The Price of Stability for Undirected Broadcast Network Design with Fair Cost Allocation is Constant

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***Abstract*—We consider broadcast network design games in undirected networks in which every player is a node wishing to receive communication from a distinguished source node s and the cost of each communication link is equally shared among the downstream receivers according to the Shapley value. We prove that the Price of Stability of such games is constant, thus closing a long-standing open problem raised in [2]. Our result is obtained by means of homogenization, a new technique that, in any intermediate state locally diverging from a given optimal solution T^*, is able to restore local similarity by exploiting cost differences between nearby players in T^*.**

***Keywords*-Network Design Games, Price of Stability, Nash equilibria**

I. Introduction

Congestion games [20] are a well established approach to model resource sharing among selfish players. In such games, a set of resources is available to a set of n players. Every player comes along with a set of strategies, each corresponding to the selection of a subset of resources. A state of the game is any combination of strategies for the players. The cost incurred by a player in a given state is defined as the sum of the costs associated to each selected resource, which depends on the number of players choosing it. The social cost of a state denotes its quality from a global perspective, which is typically defined as the sum of the players' costs or the maximum among the players' costs. By defining an elegant potential function, Rosenthal [20] has shown that the natural decentralized mechanism known as Nash dynamics, in which at each step some player performs an improving deviation by switching her strategy to a better alternative, is guaranteed to converge to a (pure) *Nash equilibrium* [19], i.e., a fixed point of such dynamic in which no player can improve her situation by unilaterally changing her selected strategy. A Nash equilibrium may not necessarily minimize the social cost. A widely used measure for quantifying the quality of equilibria, and thus the performance degradation due to the players' selfish behavior, is the *price of anarchy*, introduced by Koutsoupias and Papadimitriou [17], which is formally defined as the worst-case ratio of the social cost of a Nash equilibrium to the optimal social cost.

Network design games with fair cost allocations, introduced by Anshelevich *et al.* [2], are one of the most interesting subclasses of congestion games. In such games, we are given a graph with non-negative edge costs and, for each player, a source and a destination node. The goal of a player is to choose a path connecting her source and destination nodes. Thus, edges correspond to resources and paths connecting source and destination nodes to strategies (subsets of resources). The cost of each edge e is equally shared by all the players whose selected path contains e, i.e., according to the Shapley value [21]. A relevant and largely investigated special case of network design games occurs when all players share the same source node (*multicast games*). In this case, players are assumed as being associated to the endpoint node they wish to connect with the source. *Broadcast games* are multicast games in which there is a player associated to every node of the network.

In their seminal paper, Anshelevich *et al.* [2] raised the problem of the bad performance of Nash equilibria in network design games. The price of anarchy, in fact, is as large as the number of players even for broadcast games in undirected graph. Motivated by this issue, they started to explore the middle ground between centrally enforced solutions and completely unregulated anarchy by proposing the notion of *price of stability* (PoS), that is the ratio of the social cost of the cheapest Nash equilibrium and the social cost of an optimal solution. They argued that each local minimum of Rosenthal's potential function is a Nash equilibrium and, by comparing the social cost of the global minimum with that of an optimal solution, they obtained an upper bound of $H_n := \sum_{i=1}^{n} 1/i = O(\log n)$ on the PoS of network design games. They also provided an instance of broadcast games in directed graphs for which PoS $= H_n$, thus completely characterizing the PoS of network design games in the directed case. However, since then, the question of determining tight bounds for the case of undirected graphs has stood as a major

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open problem and after all these years is still far from being solved.

At the time of writing this paper, while no improvements on the $O(\log n)$ result by Anshelevich *et al.* [2] have been achieved for network design games, two upper bounds of $O(\log\log n)$ and $O(\log n/\log\log n)$ have been given by Fiat *et al.* [15] for broadcast games and by Li [18] for multicast games, respectively. However, the best-known lower bounds, determined by Bilò *et al.* [5], are 1.818 for broadcast games, 1.862 for multicast games and 2.245 for network design games, thus leaving a huge gap to be filled.

A recent result by Kawase and Makino [16] shows that, even in broadcast games, the social cost of the Nash equilibrium minimizing Rosenthal's potential function, which is at the basis of Anshelevich *et al.*'s approach, can be $\Omega(\sqrt{\log\log n})$ times the cost of the social optimum. Such a pathological behavior does not occur in other special cases of congestion games where this bounding technique yields tight or asymptotically tight upper bounds, see [6], [10], [11]. However, in network design games with fair cost allocation, this implies that, in order to get an $o(\sqrt{\log\log n})$ upper bound on the PoS, one has to resort on different arguments.

A. Our Contribution

In this work, we close the PoS question for broadcast games by proving the following result.

Theorem 1: The PoS of broadcast games in undirected graphs is $O(1)$.

Such a result is achieved by introducing and exploiting the new concept of *homogenization*. Roughly speaking, a state is homogeneous with respect to an optimal state T^* if the difference between the costs of any two players is upper bounded by a certain function of the set of edges connecting them in T^*. We call *homogenization process* a transformation that has the property of decreasing Rosenthal's potential starting from a given non-homogeneous state. The nice property possessed by homogeneous states is that, for each improving deviation by a player that causes the insertion of an edge e not belonging to T^*, there always exists either a subsequent improving deviation which immediately removes e from the state, or a sequence of improving deviations, that we call *absorbing process*, which is able to attract a consistent part of T^* in the current state. Thanks to the afore mentioned properties, it is possible to design an algorithm which, starting from T^*, suitably combines improving deviations, homogenization and absorbing processes so as to generate a sequence of states which ends up at a Nash equilibrium whose social cost compares nicely with that of T^*.

We stress here that the idea of constructing a Nash equilibrium of small social cost as an output of an algorithm that suitably schedules a sequence of improving deviations starting from an optimal state was already at the basis of Fiat *et al.*'s approach [15]. Our approach, however, is not a refinement of their technique, as it strongly relies on the new properties of homogeneous profiles (see the beginning of Section III for a more detailed comparison). Moreover, our homogenizing process does not consist of improving deviations, but it corresponds to a transformation globally decreasing the potential. Hence, it can be appreciated how crucial is the role of the novel concept of homogenization in the process of lowering the PoS from a super-constant to a constant factor.

B. Related Works

Christodoulou *et al.* [9] consider the case of $n = 2, 3$ and show that the PoS is $4/3$ for all the three variants of the game when $n = 2$ and that the PoS of network design games is between 1.541 and 1.65 when $n = 3$. Still for $n = 3$, Bilò and Bove [4] lower the upper bound of network design games to 1.634 and show that the PoS is 1.486 for broadcast games and it is between 1.524 and 1.532 for multicast games.

Concerning specific topologies, Fanelli at al. [13] prove that the PoS of network design games on undirected rings is $3/2$.

For the generalized setting with weighted players, Anshelevich *et al.* [2] show that pure Nash equilibria are always guaranteed to exist in the case of $n = 2$, but a negative result of Chen and Roughgarden [8] implies that this is no longer true, even in multicast games, when $n \geq 3$. As to the PoS, Chen and Roughgarden [8] give an upper bound of $O(\log W)$, while Albers [1] provides a lower bound of $\Omega(\log W/\log\log W)$, where W is the sum of the weights of all players in the game. To the best of our knowledge, no result in the directed case is known for multicast and broadcast games.

Charikar *et al.* [7] consider the quality of the states achieved after sequences of best-responses that start from the empty state, that is, the situation in which no player has performed any strategic choice yet. They show that, in multicast games, the social cost of the state obtained after a sequence of n best-responses, one for each player (the so called one-round walk), is at most $O(\log^2 n)$ times the one of an optimal state. Moreover, they prove that, if players continue performing improving deviations until a Nash equilibrium is reached, the social cost of such an equilibrium is at most $O(\log^3 n)$ times the one of an optimal state.

Strong Nash equilibria [3], i.e., Nash equilibria which are resistant to even joint deviations of coalitions of players, have been also investigated. In particular, Epstein *et al.* [12] show the existence of such equilibria under some topological properties of the underlying input graph, while Albers [1] prove an upper bound of $O(\log n)$ and a lower bound of $\Omega(\sqrt{\log n})$ on the price of anarchy of Strong Nash equilibria in undirected games.

The determination of upper bounds on the price of anarchy of strong equilibria and of equilibria reached from the empty state is fairly related to the PoS problem, as any such bound translates into a corresponding one for the PoS. Unfortunately, none of them asymptotically improves with respect to $O(\log n)$.

Recently, Feldman and Ron [14] started the study of network design games under the restriction in which edges have capacity constraints.

II. Preliminaries

Basic Game Definitions: An instance of the (undirected) *broadcast games* is defined as a tuple $\mathcal{G} = \left(G = (V,E), (w_e)_{e \in E}, s\right)$, where G is an undirected graph in which each edge $e \in E$ has a positive cost w_e, and $s \in V$ is a distinguished source node. Without loss of generality we assume that $w_e \geq 1$ for every $e \in E$. Each node $p \in V$ is associated to a player aiming at connecting s to p. In the following we will identify a player with its associated node.

Let $n = |V|$ and Σ_p denote the strategy set of player $p \in V$, that is the set of all the paths s_p connecting s to p. We often consider s_p as being the set of its contained edges. Let $S \in \times_{p \in V} \Sigma_p$ be the state in which player p chooses the strategy $s_p \in \Sigma_p$. We denote by $G(S) = (V, E(S))$ the subgraph of G containing all the edges $E(S)$ used by the players in state S, i.e., $E(S) = \bigcup_{p \in V} s_p$. Given a state S and an edge $e \in E$, let $n_e(S)$ be the number of different players using e in S, that is $n_e(S) = |\{p \in V : e \in s_p\}|$. We assume that all the players using an edge equally share its cost, i.e., for every edge e and player p using e in state S, the cost charged to p for e is $c_p^e(S) = \frac{w_e}{n_e(S)}$. The total cost incurred by player p in S is defined as the sum of the shared costs of all edges used by p, i.e., $c_p(S) = \sum_{e \in s_p} c_p^e(S) = \sum_{e \in s_p} \frac{w_e}{n_e(S)}$.

Given a state S and a strategy $t_p \in \Sigma_p$, we denote as $S_{-p} \diamond t_p = (s_1, \ldots, s_{p-1}, t_p, s_{p+1}, \ldots, s_n)$ the state obtained from S when player p unilaterally changes her strategy from s_p to t_p. A state $S_{-p} \diamond t_p$ such that $c_p(S_{-p} \diamond t_p) < c_p(S)$ is an *improving deviation* for player p in S. A state S is a (pure) *Nash equilibrium* if $c_p(S) \leq c_p(S_{-p} \diamond t_p)$ for every player $p \in V$ and strategy $t_p \in \Sigma_p$, i.e., no player possesses an improving deviation in S. We denote by $\mathcal{NE}(\mathcal{G})$ the set of all Nash equilibria of a broadcast game $\mathcal{G}$.

Potential Function: Let $\Phi : \times_{p \in V} \Sigma_p \to \mathbb{R}_+$ be the function such that

$$\Phi(S) = \sum_{e \in E} \sum_{i=1}^{n_e(S)} \frac{w_e}{i} = \sum_{e \in E} w_e H_{n_e(S)}.$$

Function Φ, originally defined by Rosenthal [20] for the general class of the congestion games, is an *exact potential function*, that is

$$\Phi(S_{-p} \diamond t_p) - \Phi(S) = c_p(S_{-p} \diamond t_p) - c_p(S) \qquad (1)$$

for each state S and strategy $t_p \in \Sigma_p$.

Social Cost: The social cost of a state S is defined as the sum of all the players' costs, i.e., $C(S) = \sum_{p \in V} c_i(S)$. Obviously $C(S) = \sum_{e \in E(S)} w_e$, that is the sum of the cost of all the edges used by some player in S. An optimal state for a broadcast game $\mathcal{G}$, denoted as $S^*(\mathcal{G})$, is a strategy profile having minimum social cost. Clearly, $G(S^*(\mathcal{G}))$ is a *minimum spanning tree* T^* of G. The *price of stability* of a broadcast game $\mathcal{G}$ is defined as $\mathrm{PoA}(\mathcal{G}) = \min_{S \in \mathcal{NE}(\mathcal{G})} \frac{C(S)}{C(S^*(\mathcal{G}))}$.

Technical Definitions and Notation:

Definition 1 (First edge): Given a player $p \in V$ using strategy s_p in state S, the *first edge* of p in S is defined as the edge in s_p incident to p.

Definition 2 (Function class): Given an edge $e \in E$, we say that e is of class $\alpha \geq 0$ ($class(e) = \alpha$) if $64^\alpha \leq w_e < 64^{\alpha+1}$. Let $E_\alpha \subseteq E$ be the set of the edges of class α.

In the following, T^* will be a fixed minimum spanning tree of G rooted at s, $S^*(\mathcal{G})$ (or simply S^*) the optimal state corresponding to T^*, and $\pi_{p,q}$ the path connecting p to q in T^*. For the sake of brevity, we often identify T^* (and its subtrees) with the corresponding set of edges.

For any subset $X \subseteq V$, let $T^*(X)$ be the subtree of T^* induced by X and $n_{\alpha,X}$ be the number of edges of class α in $T^*(X)$. A segment of T^* is any subset $X \subseteq V$ such that $T^*(X)$ is a path, i.e., all the nodes in X are consecutive in T^*. The two nodes p and q such that $T^*(X) = \pi_{p,q}$ are called the endpoints of X. We denote by $\mathcal{X}$ the family of all the *segments* in T^*.

Definition 3 (Main cycle and function τ): Consider an ordering $s = p_0, \ldots, p_{n-1}$ of the nodes according to a preorder traversal of T^* starting at s. We define the *main cycle* of T^*, denoted as $MC(T^*)$ or simply MC, the (non-simple) *directed cycle* obtained by concatenating together, for $i = 0, \ldots, n-1$, all the paths $\pi_{p_i, p_{i+1}}$ in T^* from the i-th to the $(i+1)$-th visited node, oriented from p_i to p_{i+1}, starting from s and finally coming back at s (summations on indexes i are considered modulo n). Since for each edge $\{p,q\}$ of T^* the two opposite arcs (p,q) and (q,p) occur in MC, both the number of nodes and arcs in MC is $2n-2$ and MC can be expressed as $MC = \langle \tau(0), \ldots, \tau(2n-3) \rangle$, that is as the ordered sequence of the nodes met along MC starting at s, with $\tau(0) \equiv s$ and $\tau(j)$ being the node reached after the j-th hop along MC starting at s.

Given an arc a in MC induced by an edge e of T^*, let $w_a = w_e$ and $class(a) = class(e)$ be the weight and the class of a, respectively. Then MC has total cost $2C(S^*)$.

Definition 4 (Function σ): For a given player/node p, let $\sigma(p)$ denote the first occurrence of p along MC, that is the minimum index j such that $\tau(j) \equiv p$. In the following, similarly as for segments of T^*, we provide some definitions and notation for the subpaths of MC.

An interval I of cardinality j in MC, $0 \leq j \leq 2n-2$, is any subsequence $\langle \tau(i), \ldots, \tau(i+j) \rangle$ of MC (again operations on indexes are considered modulo $2n-2$). Let $m_{\alpha,I}$ be the number of arcs of class α in the interval I, $N(I) \subseteq V$ (resp. $A(I)$) be the set of the nodes (resp. arcs) contained in I and finally $\mathcal{I}$ be the set of all the intervals of MC.

Definition 5 (Oriented intervals of MC): Given any i such that $0 \leq i < 2n-2$ and $y \geq 0$, the **right MC-interval with budget y at position i**, denoted as $I^{MC,+}(i,y)$, is the interval $I = \langle \tau(i), \ldots, \tau(i+j) \rangle \in \mathcal{I}$ of maximum cardinality (maximum j) such that $\sum_{\alpha \geq 0} 64^{\alpha+1} H^2_{m_{\alpha,I}} \leq y$. Similarly, the **left MC-interval with budget y at position i**, denoted as $I^{MC,-}(i,y)$, is the interval $I = \langle \tau(i-j), \ldots, \tau(i) \rangle \in \mathcal{I}$ of maximum cardinality such that $\sum_{\alpha \geq 0} 64^{\alpha+1} H^2_{m_{\alpha,I}} \leq y$. The full **$MC$-interval with budget y at position i** is the interval $I^{MC}(i,y) = I^{MC,-}(i,y) | I^{MC,+}(i,y)$ obtained by concatenating $I^{MC,-}(i,y)$ and $I^{MC,+}(i,y)$ if the sum of their cardinalities is less than $2n-2$, otherwise $I^{MC}(i,y) = MC =$

$\langle\tau(0),\ldots,\tau(2n-3)\rangle$.

Definition 6 (MC-boundary): Given an interval $I = \langle\tau(i),\ldots,\tau(i+j)\rangle \in \mathcal{I}$, the MC**-boundary** of I is defined as the arc $(\tau(i+j),\tau(i+j+1))$ of MC.

Definition 7 (Neighborhood): Given a player $p \in V$ and any $y \geq 0$, the **neighborhood of** p **with budget** y, denoted as $V(p,y)$, is the set of all the nodes contained in the full MC-interval with budget y at the first position in which p occurs in MC, that is $V(p,y) = N\left(I^{MC}(\sigma(p),y)\right)$.

Notice that $T^*(V(p,y))$ is a subtree of T^*.

Property 1: Since for any $I \in \mathcal{I}$ it holds that the set of edges of $T^*(N(I))$ coincides with the set $\{\{p,q\}|(p,q) \in A(I)\}$, it follows that, for any subset $X \subseteq V(p,y)$, $n_{\alpha,x} \leq m_{\alpha,I}$ for every $\alpha \geq 0$ and $\sum_{\alpha\geq 0} 64^{\alpha+1} H^2_{n_{\alpha,X}} \leq 2y$.

Given a state S, let $\Delta_{p,q}(S) = c_p(S) - c_q(S)$ be the difference between the costs of players p and q in S.

Definition 8 (Homogeneity): Given a segment $X \in \mathcal{X}$ with endpoints p and q, a state S is X**-homogeneous** if

$$\Delta_{p,q}(S) \leq 2\sum_{\alpha\geq 0} 64^{\alpha+1} H^2_{n_{\alpha,X}}.$$

S is **homogeneous** if it is X-homogeneous for every segment $X \in \mathcal{X}$.

Property 2: Given any player p and budget $y \geq 0$, if S is X-homogeneous for any $X \in \mathcal{X}$ such that $X \subseteq V(p,y)$, then $\Delta_{q,r}(S) \leq 4y$ for every $q,r \in V(p,y)$.

III. The Algorithm

Before presenting the details of our algorithm, let us first briefly discuss the main underlying idea.

One key ingredient used in [15] for reaching an equilibrium with cost $O(C(S^*)\log\log n)$ is that, when a player p performs an improving deviation introducing (as first edge) a new edge e of cost w_e not belonging to T^*, since the other players would pay at most $w_e/2$ for sharing such an edge, the state obtained when e "absorbs" all the players at distance at most $w_e/4$ from p in T^*, i.e., when such players select the concatenation of the subpath connecting them to p along T^* and the path going from p to s, has a lower potential. Then, if we partition the edges in classes of exponentially increasing values as in the previous section, since all the non-optimal edges of the same class contained in the final equilibrium will have mutual distances in T^* at least proportional to their values, the overall contribution of all the edges of the same class to the cost of the final state will be $O(C(S^*))$. This immediately gives a price of stability at most proportional to the number of non-empty classes, even if in [15] the authors could finally prove an $O(\log\log n)$ bound.

Our argument is that the insertion of a non-optimal edge in the current state is able to absorb a more consistent part of T^*. In fact, assume for the sake of simplicity that all the edges of T^* have unitary weights. Then, starting from T^*, since edges of strictly increasing congestion are traversed when going towards s in T^*, it holds $\Delta_{p,q}(S^*) \leq H_d$ for every player q at distance at most d from p in T^*. Therefore, if p introduces a non-optimal edge e, a state with a lower potential is obtained when all players within distance exponential in w_e, moving in order of distance from p (in T^*), select the strategy going through p along T^* as described above. In this way, the contribution of each class of edges in the final state would be $o(C(S^*))$, so that the overall contribution of all the classes remains constant.

The problem in the above process is that when more and more deviations are performed, the current state S tends to diverge from T^*, so that the property that two players p and q at distance d in T^* have cost difference at most H_d in general is not true any more. However, a similar argument still holds if the current state is *homogeneous*, that is if $\Delta_{p,q}(S) \leq 2H^2_d$. On the other hand, if S is not homogeneous, there is a way of *homogenizing* it, getting a new state S' with $\Phi(S') < \Phi(S)$, by adding only edges of T^* (see Lemma 1). If $c_p(S) < c_q(S) - 2H^2_d$, this is roughly obtained letting players from p to q in T^* follow p along T^*.

The remaining technicalities are set to deal with the various details and specific cases that might occur during the execution of the algorithm, like the occurrence of edges of different weights in T^*, the fact that during homogenization or absorption the costs of the involved players might decrease due to previously deviating ones, thus compromising the decrease of potential, and so forth.

We remark that, in the intermediate steps of our algorithm, states may be reached in which some players' strategies are non-simple paths. Even if they are not legal in the strict sense, they make the pseudo-code simpler and however they are removed before the end of the algorithm. Notice also that the algorithm does not necessarily give in output a homogeneous state, as it only aims at determining an equilibrium with low social cost. Anyway, a simple modification can easily enforce the final homogeneity.

Having in mind all the above arguments, we design Algorithm 1 that, starting from the optimal state S^*, constructs a sequence of states $\langle S^0 = S^*, S^1, \ldots, S^{\overline{k}}\rangle$ such that $S^{\overline{k}}$ is a Nash equilibrium of $\mathcal{G}$ with $C(S^{\overline{k}}) = O(C(S^*))$.

For any integer k, $0 \leq k < \overline{k}$, the transition from S^k to S^{k+1} occurs by means of:

- A **basic move**: an improving deviation $S^{k+1} = S^k_{-p} \diamond t_p$ such that either $t_p \subseteq E(S^k)$, or $t_p \setminus E(S^k) = \{e\}$ with $w_e \leq C(S^*)$, i.e., a deviation introducing in the current state at most a single new edge e as the first edge of the deviating player p and having cost at most equal to the social optimum. More precisely, we partition the basic moves into **critical** and **safe** ones: a basic move is critical if $t_p \setminus E(S^k) = \{e\}$ and $e \notin T^*$, otherwise it is safe.
  As we will prove in Lemma 4, among all the moves introducing multiple non-optimal edges, it is possible to restrict to this type of critical ones without affecting the global correctness. Namely, a state not admitting any basic move is guaranteed to be a Nash equilibrium.
- An **internal** e**-neutral move**: an improving deviation $S^{k+1} = S^k_{-p} \diamond t_p$ with $t_p \subseteq E(S^k)$ and $n_e(S^k) = n_e(S^{k+1})$, that is, no new edges are added to the current

Algorithm 1 Computes a cheap Nash equilibrium

1: $k \leftarrow 0$
2: $S_0 \leftarrow S^*$
3: **while** there exists a *basic move* $S^k_{-p} \diamond t_p$ for player p in S^k **do**
4: $\quad S^{k+1} \leftarrow S^k_{-p} \diamond t_p$
5: $\quad k \leftarrow k+1$
6: $\quad$ **if** $S^{k-1}_{-p} \diamond t_p$ is a *critical* basic move **then**
7: $\quad\quad$ let $e = E(S^k) \setminus E(S^{k-1})$
8: $\quad\quad$ **while** there exists a segment X not containing p such that S^k is not X-homogeneous
9: $\quad\quad\quad$ **or** there exists an *internal* e*-neutral* move **do**
10: $\quad\quad\quad$ **if** there exists an *internal* e*-neutral* move $S^k_{-p} \diamond t_p$ **then**
11: $\quad\quad\quad\quad S^{k+1} \leftarrow S^k_{-p} \diamond t_p$
12: $\quad\quad\quad$ **else**
13: $\quad\quad\quad\quad S^{k+1} \leftarrow$ Homogenize(X, S^k)
14: $\quad\quad\quad$ **end if**
15: $\quad\quad\quad k \leftarrow k+1$
16: $\quad\quad$ **end while**
17: $\quad\quad$ **if** there exists $q \in V(p, 64^{class(e)}/28)$ such that $c_q(S^k) < c_p(S^k) - \frac{2 \cdot 64^{class(e)}}{7}$ **then**
18: $\quad\quad\quad S^{k+1} \leftarrow$ Delete(p, q, e, S^k)
19: $\quad\quad$ **else**
20: $\quad\quad\quad S^{k+1} \leftarrow$ Absorbe(p, e, S^k)
21: $\quad\quad$ **end if**
22: $\quad\quad k \leftarrow k+1$
23: $\quad$ **end if**
24: **end while**
25: **return** S^k

state, and the number of players using edge e is not modified by the transition from S^k to S^{k+1}.
As we will see, in the algorithm, after a critical move inserting e, we homogenize along proper segments. Subsequently, a sequence of these moves allows to reach a profile S^k with a lower potential and such that $E(S^k) \setminus \{e\}$ is a tree.

- A **homogenization process**: given a non-homogeneous segment X of T^*, it leads to a state S^{k+1} with $\Phi(S^{k+1}) < \Phi(S^k)$, with S^{k+1} satisfying some suitable properties. See Subsection III-A for further details.
- A **deleting move**: it deals with the particular situation in which, after a critical move of player p adding edge e to the state and the subsequent homogenization and internal e-neutral moves, edge $e \notin T^*$ can be removed from the current state by only adding edges in $E(S^k) \cup T^*$. It is described in Subsection III-B.
- An **absorbing process**: when, after a critical move of player p adding edge e to the state and the subsequent homogenization and internal e-neutral moves, no deleting move exists, a particular sequence of improving deviations absorbing all the nodes in the neighborhood $V(p, 64^{class(e)}/28)$ (and also their descendants in S^k) can be performed. This process is described in Subsection III-C and corresponds to the above mentioned idea of absorbing a consistent part of T^* into the current state as a consequence of the introduction of a non-optimal edge.

A. Homogenization Process

In this subsection, we show the fundamental process being at the basis of our result: homogenization.

Notice that the states resulting from Algorithm 2 (and also the intermediate states of Algorithm 2) may contain strategies being non-simple paths. To this respect, if the strategy selected by player p crosses a same edge e more than once, we do not consider multisets of edges, but identify s_p with a set containing each edge of the non-simple path only once, and the contribution of player p to $n_e(S)$ is always 1. We would like to stress that, thanks to the internal e-neutral moves, at line 16 of Algorithm 1 all non-simple paths are removed from the state.

Lemma 1: Given a segment $X \in \mathcal{X}$ and a state S^k not X-homogeneous, the **homogenization process** of Algorithm 2 (with parameters X and S^k) returns a state S^{k+1} with $\Phi(S^{k+1}) < \Phi(S^k)$ such that (i) for every player $p \notin X$, $s_p^{k+1} = s_p^k$ and (ii) $\bigcup_{p \in X} s_p^{k+1} \subseteq \bigcup_{p \in X} s_p^k \cup T^*$.

Proof: Let q and r be the endpoints of X, with $\Delta_{q,r}(S) > 0$. Since S^k is not X-homogeneous, it follows that $\Delta_{q,r}(S^k) > 2\sum_{\alpha \geq 0} 64^{\alpha+1} H^2_{n_{\alpha,X}}$.

Consider the states $\bar{S}^i$ built by Algorithm 2 when called with parameters X and S^k. For any $i = 1, \dots, |X|-1$, the following conditions hold: (i) for every player $p \notin X$, $\bar{s}_p^i = s_p^k$; (ii) $\bigcup_{p \in X} \bar{s}_p^i \subseteq \bigcup_{p \in X} s_p^k \cup T^*$.

Assume that there exists no integer $i \in \{1, \dots, |X|-1\}$ such that $\Phi(\bar{S}^i) < \Phi(S^k)$. We arrive to the contradiction that

Algorithm 2 Homogenize(segment X, state S)

Let q and r be the endpoints of X, with $\Delta_{q,r}(S) > 0$
Let $q = p_1, \ldots, p_X = r$ be the players in the order they appear in $T^*(X)$
for $i = 1 \to |X| - 1$ **do**
    $\bar{S}^{i,0} \leftarrow S$
    for $j = 1 \to i$ **do**
        $\bar{S}^{i,j} \leftarrow \bar{S}^{i,j-1}_{-p_j} \diamond (\pi_{j,i+1} \cup s_{p_{i+1}})$
    end for
    $\bar{S}^i \leftarrow \bar{S}^{i,i}$
    if $\Phi(\bar{S}^i) < \Phi(S)$ **then**
        return $\bar{S}^i$
    end if
end for

X must be homogeneous.

For every $i = 1, \ldots, |X| - 1$, let $e_i = \{p_i, p_{i+1}\}$, w_i be the weight of e_i, and $\delta_i = \Delta_{p_i,p_{i+1}}(S^k)$. Morever, consider the sequence of the $i + 1$ states *sequence*$(i) = \langle S^k = \bar{S}^{i,0}, \bar{S}^{i,1}, \ldots, \bar{S}^{i,i} = \bar{S}^i \rangle$ determined by Algorithm 2 (with parameters X and S^k). By Equation 1, it holds $\Phi(\bar{S}^i) = \Phi(S^k) + \sum_{j=1}^{i} \big(c_{p_j}(\bar{S}^{i,j}) - c_{p_j}(\bar{S}^{i,j-1})\big)$.

Therefore, by the initial (contradicting) assumption,

$$\sum_{j=1}^{i} \big(c_{p_j}(\bar{S}^{i,j}) - c_{p_j}(\bar{S}^{i,j-1})\big) \geq 0. \tag{2}$$

In the following, Claim 1 will provide a lower bound to $\sum_{j=1}^{i} c_{p_j}(\bar{S}^{i,j-1})$ and Claim 2 an upper bound to $\sum_{j=1}^{i} c_{p_j}(\bar{S}^{i,j})$. To this aim, let us introduce some additional definitions.

For any l such that $1 \leq j \leq l \leq i+1 \leq |X|$, let $dec^l_{i,j} = \max\{0; c_{p_l}(\bar{S}^{i,0}) - c_{p_l}(\bar{S}^{i,j-1})\}$ be an upper bound on the decrease caused to $c_{p_l}(S^k)$ by the moves of players $p_1, \ldots, p_{j-1}$ in *sequence*(i). Notice that $dec^l_{i,j}$ allows to upper bound the cost of player p_l just after the deviation of p_{j-1} as $c_{p_l}(\bar{S}^{i,j-1}) \geq c_{p_l}(S^k) - dec^l_{i,j}$. Moreover, for any $e \in T^*$ and any $1 \leq j \leq i \leq |X| - 1$, let

$$\overline{dec}^e_{i,j} = \sum_{p \in V \setminus \{j\}} \max\left\{0; c^e_p(\bar{S}^{i,j-1}) - c^e_p(\bar{S}^{i,j})\right\}$$

be an upper bound to the total decrease induced by player p_j in $sequence(i)$ on the cost payed on edge e by the other players.

Claim 1: For any $i = 1, \ldots, |X| - 1$, it holds

$$\begin{aligned}\sum_{j=1}^{i} c_{p_j}(\bar{S}^{i,j-1}) \quad \geq \quad & i \cdot c_{p_{i+1}}(S^k) + \sum_{j=1}^{i}\sum_{h=j}^{i} \delta_h \\ & - \sum_{j=1}^{i}\left(\frac{1}{j}\sum_{h=j}^{i} w_h\right) - \sum_{j=1}^{i} dec^{i+1}_{i,j}.\end{aligned}$$

Proof: Given any $i = 1, \ldots, |X| - 1$, for any $j = 1, \ldots, i$, since $c_{p_j}(\bar{S}^{i,0}) = c_{p_j}(S^k) = c_{p_{i+1}}(S^k) + \sum_{h=j}^{i} \delta_h$, we obtain

$$c_{p_j}(\bar{S}^{i,j-1}) \geq c_{p_{i+1}}(S^k) + \sum_{h=j}^{i} \delta_h - dec^j_{i,j}. \tag{3}$$

Notice that $dec^j_{i,j} > 0$ only if new players use edges in $s^k_{p_j}$. Since all the players deviating in $sequence(i)$ use edges in $\pi_{p_1,p_{i+1}}$ till they reach player p_{i+1}, and then they use edges in $s^k_{p_{i+1}}$, it follows

$$\sum_{j=1}^{i} dec^j_{i,j} \leq \sum_{j=1}^{i}\sum_{h=j}^{i} \overline{dec}^{e_h}_{i,j} + \sum_{j=1}^{i} dec^{i+1}_{i,j}. \tag{4}$$

Furthermore, for any integer i and j and any $h \in \{1, \ldots, |X| - 1\}$, we can bound $\overline{dec}^{e_h}_{i,j}$ by the cost paid by player p_j for e_h in $\bar{S}^{i,j}$, i.e., $\overline{dec}^{e_h}_{i,j} \leq c^{e_h}_{p_j}(\bar{S}^{i,j})$. Thus, since $c^{e_h}_{p_j}(\bar{S}^{i,j}) \leq \frac{w_h}{j}$,

$$\sum_{h=j}^{i} \overline{dec}^{e_h}_{i,j} \leq \frac{1}{j}\sum_{h=j}^{i} w_h. \tag{5}$$

By summing up inequality (3) over all $j = 1, \ldots, i$, we obtain

$$\begin{aligned}\sum_{j=1}^{i} c_{p_j}(\bar{S}^{i,j-1}) \quad & \geq \quad i \cdot c_{p_{i+1}}(S^k) + \sum_{j=1}^{i}\sum_{h=j}^{i} \delta_h - \sum_{j=1}^{i} dec^j_{i,j} \\ & \geq \quad i \cdot c_{p_{i+1}}(S^k) + \sum_{j=1}^{i}\sum_{h=j}^{i} \delta_h \qquad (6) \\ & \qquad - \sum_{j=1}^{i}\sum_{h=j}^{i} \overline{dec}^{e_h}_{i,j} - \sum_{j=1}^{i} dec^{i+1}_{i,j} \\ & \geq \quad i \cdot c_{p_{i+1}}(S^k) + \sum_{j=1}^{i}\sum_{h=j}^{i} \delta_h \qquad (7) \\ & \qquad - \sum_{j=1}^{i}\left(\frac{1}{j}\sum_{h=j}^{i} w_h\right) - \sum_{j=1}^{i} dec^{i+1}_{i,j},\end{aligned}$$

where (6) holds by inequality (4), and (7) by inequality (5). ■

Claim 2: For any $i = 1, \ldots, |X|-1$, it holds

$$\sum_{j=1}^{i} c_{p_j}(\bar{S}^{i,j}) \leq \sum_{j=1}^{i} \left(\frac{1}{j} \sum_{h=j}^{i} w_h \right) + i \cdot c_{p_{i+1}}(S^k) - \sum_{j=1}^{i} dec_{i,j}^{i+1}.$$

Proof: Given any $i = 1, \ldots, |X|-1$, for any $j = 1, \ldots, i$, since players sequentially select the edges of T^* connecting them to p_{i+1}, we obtain

$$c_{p_j}(\bar{S}^{i,j}) \leq \frac{1}{j} \sum_{h=j}^{i} w_h + c_{p_{i+1}}(S^k) - dec_{i,j}^{i+1}. \tag{8}$$

By summing up inequality (8) over all $j = 1, \ldots, i$, we obtain the claim. ∎

We now prove the final claim, according to which, under the assumption that there exists no integer $i \in \{1, \ldots, |X|-1\}$ such that $\Phi(\bar{S}^i) < \Phi(S^k)$, S^k must be X-homogeneous: a contradiction.

Claim 3: $\Delta_{q,r}(S^k) \leq 2 \sum_{\alpha \geq 0} 64^{\alpha+1} H^2_{n_{\alpha,X}}$.

Proof: By combining inequality (2) with Claims 1 and 2, we obtain

$$\begin{aligned} 0 &\leq \sum_{j=1}^{i} \left(c_{p_j}(\bar{S}^{i,j}) - c_{p_j}(\bar{S}^{i,j-1}) \right) \\ &\leq \sum_{j=1}^{i} \left(\frac{1}{j} \sum_{h=j}^{i} w_h \right) + i \cdot c_{p_{i+1}}(S^k) - \sum_{j=1}^{i} dec_{i,j}^{i+1} \quad (9) \\ &\quad - i \cdot c_{p_{i+1}}(S^k) - \sum_{j=1}^{i} \sum_{h=j}^{i} \delta_h \\ &\quad + \sum_{j=1}^{i} \left(\frac{1}{j} \sum_{h=j}^{i} w_h \right) + \sum_{j=1}^{i} dec_{i,j}^{i+1} \\ &= 2 \sum_{j=1}^{i} \left(\frac{1}{j} \sum_{h=j}^{i} w_h \right) - \sum_{j=1}^{i} \sum_{h=j}^{i} \delta_h. \end{aligned}$$

By easy counting arguments, the inequality $2 \sum_{j=1}^{i} \left(\frac{1}{j} \sum_{h=j}^{i} w_h \right) - \sum_{j=1}^{i} \sum_{h=j}^{i} \delta_h \geq 0$ can be rewritten as

$$\sum_{j=1}^{i} j \cdot \delta_j \leq 2 \sum_{j=1}^{i} H_j \cdot w_j, \tag{10}$$

for any $i = 1, \ldots, |X|-1$.

Let $x_i = \frac{1}{i(i+1)}$ for any $i = 1, \ldots, |X|-2$ and $x_{|X|-1} = \frac{1}{|X|-1}$. It is easy to check that, for any $i = 1, \ldots, |X|-1$, it holds $\sum_{j=i}^{|X|-1} x_j = \frac{1}{i}$.

Consider the following inequalities, obtained by multiplying inequalities (10) by the coefficients x_i's.

$$x_i \sum_{j=1}^{i} j \cdot \delta_j \leq 2 x_i \sum_{j=1}^{i} H_j \cdot w_j, \quad \forall i = 1, \ldots, |X|-1.$$

By summing up over all $i = 1, \ldots, |X|-1$, we obtain

$$\sum_{i=1}^{|X|-1} \left(x_i \sum_{j=1}^{i} j \cdot \delta_j \right) \leq 2 \sum_{i=1}^{|X|-1} \left(x_i \sum_{j=1}^{i} H_j \cdot w_j \right).$$

Since, by rearranging summations, it holds

$$\begin{aligned} \sum_{i=1}^{|X|-1} \left(x_i \sum_{j=1}^{i} j \cdot \delta_j \right) &= \sum_{j=1}^{|X|-1} \left(j \cdot \delta_j \sum_{i=j}^{|X|-1} x_i \right) \\ &= \sum_{j=1}^{|X|-1} \delta_j = \Delta_{q,r}(S^k), \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{|X|-1} \left(x_i \sum_{j=1}^{i} H_j \cdot w_j \right) &= \sum_{j=1}^{|X|-1} \left(H_j \cdot w_j \sum_{i=j}^{|X|-1} x_i \right) \\ &= \sum_{j=1}^{|X|-1} \frac{H_j}{j} w_j, \end{aligned}$$

we finally obtain

$$\begin{aligned} \Delta_{q,r}(S^k) &\leq 2 \sum_{j=1}^{|X|-1} \frac{H_j}{j} w_j \\ &\leq 2 \sum_{j=1}^{|X|-1} \frac{H_j}{j} 64^{class(e_j)+1} \\ &= 2 \sum_{\alpha \geq 0} \left(64^{\alpha+1} \sum_{j=1}^{n_{\alpha,X}} \frac{H_j}{j} \right) \\ &\leq 2 \sum_{\alpha \geq 0} 64^{\alpha+1} H^2_{n_{\alpha,X}}. \end{aligned}$$

∎

Therefore, the proof of Lemma 1 is completed. ∎

B. Deleting Move

When, after a critical move of player p adding edge e to the state and the subsequent homogenization and internal e-neutral moves, a player q in a suitable neighborhood of p is paying a cost significantly smaller than the one paid by p, there exists an improving deviation $S^{k+1} = S^k_{-p} \diamond t_p$ with $t_p \subseteq E(S^k) \cup T^* \setminus \{e\}$ and $e \in s_p(S^k) \setminus T^*$. We call such a deviation a deleting move, as it removes edge $e \notin T^*$ from the current state.

Algorithm 3 Delete(players p and q, edge e, state S)

1: Let r be the closest node to p in $\pi_{p,q}$
2: **return** $S_{-p} \diamond (\{p, r\} \cup s_r)$

Lemma 2: Consider a state S^k such that p is the only player using e (as her first edge) in S^k and S^k is X-homogeneous for any segment in $\mathcal{X}$ not containing p. Moreover, assume that there exists a node q belonging to the neighborhood $V(p, 64^{class(e)}/28)$ such that $c_q(S^k) < c_p(S^k) - \frac{2 \cdot 64^{class(e)}}{7}$. Starting from state S^k, the deleting move performed by Algorithm 3 (with parameters p, q, e and S^k) returns a state S^{k+1} with $\Phi(S^{k+1}) < \Phi(S^k)$ and such that $E(S^{k+1}) \subseteq E(S^k) \cup T^* \setminus \{e\}$.

Proof: For the sake of brevity let us denote $V(p, 64^{class(e)}/28)$ simply as V_p.

First of all, notice that all edges in $T^*(V_p)$ are of class at most $class(e) - 2$, because otherwise $\sum_{\alpha \geq 0} 64^{\alpha+1} H^2_{n_{\alpha,V_p}} \geq 64^{class(e)}$: by Property 1 a contradiction to the definition of neighborhood.

Let X be the segment with endpoints q and r.

By hypothesis $c_p(S^k) > c_q(S^k) + \frac{2 \cdot 64^{class(e)}}{7}$ and S^k is X-homogeneous. Therefore, as $X \subseteq V_p$, by Property 2

$$c_r(S^k) \leq c_q(S^k) + \frac{64^{class(e)}}{7}.$$

If p selects the strategy composed by the edge $\{p, r\}$ and then by s^k_r, she pays at most

$$64^{class(e)-1} + c_r(S^k) \leq 64^{class(e)-1} + c_q(S^k) + \frac{64^{class(e)}}{7}.$$

Since $64^{class(e)-1} + \frac{64^{class(e)}}{7} < \frac{2 \cdot 64^{class(e)}}{7}$, it results $\Phi(S^{k+1}) < \Phi(S^k)$. ∎

C. Absorbing Process

Starting from a state S_k such that $E(S^k) \setminus \{e\}$ is a tree and p is the only player using e as first edge, this process is a sequence of improving deviations absorbing all the nodes in the neighborhood $V(p, 64^{class(e)}/28)$ (and also their descendants in S^k) into the current state, adding only edges of T^* and finally reaching a state S^{k+1} such that $\Phi(S^{k+1}) < \Phi(S^k)$.

Lemma 3: Given a player p, an edge e and a state S^k satisfying the following conditions: (i) $E(S^k) \setminus \{e\}$ is a tree, (ii) p is the only player using e (as her first edge) in S^k and (iii) any node q belonging to the neighborhood $V(p, 64^{class(e)}/28)$ is such that $c_q(S^k) \geq c_p(S^k) - \frac{2 \cdot 64^{class(e)}}{7}$, the **absorbing process** of Algorithm 4 (with parameters p, e and S^k) returns a state S^{k+1} with $\Phi(S^{k+1}) < \Phi(S^k)$ such that $E(S^{k+1}) \subseteq (E(S^k) \setminus E^f) \cup T^*$, where E^f is the set of the first edges of players in $V(p, 64^{class(e)}/28) \setminus \{p\}$.

Proof: Again for the sake of brevity let us denote $V(p, 64^{class(e)}/28)$ simply as V_p. First of all notice that no node $q \in V_p$ is an endpoint of some edge in s^k_p, because otherwise $c_q(S^k) \leq c_p(S^k) - 64^{class(e)} < c_p(S^k) - \frac{2 \cdot 64^{class(e)}}{7}$, thus contradicting the hypothesis. Therefore, in state $\bar{S}^{|V_p|}$ no edge in $E^f \setminus T^*$ is used by some player.

We now show that all the moves of Algorithm 4 are improving deviations, and therefore $\Phi(\bar{S}^i) < \Phi(\bar{S}^{i-1})$ for any $i = 1, \ldots, |V_p|$.

Let $dec_i = \max\{0; c_{p_i}(\bar{S}^0) - c_{p_i}(\bar{S}^{i-1})\}$ be an upper bound on the decrease caused to $c_{p_i}(S^k)$ by the moves of players $p_1, \ldots, p_{i-1}$ and of their moving descendants in $E(S^k) \setminus \{e\}$. Moreover, let

$$\overline{dec}_i = \sum_{e' \in s^k_p \setminus \{e\}} \max\left\{0; c^{e'}_p(\bar{S}^0) - c^{e'}_p(\bar{S}^{i-1})\right\}$$

be an upper bound to the decrease induced by the moves of players $p_1, \ldots, p_{i-1}$ and of their moving descendants in $E(S^k) \setminus \{e\}$ to the cost p incurs in all edges she uses in s^k_p except for e.

On the one hand, for any $i = 1, \ldots, |V_p|$, it holds

$$\begin{aligned} c_{p_i}(\bar{S}^{i-1}) = c_{p_i}(\bar{S}^0) - (c_{p_i}(\bar{S}^0) - c_{p_i}(\bar{S}^{i-1})) \quad \geq \\ \geq c_p(S^k) - \frac{2 \cdot 64^{class(e)}}{7} - dec_i. \end{aligned} \quad (11)$$

Notice that $dec_i > 0$ only if new players use edges in $s^k_{p_i}$. Since $e \notin s^k_{p_i}$, such edges can only belong either to $\left(s^k_{p_i} \setminus s^k_p\right) \cap T^*(V_p)$ or to $s^k_p \setminus \{e\}$. We can bound the contribution to dec_i due to the edges in $\left(s^k_{p_i} \setminus s^k_p\right) \cap T^*(V_p)$ as the total cost player p_i pays in S^k on such edges. More precisely, since we are considering broadcast games and $E(S^k) \setminus \{e\}$ is a tree, it can be upper bounded by $\sum_{\alpha \geq 0} 64^{\alpha+1} H_{n_{\alpha,V_p}} \leq 64^{class(e)}/14$ (see Property 1). Clearly, the contribution to dec_i due to edges in $s^k_p \setminus \{e\}$ is at most $\overline{dec}_i$. We therefore obtain

$$dec_i \leq \overline{dec}_i + \frac{64^{class(e)}}{14}. \quad (12)$$

From inequalities (11) and (12) we obtain

$$c_{p_i}(\bar{S}^{i-1}) \geq c_p(S^k) - \frac{2 \cdot 64^{class(e)}}{7} - \overline{dec}_i - \frac{64^{class(e)}}{14}. \quad (13)$$

On the other hand, for any $i = 1, \ldots, |V_p|$, since players are sorted with respect to a breadth-first order and they sequentially select the edges of T^* connecting them to p, and also considering that edge e is paid in S^k only by player p, we obtain

$$\begin{aligned} c_{p_i}(\bar{S}^i) &\leq c_p(S^k) - \overline{dec}_i - \frac{64^{class(e)}}{2} + \sum_{\alpha \geq 0} 64^{\alpha+1} H_{n_{\alpha,V_p}} \\ &\leq c_p(S^k) - \frac{64^{class(e)}}{2} - \overline{dec}_i + \frac{64^{class(e)}}{14}. \end{aligned} \quad (14)$$

By combining inequalities (13) and (14) we obtain $c_{p_i}(\bar{S}^{i-1}) \geq c_{p_i}(\bar{S}^i)$, i.e., the move of player p_i, as well as the ones of her moving descendants in $E(S^k) \setminus \{e\}$, decreases the potential function Φ. ∎

D. Proof of Theorem 1

We are now able to prove that Algorithm 1 returns a Nash equilibrium of cost proportional to the social optimum, thus implying Theorem 1. Lemma 4 (whose proof is omitted due to space constraints) focuses on the correctness of Algorithm 1, while Lemma 5 on its performance.

Lemma 4: Algorithm 1 always returns a Nash equilibrium for game $\mathcal{G}$.

Lemma 5: Let $S^{\bar{k}}$ be the state returned by Algorithm 1. Then $C(S^{\bar{k}}) = O(C(S^*))$.

Proof: The cost of the edges of $E(S^{\bar{k}})$ belonging to T^* is at most $C(S^*)$, thus in order to prove the claim it is sufficient to bound the contribution to $C(S^{\bar{k}})$ due to the edges in $E(S^{\bar{k}} \setminus T^*)$. To this aim, we observe that any such an edge $e \in E(S^{\bar{k}} \setminus T^*)$ is added by a critical move (line 7 of Algorithm 1) and remains into the final profile only if it is not removed by a deleting move (line 18), consequently generating an absorbing process (line 20).

Algorithm 4 Absorbe(player p, edge e, state S)

$V_p \leftarrow V(p, 64^{class(e)}/28)$
Let $p = p_1, \ldots, p_{|V_p|}$ be the nodes of $T^*(V_p)$ sorted with respect to a breadth-first traversal of $T^*(V_p)$ rooted at p.
$\bar{S}^0 \leftarrow S$
for $i = 1 \rightarrow |V_p|$ **do**
    $\bar{S}^i \leftarrow \bar{S}^{i-1}_{-p_i} \diamond (\pi_{p_i,p} \cup s_p)$
    for all players q belonging to the subtree of $E(S^k) \setminus \{e\}$ rooted at p_i and not belonging to its subtrees rooted at $\{p_1, \ldots, p_{|V_p|}\}$ **do**
        Let F be the set of edges connecting q to p_i in S
        $\bar{S}^i \leftarrow \bar{S}^i_{-q} \diamond (F \cup \pi_{p_i,p} \cup s_p)$
    end for
end for
return $\bar{S}^{|V_p|}$

Let $\alpha = class(e)$, I^+ (resp. I) denote the right MC-interval $I^{MC,+}(\sigma(p), 64^\alpha/28)$ (resp. the full MC-interval $I^{MC}(\sigma(p), 64^\alpha/28)$).

Let us first consider the case $I^+ \neq MC$. First of all, notice that all the arcs in I^+ are of class at most $\alpha - 2$, because otherwise $\sum_{\gamma \geq 0} 64^{\gamma+1} H^2_{m_{\gamma,I^+}} \geq 64^\alpha$: a contradiction to the definition of neighborhood. Let arc a be the MC-boundary of I^+ and $\mu = class(a)$. We charge w_e either to a or to some arcs belonging to I^+, according to the following cases:

- $\mu \geq \alpha - 1$. We charge w_e to a. Notice that if another edge e' of class α has been already accounted to a in the same way due a critical move involving another player q, then a is the MC-boundary also of the right MC-interval $I^{MC,+}(\sigma(q), 64^\alpha/28)$. Therefore, since I^+ and $I^{MC,+}(\sigma(q), 64^\alpha/28)$ intersect, e' is the first edge of a player occurring in I. Thus, by Lemma 2, it is removed from the solution after the absorbing process generated by p.
  As a consequence, the cost of at most one edge belonging to a class not being greater than $\beta + 1$ can be charged in this manner to every arc a of class β belonging to MC, that is at most $\sum_{\gamma=0}^{\beta+1} 64^{\gamma+1} < 64^{\beta+2} \leq 64^2 w_a$. By summing over all edges in MC, since MC has cost $2C(S^*)$, the total contribution to the cost of the final profile due to the edges subject to this boundary accounting is at most $2 \cdot 64^2 C(S^*) = O(C(S^*))$.
- $\mu \leq \alpha - 2$. We charge w_e to some arcs in I^+ as follows. By the maximality of the neighborhood,

$$\sum_{\gamma=0}^{\alpha-2} 64^{\gamma+1} H^2_{m_{\gamma,I^+}} + (64^{\mu+1} H^2_{m_{\mu,I^+}+1} - 64^{\mu+1} H^2_{m_{\mu,I^+}})$$
$$= \sum_{\gamma=0,\gamma\neq\mu}^{\alpha-2} 64^{\gamma+1} H^2_{m_{\gamma,I^+}} + 64^{\mu+1} H^2_{m_{\mu,I^+}+1} \geq \frac{64^\alpha}{28},$$

and since $\mu \leq \alpha - 2$ and $H^2_{i+1} - H^2_i \leq \frac{5}{4}$ for any $i \geq 0$, it follows that

$$\begin{aligned}
&\sum_{\gamma=0}^{\alpha-2} 64^{\gamma+1} H^2_{m_{\gamma,I^+}} \\
\geq\ & \frac{64^\alpha}{28} - 64^{\mu+1}(H^2_{m_{\mu,I^+}+1} - H^2_{m_{\mu,I^+}}) \\
>\ & \frac{64^\alpha}{28} - \frac{5}{4} 64^{\alpha-1} > 64^{\alpha-1}.
\end{aligned}$$

For every $\gamma = 1, \ldots, \alpha - 2$, let $r_\gamma = \frac{64^{\gamma+1} H^2_{m_{\gamma,I^+}}}{64^{\alpha-1}}$. We call a class γ *heavy* if $r_\gamma \geq \frac{1}{64^{\frac{\alpha-\gamma}{2}}}$ and *light* otherwise. We now show that there exists a heavy class β. Assume by contradiction that for every $\gamma = 0, \ldots, \alpha - 2$, $r_\gamma < \frac{1}{64^{\frac{\alpha-\gamma}{2}}}$. Then, by summing over all classes, we obtain

$$\begin{aligned}
\sum_{\gamma=0}^{\alpha-2} 64^{\gamma+1} H^2_{m_{\gamma,I^+}} &= \sum_{\gamma=0}^{\alpha-2} r_\gamma 64^{\alpha-1} \\
&\leq 64^{\alpha-1} \sum_{\gamma=1}^{\alpha-2} \frac{1}{64^{\frac{\alpha-\gamma}{2}}} < 64^{\alpha-1} :
\end{aligned}$$

a contradiction.
Now, consider a heavy class β. Since $H_i \leq 1 + \ln i$ for $i \geq 1$, it follows that

$$m_{\beta,I^+} \geq e^{\sqrt{64^{\frac{\alpha-\beta-4}{2}}}-1}.$$

We equally share the cost $w_e < 64^{\alpha+1}$ among all the m_{β,I^+} arcs of class β in the considered right MC-interval, by charging to each of them at most

$$64^{\alpha+1} \cdot \frac{1}{e^{\sqrt{64^{\frac{\alpha-\beta-4}{2}}}-1}} = 64^\beta \cdot \frac{64^{\alpha-\beta+1}}{e^{\sqrt{64^{\frac{\alpha-\beta-4}{2}}}-1}}.$$

Let a be an arc of class β to which we have charged the above quantity.
Notice that if another edge e' of class α has been already accounted to a in the same way (that is, with a also being in a heavy class with respect to some right MC-interval $I^{MC,+}(\sigma(q), 64^\alpha/28)$), then again I^+ and $I^{MC,+}(\sigma(q), 64^\alpha/28)$ intersect and thus e' is the first edge of a player q occurring in I. Therefore, by Lemma

2, e' is removed from the solution after the absorbing process of p. As a consequence, in this way every arc a of class β in MC can be partially charged for at most one edge of every class greater than $\beta+1$, that is, for at most

$$\sum_{\gamma\geq\beta+2} 64^{\beta}\cdot\frac{64^{\gamma-\beta+1}}{e^{\sqrt{64^{\frac{\gamma-\beta-4}{2}}-1}}} = 64^{\beta}\sum_{z\geq 0}\frac{64^{z+3}}{e^{\sqrt{64^{\frac{z-2}{2}}-1}}} = O(64^{\beta}) = O(w_a).$$

Therefore, by summing over all arcs in MC, recalling that MC has cost $2C(S^*)$, also the total contribution to the cost of the final state of the edges subject to this type of accounting is $O(C(S^*))$.

It remains to analyze the case in which $I^+ = MC$. Since in this case also $I = MC$ and thus the absorbing process removes from the state all the edges not belonging to T^* (except e), in the final equilibrium it can exist only a unique critical edge introduced by a player p such that $I^+ = MC$. Therefore, as by definition a basic move can introduce only edges of cost at most $C(S^*)$, the total contribution to the cost of the final state due to such a unique edge is at most $C(S^*)$. ■

IV. Conclusions

In this paper, we have shown that the price of stability of broadcast games with fair cost allocations in undirected networks is $O(1)$, thus closing the problem from an asymptotic point of view. However, the constant hidden inside the big-O notation is high compared to the currently best-known lower bound of 1.818. Hence, further effort is still required to achieve an exact characterization of the price of stability for these games.

An intriguing open problem is that of trying to exploit homogenization for improving the upper bounds also for multicast and unrestricted network design games. However, at the moment our technique does not seem to directly apply to these more general communication patterns, so that its extension remains a worth investigating issue, possibly capturing future research attention.

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