c_1 \equiv c_2 \equiv \cdots \equiv c_\kappa \pmod 2$, where c_i is the number of edges in C colored i.*

Consider all quaternary $\{\mathrm{AD}_{\kappa,\kappa}\}$-gates on domain size $\kappa \geq 3$. These gadgets have a succinct signature of type $\tau_{\text{color}} = (P_{\substack{1\,1\\1\,1}}, P_{\substack{1\,2\\1\,2}}, P_{\substack{1\,2\\2\,1}}, P_{\substack{1\,1\\2\,2}}, P_{\substack{1\,4\\2\,3}}, P_0)$, where

$$\begin{aligned}
P_{\substack{1\,1\\1\,1}} &= \{(w,x,y,z) \in [\kappa]^4 \mid w = x = y = z\},\\
P_{\substack{1\,2\\1\,2}} &= \{(w,x,y,z) \in [\kappa]^4 \mid w = x \neq y = z\},\\
P_{\substack{1\,2\\2\,1}} &= \{(w,x,y,z) \in [\kappa]^4 \mid w = y \neq x = z\},\\
P_{\substack{1\,1\\2\,2}} &= \{(w,x,y,z) \in [\kappa]^4 \mid w = z \neq x = y\},\\
P_{\substack{1\,4\\2\,3}} &= \{(w,x,y,z) \in [\kappa]^4 \mid w,x,y,z \text{ are distinct}\},\\
P_0 &= [\kappa]^4 - P_{\substack{1\,1\\1\,1}} - P_{\substack{1\,2\\1\,2}} - P_{\substack{1\,2\\2\,1}} - P_{\substack{1\,1\\2\,2}} - P_{\substack{1\,4\\2\,3}}.
\end{aligned}$$

Any quaternary signature of an $\{\mathrm{AD}_{\kappa,\kappa}\}$-gate is constant on the first five entries of τ_{color} since $\mathrm{AD}_{\kappa,\kappa}$ is domain invariant. Using Lemma IV.4, we can show that the entry corresponding to P_0 is 0.

Lemma IV.5. *Suppose $\kappa \geq 3$ is the domain size and let F be a quaternary $\{\mathrm{AD}_{\kappa,\kappa}\}$-gate with succinct signature f of type τ_{color}. Then $f(P_0) = 0$.*

By Lemma IV.5, we denote a quaternary signature f of an $\{\mathrm{AD}_{\kappa,\kappa}\}$-gate by the succinct signature $\langle f(P_{\substack{1\,1\\1\,1}}), f(P_{\substack{1\,2\\1\,2}}), f(P_{\substack{1\,2\\2\,1}}), f(P_{\substack{1\,1\\2\,2}}), f(P_{\substack{1\,4\\2\,3}})\rangle$ of type τ_{color}, which has the entry for P_0 omitted. When $\kappa = 3$, $P_{\substack{1\,4\\2\,3}}$ is empty and we define its entry in the succinct signature to be 0.

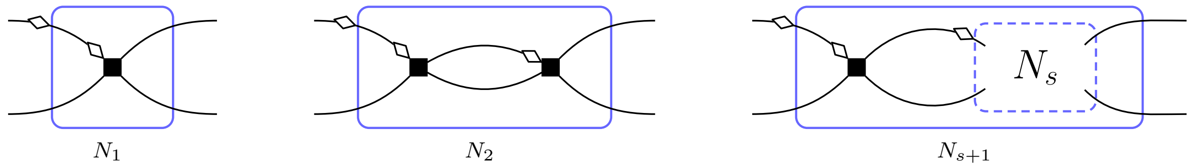

Figure 2: Recursive construction to interpolate $\langle 2, 1, 0, 1, 0\rangle$. Vertices assigned the signature of the gadget in Figure 3.

Lemma IV.6. *Let G be a connected plane graph and let G_m be the medial graph of G. Then*

$$\kappa\,\mathrm{T}(G; \kappa+1, \kappa+1) = \text{Pl-Holant}(G_m; \langle 2, 1, 0, 1, 0\rangle),$$

where the Holant problem has domain size κ and $\langle 2, 1, 0, 1, 0\rangle$ is a succinct signature of type τ_{color}.

Proof: Let $f = \langle 2, 1, 0, 1, 0\rangle$. By Lemma IV.3, we only need to prove that

$$\sum_{c \in \mathcal{C}(\vec{G}_m)} 2^{m(c)} = \text{Pl-Holant}(G_m; f), \qquad (2)$$

where the notation is from Lemma IV.3.

Each $c \in \mathcal{C}(\vec{G}_m)$ is also an edge κ-labeling of G_m. At each vertex $v \in V(\vec{G}_m)$, the four incident edges are assigned at most two distinct colors by c. If all four edges are assigned the same color, then the constraint f on v contributes a factor of 2 to the total weight. This is given by the value in the first entry of f. Otherwise, there are two different colors, say x and y. Because the orientation at v in $\vec{G}_m$ is cyclically "in, out, in, out", the coloring around v can only be of the form $xxyy$ or $xyyx$. These correspond to the second and fourth entries of f. Therefore, every term in the summation on the left-hand side of (2) appears (with the same nonzero weight) in the summation $\text{Pl-Holant}(G_m; f)$.

It is also easy to see that every nonzero term in $\text{Pl-Holant}(G_m; f)$ appears in the sum on the left-hand side of (2) with the same weight of 2 to the power of the number of monochromatic vertices. In particular, any coloring with a vertex that is cyclically colored $xyxy$ for different colors x and y does not contribute because $f(P_{\substack{1\,2\\2\,1}}) = 0$. ■

By Theorem IV.1 and Lemma IV.6, the problem $\text{Pl-Holant}(\langle 2, 1, 0, 1, 0\rangle)$ is #P-hard.

Corollary IV.7. *Suppose $\kappa \geq 3$ is the domain size. Let $\langle 2, 1, 0, 1, 0\rangle$ be a succinct quaternary signature of type τ_{color}. Then $\text{Pl-Holant}(\langle 2, 1, 0, 1, 0\rangle)$ is #P-hard.*

With this connection established, we can now show that counting edge colorings is #P-hard over planar regular graphs when the number of colors and the regularity parameter coincide.

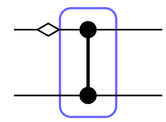

Figure 3: Quaternary gadget f used in the interpolation construction below. All vertices are assigned $\mathrm{AD}_{\kappa,\kappa}$. The bold edge represents $\kappa - 2$ parallel edges.

Theorem IV.8. *#κ-EdgeColoring is #P-hard over planar κ-regular graphs for all $\kappa \geq 3$.*

Proof: Let κ be the domain size of all Holant problems in this proof and let $\langle 2, 1, 0, 1, 0\rangle$ be a succinct quaternary signature of type τ_{color}. We reduce from $\text{Pl-Holant}(\langle 2, 1, 0, 1, 0\rangle)$ to $\text{Pl-Holant}(\mathrm{AD}_{\kappa,\kappa})$, which denotes the problem of counting edge κ-colorings in planar κ-regular graphs as a Holant problem. Then by Corollary IV.7, we conclude that $\text{Pl-Holant}(\mathrm{AD}_{\kappa,\kappa})$ is #P-hard.

Consider the gadget in Figure 3, where the bold edge represents $\kappa-2$ parallel edges. We assign $\mathrm{AD}_{\kappa,\kappa}$ to both vertices. Up to a nonzero factor of $(\kappa-2)!$, this gadget has the succinct quaternary signature $f = \langle 0, 1, 1, 0, 0\rangle$ of type τ_{color}. Consider the recursive construction in Figure 2. All vertices are assigned the signature f. Let f_s be the succinct quaternary signature of type τ_{color} for the sth gadget of the recursive construction. Then $f_1 = f$ and $f_s = M^s f_0$, where

$$M = \begin{bmatrix} 0 & \kappa-1 & 0 & 0 & 0 \\ 1 & \kappa-2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad f_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The signature f_0 is simply the succinct quaternary signature of type τ_{color} for two parallel edges. We can express M via the Jordan decomposition $M = P \Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & 1-\kappa & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\Lambda = \operatorname{diag}(\kappa-1, -1, 1, -1, 1)$. Then for $t = 2s$, we have

$$f_t = P \begin{bmatrix} \kappa-1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^t P^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= P \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} y+1 \\ y \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

where $x = (\kappa - 1)^t$ and $y = \frac{x-1}{\kappa}$.

Consider an instance Ω of $\text{Pl-Holant}(\langle 2, 1, 0, 1, 0\rangle)$ on domain size κ. Suppose $\langle 2, 1, 0, 1, 0\rangle$ appears n times in Ω. We construct from Ω a sequence of instances Ω_t of $\text{Pl-Holant}(f)$ indexed by t, where $t = 2s$ with $s \geq 0$. We obtain Ω_t from Ω by replacing each occurrence of $\langle 2, 1, 0, 1, 0\rangle$ with the gadget f_t.

As a polynomial in $x = (\kappa - 1)^t$, $\text{Pl-Holant}(\Omega_t; f)$ is independent of t and has degree at most n with integer coefficients. Using our oracle for $\text{Pl-Holant}(f)$, we can evaluate this polynomial at $n+1$ distinct points $x = (\kappa - 1)^{2s}$ for $0 \leq s \leq n$. Then via polynomial interpolation, we can recover the coefficients of this polynomial efficiently. Evaluating this polynomial at $x = \kappa + 1$ (so that $y = 1$) gives the value of $\text{Pl-Holant}(\Omega; \langle 2, 1, 0, 1, 0\rangle)$, as desired. ■

V. Dose of an Effective Siegel's Theorem

We jump into the middle of our proof for Theorem I.2. Consider the polynomial $p(x, y) \in \mathbb{Z}[x, y]$ defined by

$$p(x,y) = x^5 - (2y+1)x^3 - (y^2+2)x^2 + y(y-1)x + y^3.$$

We consider $y = \kappa + 1$ as an integer parameter $y \geq 4$, and treat $p(x, y)$ as an infinite family of quintic polynomials in x with integer coefficients. We want to show that the roots of all these quintic polynomials satisfy the lattice condition. (For $\kappa \in \{3, 4\}$, we need alternative proofs.)

Definition V.1. *Fix some $\ell \in \mathbb{N}$. We say that $\lambda_1, \lambda_2, \ldots, \lambda_\ell \in \mathbb{C} - \{0\}$ satisfy the* lattice condition *if for all $x \in \mathbb{Z}^\ell - \{\mathbf{0}\}$ with $\sum_{i=1}^{\ell} x_i = 0$, we have*

$$\prod_{i=1}^{\ell} \lambda_i^{x_i} \neq 1.$$

We suspect that for any integer $y \geq 4$, $p(x, y)$ is in fact irreducible over $\mathbb{Q}$ as a polynomial in x. We can show that this is a sufficient condition for the roots of $p(x, y)$ to satisfy the lattice condition for any integer $y \geq 4$. When considering y as an indeterminate, the bivariate polynomial $p(x, y)$ is irreducible over $\mathbb{Q}$ and the algebraic curve defined by it has genus 3, so by Theorem 1.2 in [37], $p(x, y)$ is reducible over $\mathbb{Q}$ for at most a finite number of $y \in \mathbb{Z}$.

We know of just five values of $y \in \mathbb{Z}$ for which $p(x, y)$ is reducible as a polynomial in x:

$$p(x,y) = \begin{cases} (x-1)(x^4+x^3+2x^2-x+1) & y=-1 \\ x^2(x^3-x-2) & y=0 \\ (x+1)(x^4-x^3-2x^2-x+1) & y=1 \\ (x-1)(x^2-x-4)(x^2+2x+2) & y=2 \\ (x-3)(x^4+3x^3+2x^2-5x-9) & y=3. \end{cases}$$

These five factorizations also give five integer solutions to $p(x, y) = 0$. It is a well-known theorem of Siegel [39] that an algebraic curve of genus at least 1 has only a finite number of integral points. For this curve of genus 3, Faltings' Theorem [26] says that there can be only a finite number of rational points. However these theorems are not *effective* in general. There are some effective versions of Siegel's Theorem that can be applied to our polynomial, but the best effective bound that we can find is over $10^{20,000}$ [45] and hence cannot be checked in practice.

However, it is shown in the next lemma that in fact these five are the only integer solutions. In particular, for any integer $y \geq 4$, $p(x, y)$ does not have a linear factor in $\mathbb{Z}[x]$. The following proof is based on a key auxiliary function $g_2(x, y) = \frac{y^2}{x} + y - x^2 + 1$ due to Aaron Levin [34]. We thank Aaron and also thank Bjorn Poonen [38] who suggested a similar proof.

Lemma V.2. *The only integer solutions to $p(x, y) = 0$ are $(1, -1)$, $(0, 0)$, $(-1, 1)$, $(1, 2)$, and $(3, 3)$.*

Proof sketch: Clearly these five points are solutions to $p(x, y) = 0$. Let $(a, b) \in \mathbb{Z}^2$ be a solution to $p(x, y) = 0$ with $a \neq 0$. We claim $a \mid b^2$. By definition of $p(x, y)$, clearly $a \mid b^3$. If p is a prime that divides a, then let $\text{ord}_p(a) = e$ and $\text{ord}_p(b) = f$ be the exact orders with which p divides a and b respectively. Then $f \geq 1$ since $3f \geq e$ and our claim is that $2f \geq e$. Suppose for a contradiction that $2f < e$. From $p(a, b) = 0$, we have $a^2(a^3 - 2ab - a - b^2 - 2) = -b^3 - ab(b - 1)$. The order with respect to p of the left-hand side is $\text{ord}_p\left(a^2(a^3 - 2ab - a - b^2 - 2)\right) \geq \text{ord}_p\left(a^2\right) = 2e$. Since p is relatively prime to $b - 1$, $\text{ord}_p\left(ab(b-1)\right) = e + f > 3f$, and therefore the order of the right-hand side with respect to p is $\text{ord}_p\left(-b^3 - ab(b-1)\right) = \text{ord}_p(b^3) = 3f$. However, $2e > 3f$, a contradiction. This proves the claim.

Now consider the functions $g_1(x, y) = y - x^2$ and $g_2(x, y) = \frac{y^2}{x} + y - x^2 + 1$. Whenever $(a, b) \in \mathbb{Z}^2$ is a solution to $p(x, y) = 0$ with $a \neq 0$, $g_1(a, b)$ and $g_2(a, b)$ are integers. We compute the Puiseux series expansions

$y_1(x)$ for $x \in \mathbb{R}$, $y_2(x)$ for $x > 0$, and $y_3(x)$ for $x > 0$, where

$$\begin{aligned}
y_1(x) &= x^2 + 2x^{-1} + 2x^{-2} - 6x^{-4} - 18x^{-5} + O(x^{-6}),\\
y_2(x) &= x^{3/2} - \frac{1}{2}x + \frac{1}{8}x^{1/2} - \frac{65}{128}x^{-1/2} - x^{-1}\\
&\quad - \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2}), \quad \text{and}\\
y_3(x) &= -x^{3/2} - \frac{1}{2}x - \frac{1}{8}x^{1/2} + \frac{65}{128}x^{-1/2} - x^{-1}\\
&\quad + \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2}),
\end{aligned}$$

to obtain asymptotic approximations to the roots of these polynomials with y expanded as a Puiseux series of x. These series converge to the actual roots of $p(x, y)$ for large x. The basic idea of the proof—called Runge's method—is that, for example, when we substitute $y_2(x)$ in $g_2(x, y)$, we get $g_2(x, y_2(x)) = O(x^{-1/2})$, where the multiplier in the O-notation is bounded both above and below by a nonzero constant in absolute value. Thus for large x, this cannot be an integer. However, for integer solutions (x, y) of $p(x, y)$, this must be an integer. We prove that $|x| > 16$ suffices to show this for each asymptotic approximation. For $|x| \leq 16$, one can directly check that there are no other integer solutions. ∎

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