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Non-positive curvature and the planar embedding conjecture

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***Abstract*—The planar embedding conjecture asserts that any planar metric admits an embedding into L_1 with constant distortion. This is a well-known open problem with important algorithmic implications, and has received a lot of attention over the past two decades. Despite significant efforts, it has been verified only for some very restricted cases, while the general problem remains elusive.**

In this paper we make progress towards resolving this conjecture. We show that every planar metric of non-positive curvature admits a constant-distortion embedding into L_1. This confirms the planar embedding conjecture for the case of non-positively curved metrics.

I. Introduction

If $(X, d_X), (Y, d_Y)$ are metric spaces, and $f : X \to Y$ is injective, the *distortion* of f is defined to be $\mathsf{distortion}(f) = \|f\|_{\mathsf{Lip}} \cdot \|f^{-1}\|_{\mathsf{Lip}}$, where $\|f\|_{\mathsf{Lip}} = \sup_{x \neq y \in X} \frac{d_Y(f(x), f(y))}{d_X(x,y)}$. For any metric space (X, d), we use $c_1(X, d)$ to denote the L_1 *distortion of* (X, d), i.e. the infimum over all numbers D such that X admits an embedding into L_1 with distortion D. For a graph $G = (V, E)$ we write $c_1(G) = \sup c_1(V, d)$ where d ranges over all shortest-path metrics supported on G, and for a family $\mathcal{F}$ of graphs, we write $c_1(\mathcal{F}) = \sup_{G \in \mathcal{F}} c_1(G)$. Thus for a family $\mathcal{F}$ of finite graphs, $c_1(\mathcal{F}) \leq D$ if and only if every geometry supported on a graph in $\mathcal{F}$ embeds into L_1 with distortion at most D.

In the seminal works of Linial-London-Rabinovich [20], and later Aumann-Rabani [2] and Gupta-Newman-Rabinovich-Sinclair [13], the geometry of graphs is related to the classical study of the relationship between flows and cuts.

A *multi-commodity flow instance in* G is specified by a pair of non-negative mappings $\mathsf{cap} : E \to \mathbb{R}$ and $\mathsf{dem} : V \times V \to \mathbb{R}$. We write $\mathsf{maxflow}(G; \mathsf{cap}, \mathsf{dem})$ for the value of the *maximum concurrent flow* in this instance, which is the maximal value ε such that $\varepsilon \cdot \mathsf{dem}(u, v)$ can be simultaneously routed between every pair $u, v \in V$ while not violating the given edge capacities.

A natural upper bound on $\mathsf{maxflow}(G; \mathsf{cap}, \mathsf{dem})$ is given by the *sparsity* of any cut $S \subseteq V$:

$$\frac{\sum_{uv \in E} \mathsf{cap}(u, v) |\mathbf{1}_S(u) - \mathbf{1}_S(v)|}{\sum_{u,v \in V} \mathsf{dem}(u, v) |\mathbf{1}_S(u) - \mathbf{1}_S(v)|}, \quad (1)$$

where $\mathbf{1}_S : V \to \{0, 1\}$ is the indicator function for membership in S. We write $\mathsf{gap}(G)$ for the maximum gap between the value of the flow and the upper bounds given by (1), over all multi-commodity flow instances on G. This is the *multi-commodity max-flow/min-cut gap for* G. The fundamental connection between embeddings into L_1 and multi-commodity flows is captured in the following result.

Theorem I.1 ([13], [20]). *For every graph* G, $c_1(G) = \mathsf{gap}(G)$.

In particular, combined with the techniques of [19], [20], this implies that for any graph G, there exists a $c_1(G)$-approximation for the general Sparsest Cut problem.

A. The planar embedding conjecture

It has been shown by [19], [20] that for general graphs, $c_1(G) = \Omega(\log n)$, and there has since been a lot of effort in trying to prove that $c_1(G)$ is bounded by some universal constant for interesting classes of graphs. The most well-known open case is the so-called *planar embedding conjecture*, summarized in the following:

Conjecture 1 (Planar embedding conjecture). *For every planar graph* G, $c_1(G) = O(1)$.

Despite several attempts on resolving this question, there has only been very little progress. More specifically, the work of Okamura & Seymour [21] implies that the metric induced on a *single face* of a planar graph embeds with constant distortion into L_1. In [13] it is shown that $c_1(G) = O(1)$ for any series-parallel, or outerplanar graph G. This result was extended to $O(1)$-outerplanar graphs in [7]. Chakrabarti *et al.* [6] obtained constant distortion embeddings of graphs that exclude a $(K_5 \setminus e)$-minor. Note that even the case of planar graphs of treewidth 3 remains open. We remark that the best-known upper bound on $c_1(G)$ for planar graphs is $O(\sqrt{\log n})$, due to Rao [23], while the best-known lower bound is 2, due to Lee & Raghavendra [17].

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B. Generalizations: The GNRS conjecture

Gupta, Newman, Rabinovich, and Sinclair [13] posed the following generalization of the planar embedding conjecture, which seeks to *characterize* the graph families $\mathcal{F}$ such that $c_1(\mathcal{F}) = O(1)$, which by Theorem I.1 also characterizes all graphs with multi-commodity gap bounded by some universal constant:

Conjecture 2 (GNRS conjecture [13])**.** *For every family of finite graphs $\mathcal{F}$, one has $c_1(\mathcal{F}) = O(1)$ if and only if $\mathcal{F}$ forbids some minor.*

We note that a strengthening of the GNRS conjecture for *integral* multi-commodity flows has also been considered [8]. This is a seemingly harder problem, and progress has been even more limited in this case.

At first glance, it might appear that the GNRS conjecture is a vast generalization of the planar embedding conjecture, since planar graphs exclude K_5 as a minor. Despite this, Lee & Sidiropoulos [18] have shown that the GNRS conjecture is *equivalent* to the conjunction of the planar embedding conjecture, with the manifestly simpler *k-sum embedding conjecture* summarized bellow. For a graph family $\mathcal{F}$, let $\oplus_k \mathcal{F}$ denote the closure of $\mathcal{F}$ under k-clique sums (see [18] for a more detailed exposition). We note that the case $k = 1$ is folklore, while recently progress has been reported for the case $k = 2$ by Lee and Poore [16]; even for $k = 2$ however, the problem remains open.

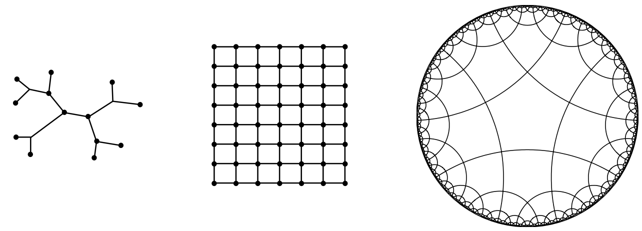
Conjecture 3 (k-sum conjecture [18])**.** *For any family of graphs $\mathcal{F}$, we have $c_1(\mathcal{F}) = O(1)$ if and only if $c_1(\oplus_k \mathcal{F}) = O(1)$ for every $k \in \mathbb{N}$.*

It is therefore apparent that the planar embedding conjecture is a major step towards determining the multi-commodity gap in *arbitrary* graphs.

C. Our results

All previous attempts on the planar embedding conjecture have been *topological* in nature, meaning that they seek to obtain constant-distortion embeddings by restricting the topology of the planar graph. As a consequence, all known methods are insufficient even for planar graphs of treewidth 3.

We depart from this paradigm by instead restricting the *geometry* of the planar metric. For any metric (X, d), we have that (X, d) is the shortest-path metric of a planar graph if and only if it can be realized as a set of points in a simply-connected (i.e. planar) surface. We say that a planar metric is *non-positively curved* if it can be realized as a set of points in a surface of non-positive curvature (see Section I-D for the definition of non-positively curved space). This leads to a natural, and very rich class of planar metrics. For instance, non-positively curved planar metrics include all trees, all regular grids (up to constant distortion), and arbitrary subsets of the hyperbolic plane $\mathbb{H}^2$.



Our main result is as follows.

Theorem I.2 (Main)**.** *There exists a universal constant $\gamma > 1$, such that every non-positively curved planar metric admits an embedding into L_1 with distortion at most γ.*

Since we are motivated by the applications of metric embeddings in computer science, we will restrict our discussion to finite metrics. We remark however that our result can be extended to obtain constant-distortion embeddings of arbitrary simply-connected surfaces of non-positive curvature into L_1.[1]

We note that embeddings of various hyperbolic spaces have been previously considered. We refer to [1], [3]–[5], [15]. However, none of the previous results captures L_1 embeddings of arbitrary non-positively curved planar metrics. In fact, it was conjectured by Chepoi [9] that any non-positively curved planar metric admits a constant-distortion embedding into L_1. Theorem I.2 resolves precisely this conjecture. Prior to our work, this was known only for some very special cases [1].

D. Preliminaries

We now review some basic definitions and notions which appear throughout the paper.

Graphs: Let G, and let $S \subseteq V(G)$. We denote by $G[S]$ the subgraphs of G induced by S, i.e. $G[S] = \left(S, E(G) \cap \binom{S}{2}\right)$. We will consider graphs with every edge having a non-negative length. We say that a graph is *unweighted* if all of its edges have unit length. Let $\mathsf{diam}(G)$ denote the diameter of G, i.e. $\mathsf{diam}(G) = \max_{x,y \in V(G)} d_G(x, y)$. We refer to a path between two vertices $x, y \in V(G)$ as a x-y path.

Cuts and L_1 embeddings: A cut of a graph G is a partition of $V(G)$ into $(S, \bar{S})$—we sometimes refer to a subset $S \subseteq V$ as a cut as well. A cut gives rise to a pseudometric; using indicator functions, we can write the cut pseudometric as $\rho_S(x, y) = |\mathbf{1}_S(x) - \mathbf{1}_S(y)|$. A central fact is that embeddings of finite metric spaces into L_1 are equivalent to sums of positively weighted cut metrics over that set (for a simple proof of this see [11]).

A *cut measure on G* is a function $\mu : 2^V \to \mathbb{R}_+$ for which $\mu(S) = \mu(\bar{S})$ for every $S \subseteq V$. Every cut measure gives rise to an embedding $f : V \to L_1$ for which

$$\|f(u) - f(v)\|_1 = \int |\mathbf{1}_S(u) - \mathbf{1}_S(v)| \, d\mu(S), \qquad (2)$$

[1]This connection was pointed out by James R. Lee.

where the integral is over all cuts $(S, \bar{S})$. Conversely, to every embedding $f : V \to L_1$, we can associate a cut measure μ such that (2) holds.

Non-positively curved spaces: We will describe our proof using the definition of non-positive curvature in the sense of Busemann. We give here a brief overview of some of the relevant terminology, and we refer the reader to [22], [24] for a more detailed exposition. A metric space (X, d) is called *geodesic* if for every pair of points there exists a geodesic joining them. We say that (X, d) is non-positively curved, if for any pair of affinely parameterized geodesics $\gamma : [a, b] \to X$, $\gamma' : [a', b'] \to X$, the map $D_{\gamma,\gamma'} : [a, b] \times [a', b'] \to \mathbb{R}$ defined by

$$D_{\gamma,\gamma'}(t, t') = d(\gamma(t), \gamma'(t))$$

is convex.[2] As we show, this property is sufficient to obtain constant-distortion embeddings of simply-connected surfaces into L_1.

Lipschitz partitions: Let (X, d) be a metric space. A distribution $\mathcal{F}$ over partitions of X is called (β, Δ)-*Lipschitz* if every partition in the support of $\mathcal{F}$ has only clusters of diameter at most Δ, and for every $x, y \in X$,

$$\Pr_{C \in \mathcal{F}}[C(x) \neq C(y)] \leq \beta \cdot \frac{d(x, y)}{\Delta}.$$

We denote by $\beta_{(X,d)}$ the infimum β such that for any $\Delta > 0$, the metric (X, d) admits a (Δ, β)-Lipschitz random partition, and we refer to $\beta_{(X,d)}$ as the *modulus of decomposability* of (X, d). The following theorem is due to Rao [23] (see also Klein, Plotkin, and Rao [14]).

Theorem I.3 ([23])**.** *For any planar graph G, we have $\beta_{(V(G),d_G)} = O(1)$.*

Stochastic embeddings: A mapping $f : X \to Y$ between two metric spaces (X, d) and (Y, d') is *non-contracting* if $d'(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$. If (X, d) is any finite metric space, and $\mathcal{Y}$ is a family of finite metric spaces, we say that (X, d) *admits a stochastic D-embedding into $\mathcal{Y}$* if there exists a random metric space $(Y, d') \in \mathcal{Y}$ and a random non-contracting mapping $f : X \to Y$ such that for every $x, y \in X$,

$$\mathbb{E}\left[d'(f(x), f(y))\right] \leq D \cdot d(x, y). \tag{3}$$

[2]We note that this notion of non-positively curved planar metrics is equivalent up to constant distortion to slightly different definitions used elsewhere. In [1], [10] an unweighted planar graph G with a fixed drawing into the plane is considered to be non-positively curved if the following holds: Assign to each inner face of G having k edges the geometry of the regular Euclidean k-gon. Then, the resulting 2-dimensional piecewise Euclidean complex is a $CAT(0)$ space (see [12]). This realization of G as a simlpy-connected surface is a constant-distortion embedding of its shortest-path metric into a planar metric which is non-positively in our sense. Conversely, every planar metric M, which is non-positively curved in our sense, embeds with constant distortion into an unweighted planar graph G satisfying the above condition.

The infimal D such that (3) holds is the *distortion of the stochastic embedding.* For a graph G and a graph family $\mathcal{F}$ we write $G \overset{D}{\rightsquigarrow} \mathcal{F}$ to denote the fact that G stochastically embeds into a distribution over graphs in $\mathcal{F}$, with distortion D. We also use the notation $G \rightsquigarrow \mathcal{F}$ to denote the fact that $G \overset{D}{\rightsquigarrow} \mathcal{F}$, for some universal constant $D \geq 1$. We will use the following fact.

Lemma I.4. *Let $\mathcal{F}$ be a family of graphs, such that every $H \in \mathcal{F}$ admits an embedding into L_1 with distortion at most $\alpha \geq 1$. Let G be a graph, such that $G \overset{\beta}{\rightsquigarrow} \mathcal{F}$, for some $\beta \geq 1$. Then, G admits an embedding into L_1 with distortion at most $\alpha\beta$.*

Let G be a graph, and let $A \subseteq V(G)$. The *dilation* of A is defined to be $\mathsf{dil}_G(A) = \max_{u,v \in V(G)} d_{G[A]}(u, v)/d_G(u, v)$. For two graphs G, G', a *1-sum* of G with G' is a graph obtained by taking two disjoint copies of G and G', and identifying a vertex $v \in V(G)$ with a vertex $v' \in V(G')$. For a graph family $\mathcal{X}$, we denote by $\oplus_1 \mathcal{X}$ the closure of $\mathcal{X}$ under 1-sums.

Lemma I.5 (Peeling lemma [18])**.** *Let G be a graph, and $A \subseteq V(G)$. Let $G' = (V(G), E')$ be a graph with $E' = E(G) \setminus E(G[A])$, and let $\beta = \beta_{(V,d_{G'})}$ be the corresponding modulus of decomposability. Then, there exists a graph family $\mathcal{F}$ such that $G \overset{D}{\rightsquigarrow} \mathcal{F}$, where $D = O(\beta \cdot \mathsf{dil}_G(A))$, and every graph in $\mathcal{F}$ is a 1-sum of isometric copies of the graphs $G[A]$ and $\{G[V \setminus A \cup \{a\}]\}_{a \in A}$.*

E. Organization

The rest of the paper is organized as follows. In Section II we show how to embed an arbitrary non-positively curved planar metric into an unweighted graph of special structure, called a *funnel*. In Section III we show how to stochastically embed a funnel into a distribution over simpler graphs, called *pyramids*. In Section IV we introduce some of the machinery that we will use when defining our embedding into L_1. More specifically, we describe the basic operation that will allow to gradually modify a cut when computing our embedding. Using this machinery, we describe our embedding in Section V. Finally, in Section VI we prove that the constructed embedding has constant distortion.

Due to lack of space, some proofs are omitted from the present extended abstract. A full version is available on ArXiv: http://arxiv.org/abs/1304.7512.

II. A CANONICAL REPRESENTATION OF NON-POSITIVELY CURVED PLANAR METRICS

In this section we show that non-positively curved planar metrics can be embedded with constant-distortion into a certain type of unweighted planar graphs that we call *funnels*. Intuitively, a funnel is obtained by taking the union of a tree having all its leaves at the same level, with a collection of

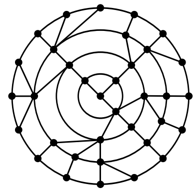


Figure 1. A funnel.

cycles, where every cycle spans all the vertices in a single layer of the tree.

Definition II.1 (Funnel)**.** *Let G be an unweighted planar graph, and let $v \in V(G)$. We say that G is a* funnel *with* basepoint *v if the following conditions are satisfied:*

(1) There exists a collection of pairwise vertex-disjoint cycles $C_1, \ldots, C_\Delta \subset G$, such that $V(G) = \bigcup_{i=1}^{\Delta} V(C_i)$. For notational convenience, we allow a cycle C_i to consist of a single vertex, in which case it has no edges. Moreover, we have $V(C_1) = \{v\}$. We refer to each C_i as a layer *of G.*

(2) For every $i \in \{2, \ldots, \Delta - 1\}$, the graph $G \setminus V(C_i)$ has exactly two connected components, one with vertex set $\bigcup_{j=1}^{i-1} V(C_j)$, and another with vertex set $\bigcup_{j=i+1}^{\Delta} V(C_j)$.

(3) For every $i \in \{2, \ldots, \Delta\}$, every $u \in V(C_i)$ has exactly one neighbor $u' \in V(C_{i-1})$. We refer to u' as the parent *of u. In particular, v is the parent of all vertices in $V(C_2)$.*

(4) For every $i \in \{1, \ldots, \Delta - 1\}$, every $w \in V(C_i)$ has at least one neighbor $w' \in V(C_{i+1})$. We refer to every such w' as a child *of w.*

Let R be a path in G between v, and a vertex $u \in V(C_\Delta)$. We say that R is a ray*. We denote by* $\mathsf{Funnels}$ *the family of all funnel graphs. Figure 1 depicts an example of a funnel.*

We will use the following two facts about metric spaces of non-positive curvature (see e.g. [22]).

Lemma II.2. *Let $(\mathcal{S}, d)$ be a geodesic metric space of non-positive curvature. Let $x^*, x, y \in \mathcal{S}$, and let $\gamma : [0, d(x, y)] \to \mathcal{S}$ be a geodesic between x, and y. Then, the function $f : [0, 1] \to \mathbb{R}$, with $f(t) = d(x^*, \gamma(t))$ is convex.*

Lemma II.3. *Let $(\mathcal{S}, d)$ be a geodesic metric space of non-positive curvature, and let $x^*, x, y \in \mathcal{S}$. Let $\gamma_x : [0, d(x^*, x)] \to \mathcal{S}$ be a geodesic between x^*, and x, and let $\gamma_y : [0, d(x^*, y)] \to \mathcal{S}$ be a geodesic between x^*, and y. Then, the function $f : [0, 1] \to \mathbb{R}$, with $f(t) = d(\gamma_x(t), \gamma_y(t))$ is non-decreasing.*

Recall that for a metric space (X, d), and some $r > 0$, an *r-net* in (X, d) is a maximal subset $X' \subseteq X$ such that for any $x, y \in X'$, we have $d(x, y) \geq r$.

Lemma II.4 (Funnel representation)**.** *Let $\mathcal{S}$ be a simply-connected surface, and let d be a non-positively curved metric on $\mathcal{S}$. Let $X \subset \mathcal{S}$ be a finite set of points. Then, (X, d) admits an embedding into a funnel with constant distortion.*

Proof: By scaling d, we may assume w.l.o.g. that the minimum distance in (X, d) is at least 1 (note that scaling d results into a metric which is still of non-positive curvature). Let $x^* \in \mathcal{S}$ be an arbitrary point. For any $x \in \mathcal{S}$, let $\gamma(x)$ denote the unique geodesic between x, and x^*. Let $r = 1/8$. For any integer $i \geq 0$, let $D_i = \{x \in \mathcal{S} : d(x^*, x) \leq ir\}$. Since d is non-positively curved, we have that for every i, the set D_i is a disk (see e.g. [22]). Let Γ_i be the cycle in $\mathcal{S}$ bounding D_i. Let $\Delta = \min\{i \in \mathbb{N} : X \subset D_i\}$. Let N_Δ be an r-net in Γ_Δ. Note that since $(\mathcal{S}, d)$ is non-positively curved, there exists a unique geodesic between any pair of points. This implies that the subspace

$$T = \bigcup_{x \in N_\delta} \gamma(x)$$

is a (simplicial) tree. For every $i \in \{0, \ldots, \Delta - 1\}$, we define an r-net N_i of Γ_i as follows. Suppose that N_{i+1} is already defined. Let Y'_i be the set of points $p \in N_\Delta$ such that $\gamma(p)$ intersects N_{i+1}. Let $N'_i = \Gamma_i \cap (\bigcup_{p \in Y'_i} \gamma(p))$. Note that for any $x \in \Gamma_i$, there exists $y \in N'_i$ such that $d(x, y) < r$. Therefore, we can set to be a maximal subset $N_i \subseteq N'_i$, such that N_i is an r-net. This concludes the definition of the sequence of subsets $N_0, \ldots, N_\Delta$. Note that $N_0 = \{x^*\}$.

We define a graph G, with $V(G) = \bigcup_{i=0}^{\Delta} N_i$. The set of edges $E(G)$ is defined as follows. For every $i \in \{1, \ldots, \Delta\}$, we add a unit-length edge $\{x, y\} \in E(G)$ for any two points $x, y \in N_i$, such that x, and y appear consecutively in a clockwise traversal of Γ_i. Moreover, for every $z \in N_i$, let $z' \in N_\Delta$ be such that $z \in \gamma(z')$. Let z'' be the point in the intersection of $\gamma(z')$ with Γ_{i-1}. If $z'' \in N_{i-1}$, then we add the unit-length edge $\{z, z''\}$. Otherwise, let w be the first point in N_{i-1} that we visit in a clockwise traversal of Γ_{i-1} starting from z''. We add the unit-length edge edge $\{z, w\}$. This concludes the definition of the graph G. One can check that G is a funnel with basepoint x^*.

We can now define an embedding $f : X \to V(G)$, by mapping every points $x \in X$ to its nearest neighbor in $V(G)$. It remains to verify that f has constant distortion. Observe that the set $V(G)$ contains a $2r$-net in D_Δ, and therefore for any $x \in X$, we have $d(x, f(x)) < 2r$. Since the minimum distance in X is at least 1, this implies that f is an injection, and for any $x, y \in X$, we have $d(x, y) = \Theta(d(f(x), f(y)))$. It therefore suffices to show that for any $x, y \in V(G)$, we have $d_G(x, y) = \Theta(d(x, y))$.

We first show that for any $x, y \in V(G)$, we have $d_G(x, y) = \Omega(d(x, y))$. To that end, it suffices to show that for any edge $\{x, y\} \in E(G)$, we have $d(x, y) = O(d_G(x, y)) = O(1)$.

We consider first case where there exists $i \in \{1, \ldots, \Delta\}$

such that $x, y \in N_i$, and x, y are consecutive in Γ_i. Let α be the arc of Γ_i between x, and y, that does not contain any other points in N_i. By the triangle inequality, there exists $z \in \alpha$, such that $d(x,z) \geq d(x,y) \geq 2$, and $d(y,z) \geq d(x,y) \geq 2$. Since N_i is an r-net in Γ_i, it follows that there exists $z' \in N_i$, such that $d(z,z') < r$. Let β be the geodesic between z, and z'. The arc β intersects either $\gamma(x)$, or $\gamma(y)$. Assume w.l.o.g. that it intersects $\gamma(x)$ at some points z''. By lemma II.2 we have that as we travel along β, the distance to x^* is a convex function. This implies that $d(x,z'') \leq d(z,z')$. We conclude that $d(x,y) \leq 2d(x,z) \leq 2(d(x,z'')+d(z'',z)) \leq 2(d(x,z'')+d(z',z)) \leq 4d(z,z') \leq 4r = O(1)$.

Next, we consider the case where $x \in N_i$, and $y \in N_{i+1}$, for some $i \in \{0,\ldots,\Delta\}$. Let y' be the point where $\gamma(y)$ intersects Γ_i. Arguing as above, we have that $d(y',x) = O(1)$. Therefore, $d(x,y) \leq d(y,y')+d(y',x) \leq r+O(1) = O(1)$. This concludes that proof that for any edge $\{x,y\} \in E(G)$, we have $d(x,y) = O(1)$, and therefore for any $x, y \in V(G)$, we have $d_G(x,y) = \Omega(d(x,y))$.

It remains to show that for any $x, y \in V(G)$, we have $d_G(x,y) = O(d(x,y))$. We consider first the case where there exists $i \in \{1,\ldots,\Delta\}$, such that $x, y \in N_i$ (the case $i = 0$ is trivial since N_0 contains only x^*). Let β be a geodesic between x, and y. By lemma II.2, we have $\beta \subset D_i$. Let x' be the unique point in $\gamma(x) \cap \Gamma_{i-1}$, and let y' be the unique point in $\gamma(y) \cap \Gamma_{i-1}$. By lemma II.3 we have $d(x',y') \leq d(x,y)$. Let x'' be the parent of x, and let y'' be the parent of y in G. Let $x' = z_1,\ldots,z_k = y'$ be the points in N_{i-1} that appear between x', and y' along Γ_{i-1}. For any $i \in \{1,\ldots,k\}$, pick a child w_i of z_i, with $w_1 = x$, and $w_k = y$. For any $i \in \{1,\ldots,k\}$, the curve β intersects $\gamma(w_i)$. By the above discussion we have that the distance between any two such consecutive intersection points is $\Omega(1)$. Therefore, $d(x,y) = \mathsf{len}(\beta) = \Omega(k)$. The x-y path in G that visits the vertices $xz_1\ldots z_k y$ in this order has length $k+2$, and therefore $d_G(x,y) = O(d(x,y))$.

Next, we consider the case where there exists $i \in \{1,\ldots,\Delta\}$, such that $x \in N_i$, and $y \in N_{i-1}$. This case is identical to the case above, by replacing y with y''. We therefore also obtain $d_G(x,y) = O(d(x,y))$ in this case.

Finally, we consider the case of arbitrary points $x, y \in V(G)$. Let β be the geodesic between x, and y. The curve β can be decomposed into consecutive segments $\beta_1,\ldots,\beta_k$, such that every such segment is contained in (the closure of) $D_i \setminus D_{i-1}$, for some $i \in \{1,\ldots,\Delta\}$. Consider such a segment β_i. There exists $j, \ell \in \{0,\ldots,\Delta\}$, with $|j-\ell| \leq 1$, and such that $x_i \in \Gamma_j$, and $y_i \in \Gamma_\ell$. Let x_i' be the nearest neighbor of x_i in N_j, and let y_i' be the nearest neighbor of y_i in N_ℓ. Since N_j is a $O(1)$-net for Γ_j, and N_ℓ is a $O(1)$-net for Γ_ℓ, we have $d(x_i',y_i') \leq d(x_i,y_i)+O(1) = O(d(x_i,y_i))$. By the above analysis we have $d_G(x_i',y_i') = O(d(x_i',y_i'))$. Therefore, we obtain $d_G(x_i,y_i) = O(d(x_i,y_i))$. We conclude that $d_G(x,y) \leq \sum_i d_G(x_i,y_i) = O(\sum_i d(x_i,y_i)) = O(d(x,y))$, as required. ■

III. CUTTING ALONG A RAY

We now show that every funnel admits a constant-distortion stochastic embedding into a distribution over simpler graphs, that we call *pyramids*. Intuitively, a pyramid is obtained by "cutting" a funnel along a ray. The structure of pyramids will simplify the exposition of the embedding into L_1 that we describe in the subsequent sections.

Definition III.1 (Pyramid)**.** *Let G be an unweighted planar graph, let $v \in V(G)$, and let $\Delta \geq 1$ be an integer. We say that G is a* pyramid *with* basepoint *v, and of* depth *Δ if the following conditions are satisfied:*

(1) There exists a collection of pairwise vertex-disjoint paths $P_1,\ldots,P_\Delta \subset G$, with $P_i = u_{i,1}\ldots u_{n_i,i}$, such that $V(G) = \bigcup_{i=1}^{\Delta} V(P_i)$. For notational convenience, we allow a path P_i to consist of a single vertex, in which case it has no edges. Moreover, we have $V(P_1) = \{v\}$. We refer to each P_i as a layer.

(2) For every $i \in \{2,\ldots,\Delta-1\}$, the graph $G \setminus V(P_i)$ has exactly two connected components, one with vertex set $\bigcup_{j=1}^{i-1} V(P_j)$, and another with vertex set $\bigcup_{j=i+1}^{\Delta} V(P_j)$.

(3) For every $i \in \{2,\ldots,\Delta\}$, every $u \in V(P_i)$ has exactly one neighbor u in $V(P_{i-1})$. We refer to this neighbor as the parent *of u. In particular, v is the parent of all vertices in $V(P_2)$.*

(4) For every $i \in \{1,\ldots,\Delta-1\}$, every $w \in V(P_i)$ at least one neighbor w' in $V(P_{i-1})$. We refer to every such w' as a child *of w.*

(5) For any $i \in \{1,\ldots,\Delta-1\}$, and for any $\{u_{i,j}, u_{i+1,j'}\}, \{u_{i,t}, u_{i+1,t'}\} \in E(G)$, we have $j \leq t \iff j' \leq t'$. In other words, the ordering of the vertices in P_{i+1} agrees with the ordering of their parents in P_i.

We say that a path R in G between v, and a vertex $u \in V(P_\Delta)$, is a ray. *We denote by* $\mathsf{Pyramids}$ *the family of all pyramid graphs. Figure 2 depicts an example of a pyramid.*

Definition III.2 (Skeleton of a pyramid)**.** *Let G be a pyramid with basepoint $v \in V(G)$. We define the* skeleton *of G to be a tree T, with $V(T) = V(G)$, with root v, and with*

$$E(T) = \left\{ \{x,y\} \in \binom{V(G)}{2} : x \text{ is the parent of } y \right\}.$$

For any $x, y \in V(G)$, we denote by nca *the nearest common ancestor of x, and y in T. We also define for any $x \in V(G)$,*

$$\mathsf{depth}(x) = d_T(v,x) + 1.$$

Figure 2 depicts an example of a skeleton.

Definition III.3 ($\prec$)**.** *For any $i \in \{1,\ldots,\Delta\}$, for any $u_{i,j}, u_{i,j'} \in V(P_i)$, with $j < j'$, we write $u_{i,j} \prec u_{i,j'}$. Moreover, for any $x, y \in V(G)$, such that x, and y do not*

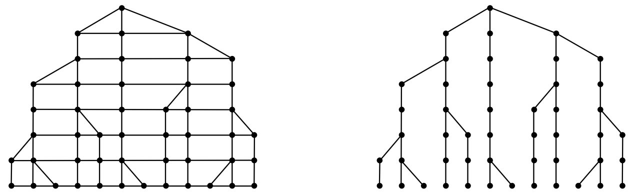


Figure 2. A pyramid (left), and its skeleton (right).

lie on the same ray, let $z = \mathsf{nca}(x, y)$*, and let* x' *(resp.* y'*) be the child of* z *in the* z*-*x *(resp.* z*-*y*) path in* T*. Then, we write* $x \prec y$ *if and only if* $x' \prec y'$*. Finally, for any* $x'', y'' \in V(G)$*, we write* $x'' \preceq y''$ *if and only if either* $x'' \preceq y''$*, or* x''*, and* y'' *lie on the same ray.*

Lemma III.4 (Pyramid representation)**.** *For every funnel* G*, we have* $G \rightsquigarrow \oplus_1\{\mathsf{Pyramids}\}$.

Proof: Let G be a funnel with basepoint $x^* \in V(G)$, and depth Δ. Let R be a ray in G. Replace $R \setminus x^*$ by a $\Delta \times 4$ grid H. Clearly, this results into an embedding of G into a funnel G' with distortion $O(1)$. Let R' be the union of the two central columns of H, and let $A = R' \cup \{x^*\}$. Observe that $\mathsf{dil}_{G'}(A) = 1$. Applying lemma I.5 on G' and the set A, we obtain a stochastic embedding of G' into a distribution of graphs $\mathcal{D}$. Since G' is planar, and $\mathsf{dil}_{G'}(A) = 1$, it follows by Theorem I.3 that the distortion of the resulting stochastic embedding is $O(1)$. Every graph in the support of $\mathcal{D}$ is obtained via 1-sums of $G'[A]$, with $G'[V \setminus A \cup \{a\}]$, for some $a \in A$. The graph $G'[A]$ is a $\Delta \times 2$ grid, with the basepoint x^* connected to the two vertices in the top row, and is therefore a pyramid. For any $a \in A$, the graph $G'[V \setminus A \cup \{a\}]$ is obtained from G' by cutting along a ray, and is therefore also a pyramid. This concludes the proof. ■

IV. Monotone cuts

In this section we describe the family of cuts, that we will use when defining our embedding into L_1. These are cuts that we call *monotone*, and intuitively correspond to sets that only cross every ray at most once. We also describe a specific "shifting" operation that will allow us to modify a cut in order to adapt to the finer geometry of a given space.

Definition IV.1 (Monotone cut)**.** *Let* G *be a pyramid with basepoint* $v \in V(G)$*, and let* $S \subseteq V(G)$*. We say that* S *is* v-monotone *(or monotone when* v *is clear from the context) if* $v \in S$*, and for any ray* R *in* G*,* $R \cap S$ *is a prefix of* R*. In particular, this implies that* $G[S]$ *is a connected subgraph (see Figure 3).*

Definition IV.2 (Boundary of a monotone cut)**.** *Let* $S \subseteq V(G)$ *be a monotone cut. We define the* vertex boundary *of* S*, denoted by* $\partial_V S$*, to be the set of all* $u \in S$*, such that all children of* u *are not in* S*. We also define the* edge boundary *of* S*, denoted by* $\partial_E S$*, to be* $\partial_E S = \{\{x, y\} \in E(G) : x, y \in \partial_V S$ *and* $\mathsf{depth}(x) = \mathsf{depth}(y)\}$*. Finally, we define the graph* $\partial S = (\partial_V S, \partial_E S)$ *(see Figure 3).*

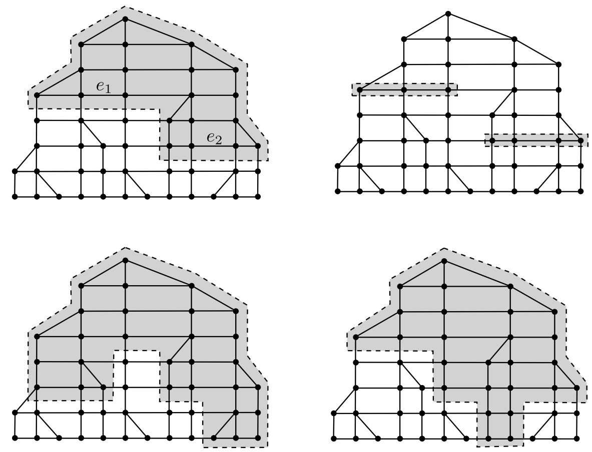


Figure 3. A monotone cut (top-left), its boundary (top-right), its odd $(2, \{e_1, e_2\})$-shift (bottom-left), and its even $(2, \{e_1, e_2\})$-shift (bottom-right).

Definition IV.3. *Let* G *be a pyramid, let* T *be the skeleton of* G*. Let* $u \in V(G)$*, and* $r \geq 0$*. Then, we denote by* $\widetilde{N}(u, r)$ *the set of all vertices* $w \in V(G)$*, such that* u *is an ancestor of* w *in* T*, and* $d_T(u, w) \leq r$*.*

Definition IV.4 (Odd/even shift of a monotone cut)**.** *Let* $S \subseteq V(G)$ *be a monotone cut, let* $r > 0$*, and* $Z \subseteq \partial_E S$*. Let* $Z = \{\{x_i, y_i\}\}_{i=1}^k$*, with*

$$x_1 \prec y_1 \preceq x_2 \prec y_2 \preceq \ldots \preceq x_k \prec y_k.$$

Let $\{V_i\}_{i=1}^{k+1}$ *be a decomposition of* $\partial_V S$*, with* $V_1 = \{u \in \partial_V S : u \preceq x_1\}$*,* $V_{k+1} = \{u \in \partial_V S : y_k \preceq u\}$*, and for any* $i \in \{2, \ldots, k\}$*,* $V_i = \{u \in \partial_V S : y_i \preceq u \preceq x_{i+1}\}$*. We define a partition* $\partial_V S = V_{odd} \cup V_{even}$*, by setting* $V_{odd} = \bigcup_{i=1}^{\lceil t/2 \rceil} V(Q_{2i-1})$*, and* $V_{even} = \bigcup_{i=1}^{\lfloor t/2 \rfloor} V(Q_{2i})$*. We define the* odd (r, Z)-shift *of* S *to be the cut* S_{odd} *given by*

$$S_{odd} = S \cup \bigcup_{u \in V_{odd}} \widetilde{N}(u, r).$$

Similarly, we define the even (r, Z)-shift *of* S *to be the cut* S_{even} *given by*

$$S_{even} = S \cup \bigcup_{u \in V_{even}} \widetilde{N}(u, r).$$

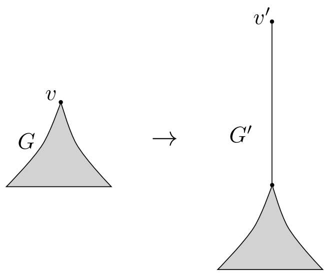
We say that a cut S' *is a* (r, Z)*-shift of* S*, if it is either the odd, or the even* (r, Z)*-shift of* S *(see Figure 3 for an example).*

V. The embedding

In this section we present a constant-distortion embedding of pyramids into L_1. Combining with lemmas I.4, II.4, &

III.4, this implies that every planar metric of non-positive curvature embeds into L_1 with constant distortion.

Let G be a pyramid, with basepoint $v \in V(G)$. Let $\Delta \geq 1$ be the depth of G, and let $\delta = \lceil \log \Delta \rceil$. It will be convenient for our exposition to isometrically embed G into a larger pyramid G', with depth $\Delta' = O(\Delta)$, as follows. The pyramid G' contains a copy of G, and a new basepoint v', that is connected to the basepoint v of G via a path of length 2Δ, resulting into a pyramid of depth $\Delta' = 3\Delta$.



We will then compute an embedding for G', and prove that its restriction on G has small distortion. We remark that our embedding will have unbounded distortion for points in G' close to v' (more precisely, pairs of vertices at distance ε from v', will be distorted by a factor of $O(1/\varepsilon)$). However, this does not affect our result, since we only case about distances in G, which lies far from v'.

Definition V.1 (Evolution of a monotone cut). *Let $r > 0$, and let $S \subseteq V(G')$ be a monotone cut. The r-*evolution *of S is a probability distribution $\mathcal{D}$ over monotone cuts, defined by the following random process. Let $Y = \{\{x, y\} \in \partial_E S : \mathsf{depth}(x) - \mathsf{depth}(\mathsf{nca}(x, y)) \in [r, 6r)\}$. Pick a random subset $Y' \subseteq Y$, by choosing every $e \in Y$ independently, with probability $1/r$. We probability $1/2$, let S' be the odd (r, Y')-shift of S, and otherwise let S' be the even (r, Y')-shift of S. The resulting random cut S' defines the distribution $\mathcal{D}$.*

Let $\mathcal{M}$ be the set of all v-monotone cuts in G'. We inductively define a sequence $\{\mu_i\}_{i=0}^{\delta+1}$, where each μ_i is a probability distribution over $\mathcal{M}$. We define μ_0 as follows. Let $P_1, \ldots, P_{\Delta'}$ be the layers of G'. For any $j \in \{1, \ldots, \Delta'\}$, let $X_i = \bigcup_{t=1}^{j} V(P_t) = \mathsf{ball}(v', i)$. Let μ_0 be the uniform distribution over the collection of cuts $X_1, \ldots, X_{\Delta'}$.

For any $i \geq 0$, given μ_i, we inductively define μ_{i+1} via the following random process: We first pick a random cut S_i according to μ_i. Let $\mathcal{D} = \mathcal{D}(S_i)$ be the $\Delta/3^i$-evolution of S_i. We pick a random cut S_{i+1} according to $\mathcal{D}$. The resulting random variable S_{i+1} defines the probability distribution μ_{i+1}.

We define the embedding f induced by the probability distribution μ_δ, and the embedding f_0 induced by the probability distribution μ_0. Finally, we set the resulting embedding to be

$$g = f \oplus f_0,$$

i.e. the concatenation of the embeddings f, and f_0. In the next section we show that the distortion of g restricted on G is bounded by some universal constant.

VI. Distortion analysis

We now analyze the distortion of the embedding g constructed in the previous section.

A. Distortion of vertical pairs of points

Lemma VI.1. *Let $u \in V(G)$, with $\mathsf{depth}(u) < \Delta'$, and let $i \in \{1, \ldots, \delta\}$. Then, $\Pr[u \in \partial_V S_i] = 1/\Delta'$.*

Proof: The proof is by induction on i. For $i = 0$, the assertion holds since μ_0 is the uniform distribution over the cuts $X_1, \ldots, X_{\Delta'}$. Suppose next that $i > 0$. Let $r = \Delta/3^{i-1}$, and let u' be the ancestor of u in T, with $d_T(u, u) = r$. Fix some S_{i-1} in the support of μ_{i-1}, and suppose that S_i is sampled from the r-evolution of S_{i-1}. This means that we first sample a set of edges Y, and for any such Y we set S_i to be the odd (r, Y)-shift of S_{i-1} with probability $1/2$, or otherwise we set S_i to be the even (r, Y)-shift of S_{i-1}. Therefore, we have have $u \in \partial_V S_i$, only if either $u \in \partial_V S_{i-1}$, or $u' \in \partial_V S_{i-1}$. Conditioned on either of these two events, and for any Y, exactly one of the odd/even shifts of S_{i-1} has u in its boundary. This implies that $\Pr[u \in \partial_V S_i] = \Pr[u \in \partial_V S_i | u \in \partial_V S_{i-1}] \cdot \Pr[u \in \partial_V S_{i-1}] + \Pr[u \in \partial_V S_i | u' \in \partial_V S_{i-1}] \cdot \Pr[u' \in \partial_V S_{i-1}] = \frac{1}{\Delta'} \cdot \frac{1}{2} + \frac{1}{\Delta'} \cdot \frac{1}{2} = 1/\Delta'$, as required. ■

Lemma VI.2. *Let $x, y \in V(G)$, such that x, y lie on the same ray. Then, $\|f(x) - f(y)\|_1 = d_G(x, y)/\Delta$.*

Proof: Let R be the ray containing both x, and y. Let R' be the subpath of R between x, and y, including x, and excluding y. By the monotonicity of $\mathcal{S}_\delta$, it follows that $\mathbf{1}_{S_\delta}(x) \neq \mathbf{1}_{S_\delta}(y)$, if and only if there exists $z \in V(R')$, such that $z \in \partial_V S_\delta$. Since these events are disjoint for different z, we obtain by lemma VI.1 that $\|f(x) - f(y)\|_1 = \Pr[\mathbf{1}_{S_\delta}(x) \neq \mathbf{1}_{S_\delta}(y)] = |V(R')|/\Delta' = d_G(x, y)/\Delta'$, as required. ■

B. Distortion of horizontal pairs of points

We now bound the distortion on pairs of vertices $x, y \in V(G)$ that lie on the same layer of G', i.e. such that $\mathsf{depth}(x) = \mathsf{depth}(y) = h$. Let $d_G(x, y) = L$. Let also $h' = \mathsf{depth}(\mathsf{nca}(x, y))$. We assume w.l.o.g. that $x \preceq y$. Let P be the subpath of P_h between x, and y.

Let

$$E_{\text{top}} = \{\{z, w\} \in E(P) : \mathsf{depth}(\mathsf{nca}(z, w)) \leq h - L/2\},$$

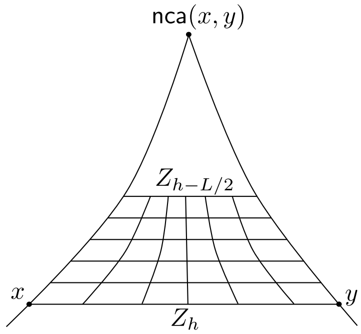
and

$$E_{\text{bottom}} = E(P) \setminus E_{\text{top}}.$$

Lemma VI.3. *$|E_{top}| \leq L$.*

Proof: Suppose, to the contrary, that $|E_{\text{top}}| > L$. For any $i \in \{h - L/2, \ldots, h\}$, let Z_i be the subpath of P_i

between the ancestor of x, and the ancestor of y in P_i. For any $e = \{z, w\} \in E_{\text{top}}$, with $z \prec w$, let R_z be a ray containing z, and let W_e be the subpath of R_e contained between $P_{h-L/2}$, and P_h.



The union of all these paths $(\bigcup_i Z_i) \cup (\bigcup_e W_e)$ forms a $(L/2+1) \times L$ grid minor in G', with x, and y being the bottom-left, and bottom-right vertices respectively. Since the x-y shortest path in G' is contained in $\mathsf{ball}(v', h)$, this implies that $d_G(x, y) > L$, which is a contradiction. ■

Let H be the subgraph of G induced on the set of vertices

$$V(H) = \{u \in V(G) : h' \leq \mathsf{depth}(u) \leq h \text{ and } x \preceq u \preceq y\}.$$

Definition VI.4 (Straight cut)**.** *Let $i \in \{1, \dots, \delta\}$, and $j \in \{1, \dots, \Delta'\}$. We say that S_i is j-*straight *if $\partial S_i \cap H \subseteq P_j$.*

Let $e = \{z, w\} \in E(P)$. We say that an edge $e' = \{z', w'\} \in E(G)$ is an *ancestor* of e, if z' is an ancestor of z in T, w' is an ancestor of w in T, and $\mathsf{depth}(z') = \mathsf{depth}(w')$.

Definition VI.5 (Bend)**.** *Let $e \in E(P)$. We say that e* bends *S_i, if the following events happen.*

(1) There exists $j \in \{1, \dots, \Delta'\}$, such that S_i is j-straight.
(2) Let $Y \subseteq \partial_E S_i$, such that S_{i+1} is the (r, Y)-shift of S_i, for some $r > 0$. Then, there exists an ancestor of e in Y.

Lemma VI.6. *Let $j \in \{h', \dots, h\}$, and let $i \in \{1, \dots, \delta\}$. Then, $\Pr[S_i \text{ is } j\text{-straight}] \leq 1/\Delta'$.*

Proof: Let z be an arbitrary vertex in $V(P_j) \cap V(H)$. Clearly, S_i can only be j-straight if $z \in \partial_V S_i$. Therefore, by lemma VI.1 we obtain $\Pr[S_i \text{ is } j\text{-straight}] \leq \Pr[z \in \partial_V S_i] = 1/\Delta'$, as required. ■

For any edge $e = \{z, w\} \in E(P)$, and for any $i \in \{1, \dots, \delta\}$, let $\mathcal{E}(e, i)$ be the conjunction of the following two events:

- $\mathcal{E}_1(e, i)$: There exists j, such that the following event, denoted by $\mathcal{E}_1(e, i, j)$, holds: Intuitively, the event $\mathcal{E}_1(e, i, j)$ describes a necessary condition such that a bend of S_i can potentially lead to a cut S_δ that separates x, and y. Formally, we have that S_i is j-straight, with
$$\Delta/3^i \leq j - \mathsf{depth}(\mathsf{nca}(z, w)) < 6\Delta/3^i, \quad (4)$$
and
$$h - j \leq 2\Delta/3^i. \quad (5)$$
- $\mathcal{E}_2(e, i)$: e bends S_i.

Lemma VI.7. *Suppose that $\mathbf{1}_{S_\delta}(x) \neq \mathbf{1}_{S_\delta}(y)$. Then, there exists $e \in E(P)$, and $i \in \{1, \dots, \delta\}$, such that the event $\mathcal{E}(e, i)$ occurs.*

Proof: Recall that by the definition of μ_0, the cut S_0 is j_0-straight, for some $j_0 \in \{1, \dots, \Delta'\}$. Since $\mathbf{1}_{S_\delta}(x) \neq \mathbf{1}_{S_\delta}(y)$, it follows that for all $j_\delta \in \{1, \dots, \Delta'\}$, the cut S_δ is not j_δ-straight. Let $i^* \in \{0, \dots, \delta - 1\}$ be the smallest integer such that for all $j \in \{1, \dots, \Delta'\}$, the cut S_{i^*+1} is not j-straight. This means that S_{i^*} is j^*-straight, for some $j^* \in \{1, \dots, \Delta'\}$. Therefore, there exists $e = \{z, w\} \in E(P)$, such that e bends S_{i^*}, which means that the event $\mathcal{E}_2(e, i^*)$ occurs. It suffices to show that the event $\mathcal{E}_1(e, i^*, j^*)$ also occurs. We have established that S_{i^*} is j^*-straight, so its remains to show that (4) & (5) hold. Condition (4) follows immediately form the fact that e bends S_{i^*}, and S_{i^*+1} is the $(Y, \Delta/3^{i^*})$-shift of S_{i^*}, with $e \in Y$. Since S_{i^*} is j^*-straight, we have $S_{i^*} \subseteq \mathsf{ball}(v', j^*)$. The cut S_δ is obtained from S_{i^*} via a sequence of (Y, r)-shifts, with exponentially decreasing values of r. This implies $S_\delta \subseteq \mathsf{ball}(v', t)$, for some $t \leq j^* + \sum_{i=i^*}^{\delta} \Delta/3^i < j^* + 2\Delta/3^*$. Since $\mathbf{1}_{S_\delta}(x) \neq \mathbf{1}_{S_\delta}(y)$, we have $t > h$, and therefore $h - j^* < 2\Delta/3^{i^*}$, which implies (5), and concludes the proof. ■

Lemma VI.8 (Expansion of horizontal pairs)**.** *Let $x, y \in V(G)$, such that $\mathsf{depth}(x) = \mathsf{depth}(y)$. Then, $\|f(x) - f(y)\|_1 = O(d(x, y)/\Delta')$.*

Proof: Let $\mathcal{E}_{\text{top}}$ denote the event that there exists $e \in E_{\text{top}}$, and $i \in \{1, \dots, \delta\}$, such that $\mathcal{E}(e, i)$ occurs. Similarly, let $\mathcal{E}_{\text{bottom}}$ denote the event that there exists $e \in E_{\text{bottom}}$, and $i \in \{1, \dots, \delta\}$, such that $\mathcal{E}(e, i)$ occurs. By lemma VI.7 we have

$$\begin{aligned}\|f(x) - f(y)\|_1 &= \Pr[\mathbf{1}_{S_\delta}(x) \neq \mathbf{1}_{S_\delta}(x)] \\ &\leq \Pr[\mathcal{E}_{\text{top}}] + \Pr[\mathcal{E}_{\text{bottom}}].\end{aligned}$$

Let us bound the two latter quantities separately.

We first bound $\Pr[\mathcal{E}_{\text{top}}]$. Let $e = \{z, w\} \in E_{\text{top}}$, and $i \in \{1, \dots, \delta\}$. Let $h' = \mathsf{depth}(\mathsf{nca}(\{z, w\}))$. Recall that by the definition of $\mathcal{E}_1(e, i, j)$, in order for $\mathcal{E}_1(e, i, j)$ to occur for some j, we must have by (4) that $j - h' \leq 6\Delta/3^i$, and by (5) that $h - j \leq 2\Delta/3^i$. We therefore obtain that $h - h' = h - j + j - h' = O(\Delta/3^i)$. Note that S_{i+1} is the (Y, r)-shift of S_i, for some $r = \Delta/3^i$, and for random some $Y \subseteq \partial_E S_i$. In order for the edge e to bend S_i, its must be the case that its unique ancestor (if it exists) in $\partial_E S_i$ is chosen in Y. Every edge in chosen in Y with probability at most $1/r$. Therefore, for any j, and for any i, we have

$$\Pr[\mathcal{E}_2(e, i) | \mathcal{E}_1(e, i, j)] \leq 3^i/\Delta.$$

Moreover, $\mathcal{E}_1(e,i,j)$ can occur only if $j \in \{h', \ldots, h\}$. For each such value $j \in \{h', \ldots, h\}$, and for any i, we have by lemma VI.6 that $\Pr[\mathcal{E}_1(e,i,j)] = O(1/\Delta)$. To summarize, we have

$$\begin{aligned}
\Pr[\mathcal{E}_{\text{top}}] &\leq \sum_{e \in E_{\text{top}}} \sum_{i \in \{1,\ldots,\delta\}} \Pr[\mathcal{E}(e,i)] \\
&\leq \sum_{e \in E_{\text{top}}} \sum_{j \in \{h',\ldots,h\}} \sum_{i \in \{1,\ldots,\delta\}} \\
&\qquad \Pr[\mathcal{E}_2(e,i) | \mathcal{E}_1(e,i,j)] \cdot \Pr[\mathcal{E}_1(e,i,j)] \\
&\leq \sum_{e \in E_{\text{top}}} \sum_{j \in \{h',\ldots,h\}} \sum_{i \in \{1,\ldots,\delta\}} \frac{3^i}{\Delta} O(1/\Delta) \\
&\leq \sum_{e \in E_{\text{top}}} \sum_{j \in \{h',\ldots,h\}} O(1/(h-h')) \cdot O(1/\Delta) \\
&\leq \sum_{e \in E_{\text{top}}} O(1/\Delta) \\
&= O(|E_{\text{top}}|/\Delta) \\
&= O(L/\Delta'). \qquad (6)
\end{aligned}$$

We next bound $\Pr[\mathcal{E}_{\text{bottom}}]$. Let $\{1, \ldots, \delta\}$, $j \in \{1, \ldots, \Delta'\}$, $e \in E_{\text{bottom}}$, such that both $\mathcal{E}_1(e,i,j)$, and $\mathcal{E}_2(e,i)$ occur. As above, let $e = \{z, w\}$, and $h' = \mathsf{depth}(\mathsf{nca}(z,w))$. Then, we must have $h' \leq j \leq h$, which implies $j - h' \leq h - h' \leq L/2$. Since S_{i+1} is the (r, Y)-shift of S_i for some $Y \subseteq E(P)$, with $r = \Delta/3^i$, we obtain that $j - h' \in [r, 6r)$, which implies $3^i \geq 2\Delta/L$. Let R_x be the ray containing x, and let χ be the unique vertex in the intersection of R_x with ∂S_i. Let also χ' be the unique vertex in the intersection of R_x with ∂S_δ. For every $i' \geq i$, the intersection of $\partial S_{i'}$ with R_x moves by at most $\Delta/3^{i'}$ along R_x, and therefore $d_T(\chi, \chi') < 2\Delta/3^{i'} = O(L)$. Since $\chi \in P_j$, and $j \in [h', h]$, it follows that $\mathsf{depth}(\chi)$ can take at most $h' - h + 1$ different values. Therefore, χ' can only lie inside a subpath $R'_x \subseteq R_x$ of length $O(h' - h)$. Applying lemma VI.1, we obtain

$$\begin{aligned}
\Pr[\mathcal{E}_{\text{bottom}}] &\leq \Pr[\chi' \in R'_x] \leq |V(R'_x)|/\Delta' \\
&= O(h' - h)/\Delta' = O(L/\Delta'). \qquad (7)
\end{aligned}$$

Combining (6) & (7) we conclude that $\|f(x) - f(y)\|_1 = O(L/\Delta') = O(d(x,y)/\Delta')$, as required. ∎

We next bound the contraction of f.

Lemma VI.9 (Contraction of horizontal pairs)**.** *Let $x, y \in V(G)$, such that $\mathsf{depth}(x) = \mathsf{depth}(y)$. Then, $\|f(x) - f(y)\|_1 = \Omega(d(x,y)/\Delta')$.*

Due to lack of space, the proof of Lemma VI.9 is given in the full version of the paper.

C. Distortion of general pairs of points

Lemma VI.10 (Embedding pyramids into L_1)**.** *There exists a universal constant $c > 1$, such that every pyramid graph admits an embedding into L_1 with distortion at most c.*

Proof: We will show that the embedding $g = f \oplus f_0$ has constant distortion on G. Let $x, y \in V(G)$. Assume w.l.o.g. that $\mathsf{depth}(x) \geq \mathsf{depth}(y)$. Let R_x be the ray containing x, and let x' be the unique vertex in R_x, with $\mathsf{depth}(x') = \mathsf{depth}(y)$. By lemmas VI.8 & VI.9 we have that there exist universal constants $\alpha > \beta > 0$, such that for any

$$\beta d_G(x', y)/\Delta \leq \|f(x') - f(y)\|_1 \leq \alpha d_G(x', y)/\Delta \qquad (8)$$

Note that

$$\begin{aligned}
d_G(x, x') &= \mathsf{depth}(x) - \mathsf{depth}(x') \\
&= \mathsf{depth}(x) - \mathsf{depth}(y) \geq d_G(x, y). \qquad (9)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|f(x) - f(y)\|_1 &\leq \|f(x) - f(x')\|_1 \\
&\quad + \|f(x') - f(y)\|_1 \qquad (10) \\
&\leq d_G(x,y)/\Delta + \alpha d_G(x', y)/\Delta \qquad (11) \\
&\leq d_G(x,y)/\Delta + \alpha d_G(x', x)/\Delta \\
&\quad + \alpha d_G(x,y)/\Delta \\
&= (\alpha + 1) d_G(x, x')/\Delta + \alpha d_G(x,y)/\Delta \\
&\leq (\alpha + 1) d_G(x,y)/\Delta \\
&\quad + \alpha d_G(x,y)/\Delta \qquad (12) \\
&= (2\alpha + 1) d_G(x,y)/\Delta, \qquad (13)
\end{aligned}$$

where (10) follows by the triangle inequality, (11) by lemma VI.2 & (8), and (12) by (9). By (13) we have

$$\begin{aligned}
\|g(x) - g(y)\|_1 &= \|f(x) - f(y)\|_1 + \|f_0(x) - f_0(y)\|_1 \\
&\leq (2\alpha + 1) d_G(x,y)/\Delta + d_G(x,y)/\Delta \\
&= (2\alpha + 2) d_G(x,y)/\Delta. \qquad (14)
\end{aligned}$$

This bounds the expansion of g. It remains to bound the contraction of g.

Let $\gamma = \frac{\beta}{4(2\alpha+1)}$. Assume first that $d_G(x,y) \geq \gamma d_G(x,y)$. We have

$$\begin{aligned}
\|g(x) - g(y)\|_1 &\geq \|f_0(x) - f_0(y)\|_1 \\
&= d_G(x', y)/\Delta \\
&\geq \gamma d_G(x,y)/\Delta \qquad (15)
\end{aligned}$$

Next, assume that $d_G(x,y) < \gamma d_G(x,y)$. We have

$$\begin{aligned}
\|g(x) - g(y)\|_1 &\geq \|f(x) - f(y)\|_1 \\
&\geq \|f(x') - f(y)\|_1 - \|f(x) - f(x')\|_1 \\
&\geq \beta \frac{d_G(x', y)}{\Delta} - (2\alpha + 1) \frac{d_G(x, x')}{\Delta} \qquad (16) \\
&> (1 - \gamma) \beta \frac{d_G(x,y)}{\Delta} - \gamma (2\alpha + 1) \frac{d_G(x,y)}{\Delta} \\
&> \frac{\beta}{2} \frac{d_G(x,y)}{\Delta} \qquad (17)
\end{aligned}$$

where (16) follows by (8) & (13). Combining (15) & (17), we obtain that for all $x, y \in V(G)$

$$\|g(x) - g(y)\|_1 \geq \frac{\beta}{4(2\alpha+1)} d_G(x,y)/\Delta. \quad (18)$$

From (14) & (18) we conclude that the distortion of g is at most $4(2\alpha+1)(2\alpha+2)/\beta = O(1)$, concluding the proof. ∎

D. Proof of the main result

Combining the above results, we can now prove our main theorem.

Proof: Proof of theorem I.2 Let (X, d) be a planar metric of non-positive curvature. Using lemma II.4, the metric (X, d) admits an embedding into some funnel G with distortion $c_1 = O(1)$. Using lemma III.4 we can find a stochastic embedding of G into a distribution $\mathcal{F}$ over pyramids with distortion $c_2 = O(1)$. By lemma VI.10 every pyramid in the support of $\mathcal{F}$ admits an embedding into L_1 with distortion $c_3 = O(1)$. Combining with lemma I.4 we obtain that G admits an embedding into L_1 with distortion c_2c_3. Therefore (X, d) admits an embedding into L_1 with distortion $\gamma = c_1c_2c_3 = O(1)$, concluding the proof. ∎

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REFERENCES

[1] Hans-Jürgen Bandelt anf Victor Chepoi. Metric graph theory and geometry: a survey. In J.E. Goodman, J. Pach, and R. Pollack, editors, *Surveys on Discrete and Computational Geometry: Twenty Years Later*, volume 453 of *Contemp. Math.*, pages 49–86. 2008.

[2] Yonatan Aumann and Yuval Rabani. An $O(\log k)$ approximate min-cut max-flow theorem and approximation algorithm. *SIAM J. Comput.*, 27(1):291–301 (electronic), 1998.

[3] Mario Bonk and Oded Schramm. Embeddings of gromov hyperbolic spaces. *Geom. Funct. Anal*, 10:266–306.

[4] Sergei Buyalo and Viktor Schroeder. A product of trees as universal space for hyperbolic groups. *arXiv.org*, September 2005.

[5] Sergei Buyalo and Viktor Schroeder. Embedding of hyperbolic spaces in the product of trees. *Geometriae Dedicata*, 113(1):75–93, 2005.

[6] Amit Chakrabarti, Alexander Jaffe, James R. Lee, and Justin Vincent. Embeddings of topological graphs: Lossy invariants, linearization, and 2-sums. In *FOCS*, pages 761–770, 2008.

[7] Chandra Chekuri, Anupam Gupta, Ilan Newman, Yuri Rabinovich, and Alistair Sinclair. Embedding k-outerplanar graphs into l_1. *SIAM J. Discrete Math.*, 20(1):119–136 (electronic), 2006.

[8] Chandra Chekuri, F. Bruce Shepherd, and Christophe Weibel. Flow-cut gaps for integer and fractional multiflows. *J. Comb. Theory, Ser. B*, 103(2):248–273, 2013.

[9] Victor Chepoi. Private communication, 2013.

[10] Victor Chepoi, Feodor F. Dragan, and Yann Vaxès. Distance and routing labeling schemes for non-positively curved plane graphs. *J. Algorithms*, 61(2):60–88, 2006.

[11] Michel Marie Deza and Monique Laurent. *Geometry of cuts and metrics*, volume 15 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 1997.

[12] Mikhail Gromov. Hyperbolic groups. In S.M. Gersten, editor, *Essays in Group Theory*, Mathematical Sciences Research Institute Publications, pages 75–263. Springer London, Limited, 1987.

[13] Anupam Gupta, Ilan Newman, Yuri Rabinovich, and Alistair Sinclair. Cuts, trees and l_1-embeddings of graphs. *Combinatorica*, 24(2):233–269, 2004.

[14] Philip N. Klein, Serge A. Plotkin, and Satish Rao. Excluded minors, network decomposition, and multicommodity flow. In *STOC*, pages 682–690, 1993.

[15] Robert Krauthgamer and James R. Lee. Algorithms on negatively curved spaces. In *FOCS*, pages 119–132, 2006.

[16] James R. Lee and Daniel Poore. On th e 2-sum embedding conjecture. In *SoCG*, 2013.

[17] James R. Lee and Prasad Raghavendra. Coarse differentiation and multi-flows in planar graphs. *Discrete & Computational Geometry*, 43(2):346–362, 2010.

[18] James R. Lee and Anastasios Sidiropoulos. On the geometry of graphs with a forbidden minor. In *STOC*, pages 245–254, 2009.

[19] Tom Leighton and Satish Rao. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. *J. ACM*, 46(6):787–832, 1999.

[20] N. Linial, E. London, and Y. Rabinovich. The geometry of graphs and some of its algorithmic applications. *Combinatorica*, 15(2):215–245, 1995.

[21] Haruko Okamura and Paul D. Seymour. Multicommodity flows in planar graphs. *Journal of Combinatorial Theory, Series B*, 31(1):75 – 81, 1981.

[22] Athanase Papadopoulos. *Metric spaces, convexity and non-positive curvature*. Irma Lectures in Mathematics and Theoretical Physics Series. European Mathematical Society Publishing House, 2005.

[23] Satish Rao. Small distortion and volume preserving embeddings for planar and euclidean metrics. In *Symposium on Computational Geometry*, pages 300–306, 1999.

[24] William P. Thurston and Silvio Levy. *Three-dimensional geometry and topology. 1 (1997)*. Princeton mathematical series. Princeton University Press, 1997.