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Interlacing Families I: Bipartite Ramanujan Graphs of All Degrees

Adam Marcus
Yale University
adam.marcus@yale.edu

Daniel A. Spielman
Yale University
daniel.spielman@yale.edu

Nikhil Srivastava
Microsoft Research, India
niksri@microsoft.com

***Abstract*—We prove that there exist infinite families of regular bipartite Ramanujan graphs of every degree bigger than 2. We do this by proving a variant of a conjecture of Bilu and Linial about the existence of good 2-lifts of every graph.**

We also establish the existence of infinite families of 'irregular Ramanujan' graphs, whose eigenvalues are bounded by the spectral radius of their universal cover. Such families were conjectured to exist by Linial and others. In particular, we prove the existence of infinite families of (c, d)-biregular bipartite graphs with all non-trivial eigenvalues bounded by $\sqrt{c-1} + \sqrt{d-1}$, for all $c, d \geq 3$.

Our proof exploits a new technique for demonstrating the existence of useful combinatorial objects that we call the "method of interlacing polynomials".

***Keywords*-Ramanujan Graph; Matching Polynomial; Lifts of Graphs**

I. Introduction

Ramanujan graphs have been the focus of substantial study in Theoretical Computer Science and Mathematics. They are graphs whose non-trivial adjacency matrix eigenvalues are as small as possible. Previous constructions of Ramanujan graphs have been sporadic, only producing Ramanujan graphs of particular degrees. In this paper, we prove a variant of a conjecture of Bilu and Linial [1], and use it to realize an approach they suggested for constructing bipartite Ramanujan graphs of every degree.

Our main technical contribution is a novel existence argument. The conjecture of Bilu and Linial requires us to prove that every graph has a signed adjacency matrix with all of its eigenvalues in a small range. We do this by proving that the roots of the *expected* characteristic polynomial of a randomly signed adjacency matrix lie in this range. In general, a statement like this is useless, as the roots of a sum of polynomials do not necessarily have anything to do with the roots of the polynomials in the sum. However, there seem to be many sums of combinatorial polynomials for which this intuition is wrong. With this in mind, we define an "interlacing family" of polynomials and then use a technique we call the "method of interlacing polynomials" to show that such families always contain a polynomial whose largest root is at most the largest root of the sum. To finish the proof, we then bound the largest root of the sum of the characteristic polynomials of the signed adjacency matrices of a graph by observing that this sum is the well-studied matching polynomial of the graph.

This paper is the first one in a series focusing on the method of interlacing polynomials. In the next paper [2], we use the method of interlacing polynomials to give a positive resolution to the Kadison–Singer problem.

II. Technical Introduction and Preliminaries

A. Ramanujan Graphs

Ramanujan graphs are defined in terms of the eigenvalues of their adjacency matrices. If G is a d-regular graph and A is its adjacency matrix, then d is always an eigenvalue of A. The matrix A has an eigenvalue of $-d$ if and only if G is bipartite. The eigenvalues of d, and $-d$ when G is bipartite, are called the *trivial* eigenvalues of A. Following Lubotzky, Phillips and Sarnak [3], we say that a d-regular graph is *Ramanujan* if all of its non-trivial eigenvalues lie between $-2\sqrt{d-1}$ and $2\sqrt{d-1}$. It is easy to construct Ramanujan graphs with a small number of vertices: d-regular complete graphs and complete bipartite graphs are Ramanujan. The challenge is to construct an infinite family of d-regular graphs that are all Ramanujan. One cannot construct infinite families of d-regular graphs whose eigenvalues lie in a smaller range: the Alon–Boppana bound (see [4]) tells us that for every constant $\epsilon > 0$, every sufficiently large d-regular graph has a non-trivial eigenvalue with absolute value at least $2\sqrt{d-1} - \epsilon$.

Lubotzky, Phillips and Sarnak [3] and Margulis [5] were the first to construct Ramanujan graphs. They built both bipartite and non-bipartite Ramanujan graphs from Cayley graphs. All of their graphs are regular and have degrees $p + 1$ where p is a prime. There have been very few other constructions of Ramanujan graphs [6], [7], [8], [9]. To the best of our knowledge, the only degrees for which infinite families of Ramanujan graphs were previously known to exist were those of the form $q + 1$ where q is a prime power. Lubotzky [10, Problem 10.7.3] asked whether there exist infinite families of Ramanujan graphs of every degree greater than 2. We resolve this conjecture in the affirmative in the bipartite case.

B. 2-Lifts

Bilu and Linial [1] suggested constructing Ramanujan graphs through a sequence of 2-lifts of a base graph. Given a graph $G = (V, E)$, a 2-lift of G is a graph that has two vertices for each vertex in V. This pair of vertices is called

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the *fibre* of the original vertex. Every edge in E corresponds to two edges in the 2-lift. If (u, v) is an edge in E, $\{u_0, u_1\}$ is the fibre of u, and $\{v_0, v_1\}$ is the fibre of v, then the 2-lift can either contain the pair of edges

$$\{(u_0, v_0), (u_1, v_1)\}\,, \text{or} \tag{1}$$

$$\{(u_0, v_1), (u_1, v_0)\}\,. \tag{2}$$

If only edge pairs of the first type appear, then the 2-lift is just two disjoint copies of the original graph. If only edge pairs of the second type appear, then we obtain the *double-cover* of G. To determine the eigenvalues of a 2-lift, Bilu and Linial introduce *signings* of the edges of G: $s : E \to \{\pm 1\}$. They place signings in one-to-one correspondence with 2-lifts by setting $s(u, v) = 1$ if edges of type (1) appear in the 2-lift, and $s(u, v) = -1$ if edges of type (2) appear.

To analyze the eigenvalues of a 2-lift, they define the signed adjacency matrix A_s to be the same as the adjacency matrix of G, except that the entries corresponding to an edge (u, v) are $s(u, v)$. They prove [1, Lemma 3.1] that the eigenvalues of the 2-lift are the union, taken with multiplicity, of the eigenvalues of the adjacency matrix of A and of the signed adjacency matrix of A. Following Friedman [11], they refer to the eigenvalues of A as the *old eigenvalues* and the eigenvalues of the signed adjacency matrix as the *new eigenvalues*. They prove that every graph of maximal degree d has a signing in which all of the new eigenvalues have absolute value at most $O(\sqrt{d \log^3 d})$. They then build d-regular expander graphs by repeatedly taking 2-lifts of a complete graph on $d + 1$ vertices.

Bilu and Linial conjectured that every d-regular graph has a signing in which all of the new eigenvalues have absolute value at most $2\sqrt{d-1}$. If one applied the corresponding 2-lifts to the d-regular complete graph, one would obtain an infinite sequence of d-regular Ramanujan graphs. We prove a weak version of Bilu and Linial's conjecture: every d-regular graph has a signing in which all of the new eigenvalues are at most $2\sqrt{d-1}$. The difference between our result and the original conjecture is that we do not control the smallest new eigenvalue. This is why we consider bipartite graphs. The eigenvalues of the adjacency matrices of bipartite graphs are symmetric about zero (see, for example, [12, Theorem 2.4.2]) So, a bound on the smallest non-trivial eigenvalue follows from a bound on the largest. We also use the fact that a 2-lift of a bipartite graph is also bipartite. By applying the corresponding 2-lifts to the d-regular complete bipartite graph, we obtain an infinite sequence of d-regular bipartite Ramanujan graphs.

C. Irregular Ramanujan Graphs and Universal Covers

We say that a bipartite graph is (c, d)-biregular if all vertices on one side of the bipartition have degree c and all vertices on the other side have degree d. The adjacency matrix of a (c, d)-biregular graph always has eigenvalues $\pm\sqrt{cd}$; these are its trivial eigenvalues. Feng and Li [13] (see also [14]) prove a generalization of the Alon–Boppana bound that applies to (c, d)-biregular graphs: for all $\epsilon > 0$, all sufficiently large (c, d)-biregular graphs have a non-trivial eigenvalue with absolute value at least $\sqrt{c-1} + \sqrt{d-1} - \epsilon$. Thus, we say that a (c, d)-biregular graph is Ramanujan if all of its non-trivial eigenvalues are at most $\sqrt{c-1} + \sqrt{d-1}$. We prove the existence of infinite families of (c, d)-biregular Ramanujan graphs for all $c, d \geq 3$.

The regular and biregular Ramanujan graphs discussed above are actually special cases of a more general phenomenon. To describe it, we will require a construction known as the *universal cover*. The universal cover of a graph G is the infinite tree T such that every connected lift of G is a quotient of the tree (see, e.g., [15, Section 6]). It can be defined concretely by first fixing a "root" vertex v_0, and then placing one vertex in T for every non-backtracking walk $(v_0, v_1, \ldots, v_\ell)$ starting at v_0, where a walk is non-backtracking if $v_{i-1} \neq v_{i+1}$ for all i. Two vertices of T are adjacent if and only if one is a simple extension of another, i.e., the edges of T are all of the form $(v_0, v_1, \ldots, v_\ell) \sim (v_0, v_1, \ldots, v_\ell, v_{\ell+1})$. The universal cover of a graph is unique up to isomorphism, independent of the choice of v_0

The adjacency matrix A_T of the universal cover T is an infinite-dimensional symmetric matrix. We will be interested in the *spectral radius* $\rho(T)$ of T, which may be defined[1] as:

$$\rho(T) := \sup_{\|x\|_2 = 1} \|A_T x\|_2 \tag{3}$$

where $\|x\|_2^2 := \sum_{i=1}^{\infty} x(i)^2$ whenever the series converges. Naturally, the spectral radius of a finite tree is defined to be the norm of its adjacency matrix.

With these notions in hand, we can state the definition of an irregular Ramanujan graph. As before, the largest (and smallest, in the bipartite case) eigenvalues of finite adjacency matrices are considered trivial. Greenberg [17] (see also [18]) showed that for every $\epsilon > 0$ and every infinite family of graphs that have the same universal cover T, all sufficiently large graphs in the family have a non-trivial eigenvalue with absolute value at least $\rho(T) - \epsilon$. Following Hoory, Linial, and Wigderson [15, Definition 6.7], we therefore define an arbitrary graph to be Ramanujan if all of its non-trivial eigenvalues are smaller in absolute value than the spectral radius of its universal cover.

The universal cover of any d-regular graph is the infinite d-ary tree, whereas the universal cover of any (c, d)-biregular graph is the infinite $(c, d)-$biregular tree in which the degrees alternate between c and d on every other level [14]. The former tree is known to have spectral

[1]In functional analysis, the spectral radius of an infinite-dimensional operator A is traditionally defined to be the largest λ for which $(A - \lambda I)$ is unbounded. However, in the case of self-adjoint operators, this definition is equivalent to the one presented here (see, for example, Theorem VI.6 in [16]).

radius $2\sqrt{d-1}$ while the latter has a spectral radius of $\sqrt{c-1}+\sqrt{d-1}$ (see [19], [14]). Thus, a definition based on universal covers generalizes both the regular and biregular definitions of Ramanujan graphs, and the bound of Greenberg generalizes both the Alon-Boppana and Feng-Li bounds.

In this general setting, we show that every bipartite graph G has a 2-lift in which all of the new eigenvalues are less than the spectral radius of its universal cover. Applying these 2-lifts inductively to any finite irregular bipartite Ramanujan graph yields an infinite family of irregular bipartite Ramanujan graphs whose degree distribution matches that of the initial graph (since taking a 2-lift simply doubles the number of vertices of each degree). In particular, applying them to the (c, d)-biregular complete bipartite graph yields an infinite family of (c, d)-biregular Ramanujan graphs. As far as we know, infinite families of irregular Ramanujan graphs were not known to exist prior to this work.

D. Related Work

There have been numerous studies of random lifts of graphs. For some results on the spectra of random lifts, we point the reader to [20], [21], [22], [23], [24], [25]. Friedman [26] has proved that almost every d-regular graph almost meets the Ramanujan bound: he shows that for every $\epsilon > 0$ the absolute value of all the non-trivial eigenvalues of almost every sufficiently large d-regular graph are at most $2\sqrt{d-1}+\epsilon$. In the irregular case, Puder [27] has shown that with high probability a high-order lift of a graph G has new eigenvalues that are bounded in absolute value by $\sqrt{3}\rho$, where ρ is the spectral radius of the universal cover of G.

We remark that constructing bipartite Ramanujan graphs is at least as easy as constructing non-bipartite ones: the double-cover of a d-regular non-bipartite Ramanujan graph is a d-regular bipartite Ramanujan graph. For many applications of expander graphs, we refer the reader to [15]. For those applications of expanders that just require upper bounds on the second eigenvalue, one can use bipartite Ramanujan graphs. Some applications actually require bipartite expanders, while others require the non-bipartite ones. For example, the explicit constructions of error correcting codes of Sipser and Spielman [28] require non-bipartite expanders, while the improvements of their construction [29], [30], [31] require bipartite Ramanujan expanders.

III. 2-Lifts and The Matching Polynomial

For a graph G, let m_i denote the number of matchings in G with i edges. Set $m_0 = 1$. Heilmann and Lieb [32] defined the *matching polynomial* of G to be the polynomial

$$\mu_G(x) \stackrel{\text{def}}{=} \sum_{i\geq 0} x^{n-2i}(-1)^i m_i,$$

where n is the number of vertices in the graph. They proved two remarkable theorems about the matching polynomial that we will exploit in this paper. It is worth mentioning that the proofs of these theorems are elementary and short, relying only on simple recurrence formulas for the matching polynomial.

Theorem III.1 (Theorem 4.2 in [32])**.** *For every graph G, $\mu_G(x)$ has only real roots.*

Theorem III.2 (Theorem 4.3 in [32])**.** *For every graph G of maximum degree d, all of the roots of $\mu_G(x)$ have absolute value at most $2\sqrt{d-1}$.*

The preceding theorems will allow us to prove the existence of infinite families of d-regular bipartite Ramanujan graphs. To handle the irregular case, we will require a refinement of these results due to Godsil. This refinement uses the concept of a *path tree*, which was also introduced by Godsil (see [33] or [12, Section 6]). Recall that a path in G is a walk that does not visit any vertex twice.

Definition III.3. Given a graph G and a vertex u, the path tree $P(G, u)$ contains one vertex for every path in G (with distinct vertices) that starts at u. Two paths are adjacent if one is a simple extension of the other, i.e., all the edges of $P(G, u)$ are all of the form $(u, v_1, \ldots, v_\ell) \sim (u, v_1, \ldots, v_\ell, v_{\ell+1})$.

The path tree provides a natural relationship between the roots of the matching polynomial of a graph and the spectral radius of its universal cover:

Theorem III.4 ([33])**.** *Let $P(G, u)$ be a path tree of G. Then the matching polynomial of G divides the characteristic polynomial of the adjacency matrix of $P(G, u)$. In particular, all of the roots of $\mu_G(x)$ are real and have absolute value at most $\rho(P(G, u))$.*

Lemma III.5. *Let G be a graph and let T be its universal cover. Then the roots of $\mu_G(x)$ are bounded in absolute value by $\rho(T)$.*

Proof: Let u be any vertex of G and let P be the path tree rooted at u. Since the paths that correspond to the vertices of P are themselves non-backtracking walks (as defined in Section II-C), P is a finite induced subgraph of the universal cover T, and A_P is a finite submatrix of A_T. By Theorem III.4, the roots of μ_G are bounded by

$$\begin{aligned}\|A_P\|_2 &= \sup_{\|x\|_2=1} \|A_P x\|_2\\ &\leq \sup_{\|y\|_2=1,\mathrm{supp}(y)\subset P} \|A_T y\|_2\\ &\leq \sup_{\|y\|_2=1} \|A_T y\|_2 = \rho(T),\end{aligned}$$

as desired. □

We remark that one can directly prove an upper bound of $2\sqrt{d-1}$ on the spectral radius of a path tree of d-regular graph and an upper bound of $\sqrt{c-1}+\sqrt{d-1}$ on the spectral radius of a path tree of a (c, d)-regular bipartite graph without considering infinite trees. We point the reader to Section 5.6 of Godsil's book [12] for an elementary argument.

We now recall a theorem of Godsil and Gutman which says that the matching polynomial of a graph equals the expected characteristic polynomial of random signing of the adjacency matrix of that graph. To associate a signing of the edges of G with a vector in $\{\pm 1\}^m$, we choose an arbitrary ordering of the m edges of G, denote the edges by $e_1, \ldots, e_m$, and a signing of these edges by $s \in \{\pm 1\}^m$. We then let A_s denote the signed adjacency matrix corresponding to s, and define $f_s(x) = \det[xI - A_s]$ to be characteristic polynomial of A_s.

Theorem III.6 (Corollary 2.2 of Godsil and Gutman [34])**.**

$$\mathbb{E}_{s \in \{\pm 1\}^m} \left[f_s(x) \right] = \mu_G(x).$$

For the convenience of the reader, we present a simple proof of this theorem in the Appendix.

To prove that a good lift exists, it suffices, by Theorems III.2 and III.6, to show that there is a signing s so that the largest root of $f_s(x)$ is at most the largest root of $\mathbb{E}_{s \in \{\pm 1\}^m} \left[f_s(x) \right]$. To do this, we prove that the polynomials $\{f_s(x)\}_{s \in \{\pm 1\}^m}$ are what we call an interlacing family. We define interlacing families and examine their properties in the next section.

IV. Interlacing Families

Definition IV.1. We say that a polynomial $g(x) = \prod_{i=1}^{n-1} (x - \alpha_i)$ *interlaces* a polynomial $f(x) = \prod_{i=1}^{n} (x - \beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n$$

We say that polynomials $f_1, \ldots, f_k$ have a *common interlacing* if there is a polynomial g so that g interlaces f_i for each i.

Let $\beta_{i,j}$ be the jth smallest root of f_i. The polynomials $f_1, \ldots, f_k$ have a common interlacing if and only if there are numbers $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ so that $\beta_{i,j} \in [\alpha_{j-1}, \alpha_j]$ for all i and j. The numbers $\alpha_1, \ldots, \alpha_{n-1}$ come from the roots of the polynomial g, and α_0 (α_n) can be chosen to be any number that is smaller (larger) than all of the roots of all of the f_i.

Lemma IV.2. *Let $f_1, \ldots, f_k$ be polynomials of the same degree that are real-rooted and have positive leading coefficients. Define*

$$f_\emptyset = \sum_{i=1}^{k} f_i.$$

If $f_1, \ldots, f_k$ have a common interlacing, then there exists an i so that the largest root of f_i is at most the largest root of $f_\emptyset$.

Proof: Let the polynomials be of degree n. Let g be a polynomial that interlaces all of the f_i, and let α_{n-1} be the largest root of g. As each f_i has a positive leading coefficient, it is positive for sufficiently large x. As each f_i has exactly one root that is at least α_{n-1}, each f_i is non-positive at α_{n-1}. So, $f_\emptyset$ is also non-positive at α_{n-1}, and eventually becomes positive. This tells us that $f_\emptyset$ has a root that is at least α_{n-1}, and so its largest root is at least α_{n-1}. Let β_n be this root.

As $f_\emptyset$ is the sum of the f_i, there must be some i for which $f_i(\beta_n) \geq 0$. As f_i has at most one root that is at least α_{n-1}, and $f_i(\alpha_{n-1}) \leq 0$, the largest root of f_i is it at least α_{n-1} and at most β_n. □

One can show that the assumptions of the lemma imply that $f_\emptyset$ is itself a real-rooted polynomial. However, we will not require this fact.

If the polynomials do not have a common interlacing, the sum may not be real rooted: consider $(x+1)(x+2) + (x-1)(x-2)$. Even if the sum of two polynomials is real rooted, the conclusion of Lemma IV.2 may fail to hold if the interval containing the largest roots of each polynomial overlaps the interval containing their second-largest roots. For example, consider the sum of the polynomials $(x+5)(x-9)(x-10)$ and $(x+6)(x-1)(x-8)$. It has roots at approximately -5.3, 6.4, and 7.4, whence its largest root is smaller than the largest root of both polynomials of which it is the sum.

Definition IV.3. Let $S_1, \ldots, S_m$ be finite sets and for every $s_1 \in S_1, \ldots, s_m \in S_m$ let $f_{s_1,\ldots,s_m}(x)$ be a real-rooted degree n polynomial with positive leading coefficient. For every partial assignment $s_1 \in S_1, \ldots, s_k \in S_k$, define

$$f_{s_1,\ldots,s_k} \stackrel{\text{def}}{=} \sum_{s_{k+1} \in S_{k+1}, \ldots, s_m \in S_m} f_{s_1,\ldots,s_k,s_{k+1},\ldots,s_m},$$

as well as

$$f_\emptyset \stackrel{\text{def}}{=} \sum_{s_1 \in S_1, \ldots, s_m \in S_m} f_{s_1,\ldots,s_m}.$$

We say that the polynomials $\{f_{s_1,\ldots,s_m}\}_{s_1,\ldots,s_m}$ form an *interlacing family* if for all $k = 0, \ldots, m-1$, and all $s_1 \in S_1, \ldots, s_k \in S_k$, the polynomials

$$\{f_{s_1,\ldots,s_k,t}\}_{t \in S_{k+1}}$$

have a common interlacing.

Theorem IV.4. *Let $S_1, \ldots, S_m$ be finite sets and let $\{f_{s_1,\ldots,s_m}\}$ be an interlacing family of polynomials. Then, there exits some $s_1, \ldots, s_m \in S_1 \times \cdots \times S_m$ so that the largest root of $f_{s_1,\ldots,s_m}$ is less than the largest root of $f_\emptyset$.*

Proof: From the definition of an interlacing family, we know that $f_{s_1}, \ldots, f_{s_k}$ have a common interlacing and that their sum is $f_\emptyset$. So, Lemma IV.2 tells us that one of the polynomials $\{f_{s_1}\}_{s_1 \in S_1}$ has largest root at most the largest root of $f_\emptyset$. We now proceed inductively. For any $s_1, \ldots, s_k$, we know that the polynomials $f_{s_1,\ldots,s_k,s_{k+1}}$ for $s_{k+1} \in S_{k+1}$ have a common interlacing and that their sum is $f_{s_1,\ldots,s_k}$. So, for some choice of s_{k+1} the largest root of the polynomial $f_{s_1,\ldots,s_{k+1}}$ is at most the largest root of $f_{s_1,\ldots,s_k}$. □

We will prove that the polynomials $\{f_s\}_{s\in\{\pm1\}^m}$ defined in Section III are an interlacing family. Our proof will use the following result, which seems to have been discovered a number of times. It appears as Theorem 2.1 of Dedieu [35] and (essentially) as Theorem $2'$ of Fell [36]. In the case that the roots of f and g are distinct, it appears as Proposition 1.35 in Fisk [37].

Lemma IV.5. *Let f and g be (univariate) polynomials of the same degree such that, for all $\lambda \in [0,1]$, $\lambda f + (1-\lambda) g$ is real rooted. Then f and g have a common interlacing.*

V. The Main Result

Our proof that the polynomials $\{f_s\}_{s\in\{\pm1\}^m}$ are an interlacing family relies on the following generalization of the fact that the matching polynomial is real-rooted. It amounts to saying that if we pick each sign independently with any probabilities, then the resulting polynomial is still real-rooted.

Theorem V.1. *Let $p_1, \ldots, p_m$ be numbers in $[0,1]$. Then, the following polynomial is real-rooted*

$$\sum_{s\in\{\pm1\}^m} \left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1}(1-p_i)\right) f_s(x).$$

We will prove this theorem using machinery that we develop in Section VI.

Theorem V.2. *The polynomials $\{f_s\}_{s\in\{\pm1\}^m}$ are an interlacing family.*

Proof: We will show that for every $0 \le k \le m-1$, every partial assignment $s_1 \in \pm1, \ldots, s_k \in \pm1$, and every $\lambda \in [0,1]$, the polynomial

$$\lambda f_{s_1,\ldots,s_k,1}(x) + (1-\lambda) f_{s_1,\ldots,s_k,-1}(x)$$

is real-rooted. The theorem will then follow from Lemma IV.5.

To show that the above polynomial is real-rooted, we apply Theorem V.1 with $p_{k+1} = \lambda$, $p_{k+2}, \ldots, p_m = 1/2$, and $p_i = (1+s_i)/2$ for $1 \le i \le k$. □

Theorem V.3. *Let G be a graph with adjacency matrix A and universal cover T. Then there is a signing s of A so that all of the eigenvalues of A_s are at most $\rho(T)$. In particular, for d-regular graphs, the eigenvalues of A_s are at most $2\sqrt{d-1}$.*

Proof: The first statement follows immediately from Theorems IV.4 and V.2 and Lemma III.5. The second statement follows by noting that the universal cover of a d-regular graph is a d-regular tree, which has spectral radius at most $2\sqrt{d-1}$, or by directly appealing to Theorem III.2. □

Theorem V.4. *For every $d \ge 3$ there is an infinite sequence of d-regular bipartite Ramanujan graphs.*

Proof: The complete bipartite graph of degree d is Ramanujan. By Lemma 3.1 of [1] and Theorem V.3, for every d-regular bipartite Ramanujan graph G, there is a 2-lift in which every non-trivial eigenvalue is at most $2\sqrt{d-1}$. As the 2-lift of a bipartite graph is bipartite, and the eigenvalues of a bipartite graph are symmetric about 0, this 2-lift is also a regular bipartite Ramanujan graph.

Thus, for every d-regular bipartite Ramanujan graph G, there is another d-regular bipartite Ramanujan graph with twice as many vertices. □

Theorem V.5. *For every $c, d \ge 3$, there is an infinite sequence of (c,d)-biregular bipartite Ramanujan graphs.*

Proof: Let $K_{c,d}$ be the complete bipartite graph with c vertices on one side and d on the other. The adjacency matrix of this graph has rank 2, so its non-trivial eigenvalues are zero and it is Ramanujan.

We will construct an infinite sequence of (c,d)-biregular bipartite graphs. Let G be any (c,d)-biregular bipartite Ramanujan graph. As mentioned in Section II-C, the universal cover of G is the infinite (c,d)-biregular tree, which has spectral radius $\sqrt{c-1} + \sqrt{d-1}$. Thus, Theorem V.3 tells us that there is a 2-lift of G with all new eigenvalues at most $\sqrt{c-1} + \sqrt{d-1}$. As this graph is bipartite, all of its non-trivial eigenvalues have absolute value at most $\sqrt{c-1} + \sqrt{d-1}$. So, the resulting 2-lift is a larger (c,d)-biregular bipartite Ramanujan graph. □

To conclude the section, we remark that repeated application of Theorem V.3 can be used to generate an infinite sequence of irregular Ramanujan graphs from *any* finite irregular bipartite Ramanujan graph, since all of the lifts produced will have (by definition, since they are connected) the same universal cover. In contrast, Lubotzky and Nagnibeda [38] have shown that there exist infinite trees that cover infinitely many finite graphs but such that none of the finite graphs are Ramanujan.

VI. Real Stable Polynomials

In this section we will establish the real-rootedness of a class of polynomials which includes the polynomials of Theorem V.1. We will do this by considering a multivariate generalization of real-rootedness called *real stability* (see, e.g., the surveys [39], [40]). In particular, we will show that the univariate polynomials we are interested in are the images, under a well-behaved linear transformation, of a multivariate real stable polynomial.

Definition VI.1. A multivariate polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ is called *real stable* if it is the zero polynomial or if

$$f(z_1, \ldots, z_n) \neq 0$$

whenever the imaginary part of every z_i is strictly positive.

Note that a real stable polynomial has real coefficients, but may be evaluated on complex inputs.

We begin by considering certain determinantal polynomials whose real stability is guaranteed by the following lemma, which may be found in Borcea and Brändén [41, Proposition 2.4].

Lemma VI.2. *Let $A_1, \ldots, A_m$ be positive semidefinite matrices. Then*

$$\det\left[z_1 A_1 + \cdots + z_m A_m\right]$$

is real stable.

Real stable polynomials enjoy a number of useful closure properties. In particular, it is easy to see that if $f(x_1, \ldots, x_k)$ and $g(y_1, \ldots y_j)$ are real stable then $f(x_1, \ldots, x_k)g(y_1, \ldots, y_j)$ is real stable. One can also check that the real stability of $f(x_1, \ldots, x_k)$ implies the real stability of $f(x_1, \ldots, x_{k-1}, c)$ for every $c \in \mathbb{R}$ (see, e.g., Lemma 2.4 in [40]). For a variable x_i, we let Z_{x_i} be the operator on polynomials induced by setting this variable to zero.

In [42], Borcea and Brändén characterize an entire class of differential operators that preserve real stability. To simplify notation, we will let ∂_{z_i} denote the operation of partial differentiation with respect to z_i. For $\alpha, \beta \in \mathbb{N}^n$, we use the notation

$$z^\alpha = \prod_{i=1}^n z_i^{\alpha_i} \quad \text{and} \quad \partial^\beta = \prod_{i=1}^n (\partial_{z_i})^{\beta_i}.$$

Theorem VI.3 (Theorem 1.3 in [42])**.**
Let $T : \mathbb{R}[z_1, \ldots, z_n] \to \mathbb{R}[z_1, \ldots, z_n]$ be an operator of the form

$$T = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha,\beta} z^\alpha \partial^\beta$$

where $c_{\alpha,\beta} \in \mathbb{R}$ and $c_{\alpha,\beta}$ is zero for all but finitely many terms. Define

$$F_T(z, w) = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha w^\beta$$

Then T preserves real stability if and only if $F_T(z, -w)$ is real stable.

We will use a special case of this result.

Corollary VI.4. *For non-negative real numbers p and q and variables u and v, the operator $T = 1 + p\partial_u + q\partial_v$ preserves real stability.*

Proof: We just need to show that the polynomial $1 - pu - qv$ is real stable. To see this, consider u and v with positive imaginary parts. The imaginary part of $1 - pu - qv$ will then be negative, and so cannot be zero. □

We now show how operators of the preceding kind can be used to generate the expected characteristic polynomials that appear in Theorem V.1.

Lemma VI.5. *For an invertible matrix A, vectors a and b, and a number $p \in [0, 1]$,*

$$\begin{aligned} Z_u Z_v (1 + p\partial_u + (1-p)\partial_v) \det\left[A + uaa^T + vbb^T\right] \\ = p \det\left[A + aa^T\right] + (1-p)\det\left[A + bb^T\right]. \end{aligned}$$

Proof: The matrix determinant lemma (see, e.g., [43]) states that for every nonsingular matrix A and every real number t,

$$\det\left[A + taa^T\right] = \det[A]\,(1 + ta^T A^{-1} a).$$

One consequence of this is Jacobi's formula for the derivative of the determinant:

$$\partial_t \det\left[A + taa^T\right] = \det[A]\,(a^T A^{-1} a).$$

This formula implies that

$$\begin{aligned} Z_u Z_v (1 + p\partial_u + (1-p)\partial_v) \det\left[A + uaa^T + vbb^T\right] \\ = \det[A]\left(1 + p(a^T A^{-1} a) + (1-p)(b^T A^{-1} b)\right). \end{aligned}$$

By the matrix determinant lemma, this equals

$$p \det\left[A + aa^T\right] + (1-p)\det\left[A + bb^T\right].$$

□

Using these tools, we prove our main technical result on real-rootedness.

Theorem VI.6. *Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ be vectors in $\mathbb{R}^n$, and let $p_1, \ldots, p_m$ be real numbers in $[0, 1]$. Then every (univariate) polynomial of the form*

$$\begin{aligned} P(x) \stackrel{\text{def}}{=} \sum_{S \subseteq [m]} \left(\prod_{i \in S} p_i\right) \left(\prod_{i \notin S} 1 - p_i\right) \\ \cdot \det\left[xI + \sum_{i \in S} a_i a_i^T + \sum_{i \notin S} b_i b_i^T\right] \end{aligned}$$

is real-rooted.

Proof: Let $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$ be formal variables and define

$$\begin{aligned} Q(x, u_1, \ldots, u_m, v_1, \ldots, v_m) \\ = \det\left[xI + \sum_i u_i a_i a_i^T + \sum_i v_i b_i b_i^T\right]. \end{aligned}$$

Lemma VI.2 implies that Q is real stable.

We claim that we can rewrite P as

$$P(x) = \left(\prod_{i=1}^{m} Z_{u_i} Z_{v_i} T_i\right) Q(x, u_1, \ldots, u_m, v_1, \ldots, v_m),$$

where $T_i = 1 + p_i \partial_{u_i} + (1 - p_i)\partial_{v_i}$. To see this, we prove by induction on k that

$$\left(\prod_{i=1}^{k} Z_{u_i} Z_{v_i} T_i\right) Q(x, u_1, \ldots, u_m, v_1, \ldots, v_m)$$

equals

$$\sum_{S \subseteq [k]} \left(\prod_{i \in S} p_i\right) \left(\prod_{i \in [k] \setminus S} 1 - p_i\right) \cdot \det\left[xI + \sum_{i \in S} a_i a_i^T + \sum_{i \in [k] \setminus S} b_i b_i^T + \sum_{i > k} u_i a_i a_i^T + v_i b_i b_i^T\right].$$

The base case ($k = 0$) is trivially true, as it is the definition of Q. The inductive step follows from Lemma VI.5. The case $k = m$ is exactly the claimed identity.

Starting with Q (a real stable polynomial) we can then apply Corollary VI.4 and the closure of real stable polynomials under the restrictions of variables to real constants to see that each of the polynomials above, including $P(x)$, is also real stable. As $P(x)$ is real stable and has one variable, it is real-rooted. □

Alternatively, one can prove Theorem VI.6 by observing that P is a *mixed characteristic polynomial* and then applying results of our second paper in this series [2].

Proof of Theorem V.1: Let d be the maximum degree of G. We need to prove that the polynomial

$$\sum_{s \in \{\pm 1\}^m} \left(\prod_{i : s_i = 1} p_i\right) \left(\prod_{i : s_i = -1} (1 - p_i)\right) \det\left[xI - A_s\right]$$

is real-rooted. This is equivalent to proving that the the following polynomial is real-rooted

$$\sum_{s \in \{\pm 1\}^m} \left(\prod_{i : s_i = 1} p_i\right) \left(\prod_{i : s_i = -1} (1 - p_i)\right) \det\left[xI + dI - A_s\right], \tag{4}$$

as their roots only differ by d.

We now observe that the matrix $dI - A_s$ is a signed Laplacian matrix of G plus a nonnegative diagonal matrix. For each edge (u, v), define the rank 1-matrices

$$L_{u,v}^{1} = (e_u - e_v)(e_u - e_v)^T, \quad \text{and}$$
$$L_{u,v}^{-1} = (e_u + e_v)(e_u + e_v)^T,$$

where e_u is the elementary unit vector in direction u. Consider a signing s and let $s_{u,v}$ denote the sign it assigns to edge (u, v). Since the original graph had maximum degree d, we have

$$dI - A_s = \sum_{(u,v) \in E} L_{u,v}^{s_{u,v}} + D,$$

for some nonnegative diagonal matrix $D = \sum_{u \in V} d_u e_u e_u^T$, which does not depend on s and which is zero when G is regular. If we now set $a_{u,v} = (e_u - e_v)$ and $b_{u,v} = (e_u + e_v)$, we can express the polynomial in (4) as

$$\sum_{s \in \{\pm 1\}^m} \left(\prod_{i : s_i = 1} p_i\right) \left(\prod_{i : s_i = -1} (1 - p_i)\right) \cdot \det\left[xI + D + \sum_{s_{u,v} = 1} a_{u,v} a_{u,v}^T + \sum_{s_{u,v} = -1} b_{u,v} b_{u,v}^T\right].$$

The fact that this polynomial is real-rooted now follows from Theorem VI.6, by creating auxiliary p_i's that are all equal to one to correspond to any fixed $d_u e_u e_u^T$ terms. □

VII. Conclusion

We conclude by drawing an analogy between our proof technique and the probabilistic method, which relies on the fact that for every random variable $X : \Omega \to \mathbb{R}$, there is an $\omega \in \Omega$ for which $X(\omega) \leq \mathbb{E}[X]$. We have shown that for certain special polynomial-valued random variables $P : \Omega \to \mathbb{R}[x]$, there must be an ω with $\lambda_{max}(P(\omega)) \leq \lambda_{max}(\mathbb{E}[P])$. In fact it is possible to define interlacing families in greater generality than we have done here, using probabilistic notation. In particular, we call a polynomial-valued random variable P *useful* if P is deterministic or there exist disjoint non-trivial events $E_1, \ldots, E_k$ with $\sum_{i \leq k} \Pr[E_i] = 1$ such that the polynomials $\{\mathbb{E}[P|E_i]\}_{i \leq k}$ have a common interlacing and each polynomial $\mathbb{E}[P|E_i]$ is itself useful. The conclusion of Theorem IV.4 continues to hold for this definition, and we suspect it will be useful in non-product settings. In the case of this paper, the events E_i are particularly simple: they correspond to setting one sign of a lift to be $+1$ or -1, and the resulting sequence of polynomials $f_\emptyset, f_{s_1}, \ldots, f_{s_1,\ldots,s_m}$ forms a martingale (a fact that we do not use, but may be interesting in its own right).

Like many applications of the probabilistic method, our proof does not yield a polynomial-time algorithm. In the particular case of random lifts, the polynomial $f_\emptyset$ is itself a matching polynomial, which is $\#P$-hard to compute in general. It would certainly be interesting to find computationally efficient analogues of our method.

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Appendix

Let $\mathrm{sym}(S)$ denote the set of permutations of a set S and let $(-1)^\pi$ denote the sign of a permutation π (i.e., the number of inversions in π). Expanding the determinant as a sum over permutations $\sigma \in \mathrm{sym}([n])$, we have

$$
\begin{aligned}
&\mathbb{E}_s\left[\det(xI - A_s)\right] \\
&= \mathbb{E}_s\left[\sum_{\sigma\in\mathrm{sym}([n])} (-1)^\sigma \prod_{i=1}^n (xI - A_s)_{i,\sigma(i)}\right] \\
&= \sum_{k=0}^n x^{n-k} \sum_{S\subset[n],|S|=k} \sum_{\pi\in\mathrm{sym}(S)} \mathbb{E}_s\left[(-1)^\pi \prod_{i\in S} (A_s)_{i,\pi(i)}\right] \\
&\qquad \text{where } \pi \text{ denotes the part of } \sigma \text{ with } \sigma(i) \neq i \\
&= \sum_{k=0}^n x^{n-k} \sum_{S\subset[n],|S|=k} \sum_{\pi\in\mathrm{sym}(S)} \mathbb{E}_s\left[(-1)^\pi \prod_{i\in S} s_{i,\pi(i)}\right].
\end{aligned}
$$

Observe that since the s_{ij} are independent with $\mathbb{E}\left[s_{ij}\right] = 0$, only those products which contain even powers (0 or 2) of the s_{ij} survive. Thus, we may restrict our attention to the permutations π which contain only orbits of size two. These are just the perfect matchings on S. There are no perfect matchings when $|S|$ is odd; otherwise, each matching consists of $|S|/2$ inversions. Since $\mathbb{E}_s\left[s_{ij}^2\right] = 1$, we are left with

$$
\begin{aligned}
&\mathbb{E}_s\left[\det(xI - A_s)\right] \\
&= \sum_{k=0}^n x^{n-k} \sum_{|S|=k} \sum_{\text{matching } \pi \text{ on } S} (-1)^{|S|/2} \cdot 1 \\
&= \mu_G(x),
\end{aligned}
$$

as desired.