Representative sets and irrelevant vertices: New tools for kernelization

Stefan Kratsch
Utrecht University
Utrecht, the Netherlands
Email: s.kratsch@uu.nl

Magnus Wahlström
Max-Planck-Institute for Informatics
Saarbrücken, Germany
Email: wahl@mpi-inf.mpg.de

***Abstract*—The existence of a polynomial kernel for Odd Cycle Transversal was a notorious open problem in parameterized complexity. Recently, this was settled by the present authors (Kratsch and Wahlström, SODA 2012), with a randomized polynomial kernel for the problem, using matroid theory to encode flow questions over a set of terminals in size polynomial in the number of terminals (rather than the total graph size, which may be superpolynomially larger).**

In the current work we further establish the usefulness of matroid theory to kernelization by showing applications of a result on representative sets due to Lovász (Combinatorial Surveys 1977) and Marx (TCS 2009). We show how representative sets can be used to give a polynomial kernel for the elusive Almost 2-sat problem (where the task is to remove at most k clauses to make a 2-CNF formula satisfiable), solving a major open problem in kernelization.

We further apply the representative sets tool to the problem of finding irrelevant vertices in graph cut problems, that is, vertices which can be made undeletable without affecting the status of the problem. This gives the first significant progress towards a polynomial kernel for the Multiway Cut problem; in particular, we get a polynomial kernel for Multiway Cut instances with a bounded number of terminals.

Both these kernelization results have significant spin-off effects, producing the first polynomial kernels for a range of related problems.

More generally, the irrelevant vertex results have implications for covering min-cuts in graphs. In particular, given a directed graph and a set of terminals, we can find a set of size polynomial in the number of terminals (a cut-covering set) which contains a minimum vertex cut for every choice of sources and sinks from the terminal set. Similarly, given an undirected graph and a set of terminals, we can find a set of vertices, of size polynomial in the number of terminals, which contains a minimum multiway cut for every partition of the terminals into a bounded number of sets. Both results are polynomial time. We expect this to have further applications; in particular, we get direct, reduction rule-based kernelizations for all problems above, in contrast to the indirect compression-based kernel previously given for Odd Cycle Transversal.

All our results are randomized, with failure probabilities which can be made exponentially small in the size of the input, due to needing a representation of a matroid to apply the representative sets tool.

***Keywords*-kernelization; parameterized complexity; matroids; graph cuts; multiway cut; almost 2-sat**

I. INTRODUCTION

Polynomial kernelization is a formalization of the notion of efficient polynomial-time preprocessing, or more generally of efficient instance simplification and data reduction. Such reduction steps are commonly applied in practice, see, e.g., the well-known CPLEX integer programming package, or many state-of-the-art SAT solvers. However, to study this theoretically, one needs a notion of the *hardness* of an instance beyond the instance size, e.g., the length of a certificate [1] or a more generic *parameter* associated with the input (cf. [2], [3]). Informally, a kernelization is a polynomial-time reduction of an input instance, with parameter value k, to an equivalent instance of the same problem, the *kernel*, with total output size bounded as a function of k; a problem has a *polynomial kernel* if the size bound is polynomial in k. This turns out to be a robust and interesting notion, and there is much work on both upper and lower bounds for the existence of, or best possible size of, a polynomial kernel for various problems; see [4], [5] and [6]–[8]. A very recent breakthrough was achieved by Drucker [9], who proved, among other things, that the AND-distillation conjecture of Bodlaender et al. [6] holds assuming that NP does not admit non-uniform statistical zero-knowledge proofs (a weaker assumption than the usual NP $\not\subseteq$ coNP/poly).

Among the problems for which the existence of polynomial kernels is still open, one can identify two major groups. The first group is centered around the ALMOST 2-SAT problem: Given a 2-CNF formula $\mathbb{F}$ and an integer k, can you remove at most k clauses to make $\mathbb{F}$ satisfiable (or, equivalently, find an assignment under which at most k clauses are not satisfied)? This is a natural, expressive problem which (at least for purposes of parameterized complexity and kernelization) captures several problems of independent interest. For one thing, it directly expresses ODD CYCLE TRANSVERSAL (OCT); the existence of a polynomial kernel for OCT was a long-standing open problem, only recently solved by the present authors [10]. Less directly, a polynomial kernel for ALMOST 2-SAT has been shown to imply the same for VERTEX COVER ABOVE MATCHING, KÖNIG VERTEX DELETION for graphs with perfect matchings, and the RHORN-BACKDOOR DELETION SET problem from practical SAT solving (cf. [11], [12]), among other problems; see [13]–[15]. We add to the list VERTEX COVER ABOVE LP, i.e., VERTEX COVER parameterized by the size of the LP gap. For all of these problems, no polynomial kernel was

DOI 10.1109/FOCS.2012.46

previously known.

The second group of open problems represents the class of *graph cut* problems. This is a wide class, where little is known regarding polynomial kernelization; problems for which polynomial kernelization is open include DIRECTED FEEDBACK VERTEX SET (arguably one of the biggest open problems in kernelization; see [16]), MULTIWAY CUT, and MULTICUT under various parameterizations, as well as GROUP FEEDBACK ARC/VERTEX SET, which again generalizes OCT.

In this paper, we show polynomial kernels for ALMOST 2-SAT, and for a collection of graph cut problems, including MULTIWAY CUT with a constant number of terminals and MULTICUT with a constant number of cut requests. We also show results about covering min-cuts and multiway cuts through a set of terminals using few vertices, which should be of independent interest. We make use of a lemma on *representative sets* from matroid theory, due to Lovász [17] and Marx [18]. In particular, we show how to apply the lemma in *irrelevant vertex* arguments, i.e., how to use it to find vertices in cut problems which can be made undeletable without affecting the outcome. All our results are randomized, with failure probabilities which can be made exponentially small in the input size.

Related work. ALMOST 2-SAT (also known as MIN 2CNF DELETION) was showed to be FPT, runtime $\mathcal{O}^*(15^k)$, by Razgon and O'Sullivan [19]; this has been improved to $\mathcal{O}^*(9^k)$ [13], $\mathcal{O}^*(4^k)$ [20], and $\mathcal{O}^*(2.6181^k)$ [15]. It has an $\mathcal{O}(\sqrt{\log n})$-approximation by Agarwal et al. [21], and no constant factor approximation under the unique games conjecture [22].

Graph cut problems have been a catalyst for the development of new techniques in parameterized complexity, including the now ubiquitous *iterative compression* technique [23], [24], the notion of *important separators* [25], and the *shadow removal* technique [26]. Our focus here is on MULTIWAY CUT(k), first showed to be FPT by Marx [25]. The currently fastest algorithm [20], runtime $\mathcal{O}^*(2^k)$, uses an LP approach based on work of Guillemot [27]; we also use some insights of the latter.

As for polynomial kernelization of graph cut problems, in joint work with Cygan, Pilipczuk, and Pilipczuk [28] the present authors show, amongst others, that MULTICUT(k) and DIRECTED 2-MULTIWAY CUT(k) do not admit polynomial kernels unless the polynomial hierarchy collapses. Apart from this, and previous work [10] for ODD CYCLE TRANSVERSAL, little is known about kernels for cut or feedback problems beyond the kernelizations for FEEDBACK VERTEX SET, e.g., [29].

Matroids have seen little use as tools in parameterized algorithms (though see [30]), and only few papers address problems *on* matroids. However, recent work of Marx [18] on a parameterized matroid intersection problem also provides some results that are used in the current paper. Regarding kernelization, to our best knowledge, previous work of the present authors [10] is the first and so far only application of matroid theory, using it to encode terminal cut functions of a (large) graph into small space.

Irrelevant vertex arguments are a central part of the DISJOINT PATHS algorithm of Robertson and Seymour [31], which lies behind the celebrated FPT algorithm for testing graph minors. However, the arguments used by Robertson and Seymour to locate irrelevant vertices are very different from those used in this paper (and the resulting bounds are far from polynomial).

Moitra [32] defined and constructed vertex cut sparsifiers, which, given a graph G and a set of terminals X, approximate the values of all terminal cuts in G using (capacitated) edges on vertex set X only. This has lead to a sequence of follow-up work; closest to our setting is Chuzhoy [33], who gives a constant-factor approximation result using $\mathcal{O}(C^3)$ vertices, where C is the total capacity of the terminals X (assuming every edge has capacity at least one). The present work differs from hers in that, on the one hand, we do not consider weighted edges; on the other hand, our constructions are exact, run in polynomial time in both n and k, and also cover directed graphs (and vertex deletions); see Theorem 3, below, and Corollary 3 of Section V-C. Chuzhoy also covers the more general case of flow sparsifiers (see [34]).

Our results. We show several applications of the representative sets lemma of Lovász [17] and Marx [18] to polynomial kernelization, producing the first polynomial kernels for a range of important problems. First, we study DIGRAPH PAIR CUT, a constrained graph cut problem designed to capture (the iterative compression form of) ALMOST 2-SAT. We show that the representative sets lemma can be used to simplify DIGRAPH PAIR CUT down to a cut problem involving a polynomial number of terminals; from here, we can get a polynomial kernel either by compression methods as in [10] or a direct kernel by the cut-covering sets given below. We get the following.

Theorem 1. ALMOST 2-SAT *with a bound k on the solution size has a polynomial-time randomized compression into size $\tilde{\mathcal{O}}(k^6)$, with one-sided error probability $\mathcal{O}(2^{-k})$ and false positives only, and a randomized kernel with $\mathcal{O}(k^6)$ variables and failure probability $\mathcal{O}(2^{-n})$.*

As mentioned above, this gives the first polynomial kernels for a range of problems. Next, we apply the representative sets lemma to the search for irrelevant vertices for graph cut problems. This gives two sets of results. The first relates directly to polynomial kernelization.

Theorem 2. *The following kernelizations are possible:* MULTIWAY CUT WITH DELETABLE TERMINALS(k), *with $\mathcal{O}(k^3)$ vertices; s-*MULTIWAY CUT(k), *with $\mathcal{O}(k^{s+1})$ vertices; s-*MULTICUT(k), *with $\mathcal{O}(k^{\lceil\sqrt{2s}\rceil+1})$ vertices;*

GROUP FEEDBACK VERTEX SET(k), *for a group of s elements, with $\mathcal{O}(k^{2s+2})$ vertices. All results are randomized, with failure probability exponentially small in n.*

Finally, as a second set of irrelevant vertex results, we get interesting conclusions about covering min-cuts and multiway cuts in graphs.

Theorem 3. *Let $G = (V, E)$ be a digraph and let $S, T \subseteq V$. Let r denote the size of a minimum (S, T)-vertex cut (which may intersect S and T). There exists a set $Z \subseteq V$, $|Z| = \mathcal{O}(|S| \cdot |T| \cdot r)$, such that for any $A \subseteq S$ and $B \subseteq T$, it holds that Z contains a minimum (A, B)-vertex cut. We can find such a set in randomized polynomial time with failure probability $\mathcal{O}(2^{-n})$.*

Theorem 4. *Let $G = (V, E)$ be an undirected graph and $X \subseteq V$. For any s, there exists a set $Z \subseteq V$, $|Z| = \mathcal{O}(|X|^{s+1})$, such that for any partition $\mathcal{X} = (X_1, \ldots, X_s)$ with pairwise disjoint subsets of X, it holds that Z contains a minimum multiway cut of $\mathcal{X}$ (i.e., a minimum cut C such that no pairs of sets X_i, X_j are connected to each other in $G - C$). We can find such a set in randomized polynomial time with failure probability $\mathcal{O}(2^{-n})$.*

Organization. Our paper is organized as follows. Section II contains preliminaries, and Section III presents the matroid theory tools we use. Section IV gives the first application, in the form of a polynomial kernel for ALMOST 2-SAT, and Section V gives irrelevant vertex-type consequences, yielding polynomial kernels for variants of MULTIWAY CUT and the cut-covering sets of Theorems 3 and 4. Section VI concludes the paper. Most proofs are deferred to the full version of this work [35].

II. PRELIMINARIES

Parameterized complexity and kernelization. A *parameterized problem* is a language $\mathcal{Q} \subseteq \Sigma^* \times \mathbb{N}$; the second component of instances (x, k) is called the parameter (cf. [2], [3]). A parameterized problem is *fixed-parameter tractable* (FPT) if there is an algorithm A and a computable function $f : \mathbb{N} \to \mathbb{N}$ such that A decides $(x, k) \in \mathcal{Q}$ in time $f(k)|x|^{\mathcal{O}(1)}$. A *kernelization of* $\mathcal{Q}$ is a polynomial-time computable mapping $K : \Sigma^* \times \mathbb{N} \to \Sigma^* \times \mathbb{N} : (x, k) \mapsto (x', k')$ such that $(x, k) \in \mathcal{Q}$ if and only if $(x', k') \in \mathcal{Q}$ and with $|x'|, k' \leq h(k)$ where h is a computable function; h is called the *size* of the kernel and K is a *polynomial kernelization* if $h(k)$ is polynomially bounded.

All kernelization results in this paper are randomized, i.e., there is a (small) chance for the reduced instance not to be equivalent to the input. In all cases, the failure is either one-sided, with false positives only, or occurs with probability exponentially small in the input size. The former type of kernels were called coRP-kernels in previous work [10]; see [10] for a brief discussion on why they are compatible with the lower bound framework [6], [7], and refer to [1], [7] for more on randomized compression. The latter type is easily seen to be computable in non-uniform polynomial time, compatible with the exclusion of non-uniform compression [7].

Matroids. A *matroid* is a pair $M = (E, \mathcal{I})$, where E is the *ground set* and $\mathcal{I} \subseteq 2^E$ a collection of *independent sets*, such that: (i) $\emptyset \in \mathcal{I}$; (ii) if $I_1 \subseteq I_2$ and $I_2 \in \mathcal{I}$, then $I_1 \in \mathcal{I}$; and (iii) if $I_1, I_2 \in \mathcal{I}$ and $|I_2| > |I_1|$, then there exists some $x \in (I_2 \setminus I_1)$ such that $I_1 \cup \{x\} \in \mathcal{I}$. A set $I \subseteq E$ is *independent* if $I \in \mathcal{I}$, and *dependent* otherwise. A set $B \in \mathcal{I}$ is a *basis* of M if no superset of B is independent; a matroid may equivalently be defined by its set of bases. For a subset $X \subseteq E$, the *rank* $r(X)$ of X is the largest cardinality of an independent set $I \subseteq X$. The rank of M is $r(M) := r(E)$.

Let A be a matrix over a field $\mathbb{F}$ and E be the set of columns of A. Let $\mathcal{I}$ be the set of all sets $X \subseteq E$ of columns that are linearly independent over $\mathbb{F}$ (as vectors). Then $(E, \mathcal{I})$ defines a matroid M, and we say that A *represents* M. A matroid is *representable* (over a field $\mathbb{F}$) if there is a matrix (over $\mathbb{F}$) that represents it. A matroid representable over some field is called *linear*. In this work, we will concern ourselves only with linear matroids.

Gammoids. Let $D = (V, A)$ be a digraph and let $S, T \subseteq V$. The set T is *linked to* S if there exist $|T|$ vertex-disjoint paths from S to T; this allows paths of length zero, e.g., any set is linked to itself. Given any digraph $D = (V, A)$ with source vertices $S \subseteq V$, the sets $T \subseteq V$ which are linked to S in D form a matroid, a so-called *gammoid* [36] (see also [37], [38]); we refer to it as (D, S). Marx [18] gave a randomized polynomial-time procedure for finding a representation of a gammoid. The error probability can be made exponentially small in the size of the graph. By standard arguments, advice polynomial in n is sufficient to derandomize this step.

Theorem 5 ([18], [36]). *Let $D = (V, A)$ be a directed graph, and let $S \subseteq V$. The subsets $T \subseteq V$ which are linked to S form the independent sets of a matroid over V. Furthermore, a representation of this matroid can be obtained in randomized polynomial time with one-sided error.*

Throughout the paper, (A, B)-cuts may intersect A and B, unless otherwise noted. We also create *sink-only copies* of vertices; a sink-only copy of $v \in V$, in $D = (V, A)$, is a parallel copy v' of v which retains only the incoming edges of v (in the undirected case all edges are oriented inwards). These will be used, effectively, to require two paths to a vertex $v \in X$ in a set X linked to S. Note that adding v' to the graph has no effect on any independent (linked) set not containing v'.

III. Tools from matroid theory

Representative sets. The notion of *representative sets* plays an essential role in the paper.

Definition 1. *Given a matroid $M = (E, \mathcal{I})$ and a collection $\mathcal{S}$ of subsets of E, we say that a subcollection $\mathcal{S}^* \subseteq \mathcal{S}$ is* r-representative *for $\mathcal{S}$ if the following holds: for every set $Y \subseteq E$ of size at most r, if there is a set $X \in \mathcal{S}$ disjoint from Y with $X \cup Y \in \mathcal{I}$, then there is a set $X^* \in \mathcal{S}^*$ disjoint from Y with $X^* \cup Y \in \mathcal{I}$.*

We will use *representative set* without specifying r to mean an $(r(M)-s)$-representative set. The following result is due to Marx [18], building on Lovász [17].

Lemma 1 ([17], [18]). *Let M be a linear matroid of rank $r+s$, and let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a collection of independent sets, each of size s. If $|\mathcal{S}| > \binom{r+s}{s}$, then there is a set $S_i \in \mathcal{S}$ such that $\mathcal{S} \setminus \{S_i\}$ is r-representative for $\mathcal{S}$. Furthermore, given a representation A of M, we can find such a set S_i in time $(m + ||A||)^{\mathcal{O}(1)}$ (note that terms polynomial in $\binom{r+s}{s}$ are bounded by $m^{\mathcal{O}(1)}$).*

Closest cuts and gammoid rank. Our usage of representative sets and Lemma 1 is centered around the concept of *closest sets*, defined as follows. Let $D = (V, A)$ be a digraph, and $S \subseteq V$. A set $X \subseteq V$ is *closest to* S if X is the unique (S, X)-min-cut (or, if S and X are not disjoint, $X \setminus S$ is the unique $(S \setminus X, X \setminus S)$-min-cut). If so, we say that X is a *closest set*. For any set of vertices X, the *induced closest set* $C(X)$ is the unique (S, X)-min-cut which is closest to S; this is well-defined by the submodularity of cuts, and can be found in polynomial time. If X is a closest set, then $C(X) = X$. Note that a closest set does not need to be a cut; there may not be any vertices except for X which are separated from S by X.

Closest sets are a natural notion, but do not seem to have a fixed name; e.g., they occur in the bipedal stage of the Multicut(k) algorithm of Marx and Razgon [26]. There are also similarities between closest sets and the concept of *important separators* [25]: for any set X closest to S which separates some vertex $v \notin X$ from S, there is a corresponding important separator. However, the change of focus from separation to closeness means that the concepts behave differently.

We make the following observations connecting closest sets to the rank function of a gammoid.

Proposition 1 ($*$[1]). *Let $D = (V, A)$ be a digraph with a set of source vertices $S \subseteq V$, and let $X \subseteq V$. Let D' be the result of adding a sink-only copy x' for every vertex $x \in X$. The following hold.*

1) *The set X is closest to S in D if and only if $X + x'$ is independent in the gammoid (D', S) for every $x \in X \setminus S$.*
2) *Let X_B be a maximal independent subset of X. A vertex v is reachable from S in $D - C(X)$ if and only if $X_B + v$ is independent in the gammoid (D', S).*

In particular, any (S,T)-cut X which is not a closest set, i.e., such that $X + x'$ is dependent in the gammoid (D', S) for some $x \in X \setminus S$, can be replaced by a different (S,T)-cut Z of at most the same size which does not contain x.

[1]Proofs of statements marked with $*$ are postponed to the full version of the paper [35].

IV. Representative sets: A polynomial kernel for Almost 2-SAT

In this section we show how to obtain polynomial kernels via representative objects. We consider a problem that we call Digraph Pair Cut, which captures the iterative compression version of Almost 2-SAT. We show a polynomial kernel for Digraph Pair Cut, using representative sets and a gammoid representation of graph cuts. Polynomial kernels for Almost 2-SAT and related problems follow via kernelization-preserving reductions.

We now study the Digraph Pair Cut problem, to provide an $\mathcal{O}^*(2^k)$ time algorithm and a randomized polynomial kernelization for it. Given a digraph $D = (V, A)$ with a source vertex s, we say that a pair $p = \{u, v\}$, $u, v \in V$, is *reachable* in D if both u and v are reachable from s (not necessarily via disjoint paths). The problem is then defined as follows.

Digraph Pair Cut	**Parameter:** k.
Input: A digraph $D = (V, A)$ with source vertex $s \in V$, a set of pairs $P \subseteq \binom{V}{2}$, and an integer k.	
Question: Is there a set $X \subseteq V \setminus \{s\}$ with $|X| \le k$ such that no pair in P is reachable in $D - X$?	

Note that Digraph Pair Cut can be seen as a cut-based generalization of Vertex Cover. Specifically, if the graph D is an n-point star with s in the center, then the input is equivalent to a Vertex Cover instance with one edge for every pair in P.

Theorem 6 ($*$). *The* Digraph Pair Cut *problem can be solved in time $\mathcal{O}^*(2^k)$.*

It is easy to see that optimal solutions for Digraph Pair Cut can be assumed to be closest to the source vertex s. We show that there exists a representative subset P^* of the pairs P, of size $\mathcal{O}(k^2)$, which determines whether or not any pair from P is reachable from s under some cut X closest to s. Given P^*, and a gammoid encoding of the cut function between s and the vertices occurring in P^*, we get a polynomial kernel for Digraph Pair Cut. Note that the bound on $|P^*|$ is tight even in the case of Vertex Cover, where $\mathcal{O}(k^2)$ edges is optimal [8].

Lemma 2 ($*$). *Let $D = (V, A)$ be a digraph, $s \in V$, k an integer, and $P \subseteq \binom{V}{2}$ a set of vertex pairs. In randomized* *polynomial time (with failure probability exponentially small in the input size) we can find a set of $\mathcal{O}(k^2)$ pairs $P^* \subseteq P$ (the* representative pairs*), such that for any set $X \subseteq V \setminus \{s\}$ closest to s of at most k vertices, the graph $D - X$ contains a reachable pair $p \in P$ if and only if it contains a reachable pair $p^* \in P^*$.*

Proof (sketch): Proposition 1 implies that, for any cut X closest to s, a vertex v is reachable from s in $G - X$ if and only if $X + v$ is independent in the gammoid (D, S), where S consists of $k + 1$ copies of s.

By studying a matroid M consisting of two disjoint copies of (D, S), we can extend this fact to pairs of vertices. We use Lemma 1 to generate a set of representative pairs $P^* \subseteq P$ of size $\mathcal{O}(k^2)$ with respect to independent sets of size at most $2k$ in M. Now if P contains a pair that extends an independent set X', with $|X'| \leq 2k$, then P^* contains such a pair too; the crucial sets X' in M are those which correspond to two copies of some closest cut X. This completes the proof since, by Proposition 1, reachability with respect to closest cuts is equivalent to extension of the corresponding independent set. ∎

We note that a generalization from pairs to q-tuples still holds. If applied to q-tuples, the lemma uses polynomial time to reduce a given set of q-tuples to a set of $\mathcal{O}(k^q)$ representative q-tuples. This implies a polynomial kernel for the q-ary generalization of DIGRAPH PAIR CUT, where the input contains a set of q-tuples Q, and the task is to separate at least one member of every q-tuple from s. We call this variant DIGRAPH q-TUPLE CUT. Again, note that a bound of $\mathcal{O}(k^q)$ is tight for q-tuples: it is straightforward to reduce from q-HITTING SET for which total size of $\mathcal{O}(k^{q-\epsilon})$ for any $\epsilon > 0$ is excluded unless NP $\subseteq$ coNP/poly [8].

Now, let (D, s, P, k) be an instance of DIGRAPH PAIR CUT, and let P^* be a set of representative pairs of P. Clearly, it is enough to find a set X of size k such that no pair in P^* is reachable in $D - X$; we show that, in turn, the existence of such a set can be encoded into a gammoid on s and $\bigcup P^*$.

Theorem 7 (∗)**.** *There is a randomized polynomial-time compression algorithm for* DIGRAPH PAIR CUT *which given an instance $I = (D, s, P, k)$ and a positive real ϵ computes a compressed representation of I of size $\tilde{\mathcal{O}}(k^3(k + \log 1/\epsilon))$. The success probability is at least $1 - \epsilon$, and in the remaining cases the output encodes a positive instance.*

As a corollary, by standard arguments (cf. [39]), we get a polynomial coRP-kernelization for DIGRAPH PAIR CUT. A kernelization-preserving reduction from ALMOST 2-SAT to DIGRAPH PAIR CUT (omitted in this version of the paper) now finishes the kernelization of ALMOST 2-SAT, and the first half of Theorem 1. Further reductions give polynomial kernels for additional problems. Many of these were previously known; see [13], [14], [40].

Corollary 1 (∗)**.** *The following problems have randomized polynomial kernels:* VERTEX COVER ABOVE MATCHING*,* VERTEX COVER PARAMETERIZED BY KÖNIG DELETION SET*,* KÖNIG VERTEX DELETION *restricted to input graphs having perfect matchings, and* RHORN-BACKDOOR DELETION SET*.*

The result on VERTEX COVER PARAMETERIZED BY KÖNIG DELETION SET generalizes the best previously known structural kernelization results for VERTEX COVER [41].

We also provide a new reduction result in the form of a PPT from VERTEX COVER ABOVE LP to VERTEX COVER ABOVE MAXIMUM MATCHING, implying that the two problems are PPT-equivalent, and that VERTEX COVER ABOVE LP has a polynomial kernel.

Corollary 2 (∗)**.** VERTEX COVER ABOVE LP *admits a randomized polynomial kernelization.*

V. FINDING IRRELEVANT VERTICES: POLYNOMIAL KERNELS FOR CUT PROBLEMS

In this section we extend our scope by showing how to use representative sets for the identification of irrelevant vertices in terminal cut problems. In this setting, a vertex is said to be *irrelevant* if there is at least one optimal solution which does not contain the vertex. Note that it is well possible that some irrelevant vertices are needed to build an optimal solution, but that any single one of them can be avoided.

As a warm-up result, we will consider the MULTIWAY CUT WITH DELETABLE TERMINALS(k) problem, where the solution is allowed to contain terminals as well. For this simpler problem we are able to give a representative set characterization that covers not only all non-irrelevant vertices, but in fact contains an optimal solution. Thus a single iteration of the representative sets tool will produce a (randomized) polynomial kernel with $\mathcal{O}(k^3)$ vertices.

Next, we move on to the main focus of the section, namely multiway cuts with a bounded number of terminals or partitions. We present a randomized polynomial kernel with $\mathcal{O}(k^{s+1})$ vertices for the s-MULTIWAY CUT(k) problem, where the number of terminals is bounded by the constant s. For this problem we need a more "traditional" irrelevant vertex approach where only a single vertex is removed in each iteration.

The approach has further consequences, in the form of the cut-covering sets of Theorems 3 and 4 – we find a small set of vertices which simultaneously contains an optimal solution for *all* min-cut questions involving T (and a similar conclusion for multiway cuts). In particular, this result lets us replace the gammoid encoding of graph cuts of [10] by a "proper" kernelization rule.

Finally, we mention two further applications, namely polynomial kernels for s-MULTICUT(k) and Γ-FEEDBACK VERTEX SET(k) for groups Γ with at most s elements.

A. Multiway Cut with deletable terminals

In this section we focus on the MULTIWAY CUT WITH DELETABLE TERMINALS(k) problem (DT-MWC(k)), where the task is to separate a set T of terminals by deleting at most k vertices (including terminals). It can be easily seen to be equivalent to MULTIWAY CUT restricted to terminals of degree one. The problem is NP-hard by a simple reduction from VERTEX COVER.[2]

Let (G, T, k) be an instance of DT-MWC(k), with $G = (V, E)$ and $T \subseteq V$. We will use the tool of representative sets to identify a set of $\mathcal{O}(k^3)$ *representative vertices* V^* such that $V^* \cup T$ contains an optimal solution. But first, using a result of Guillemot [27], we limit $|T|$ in terms of k.

Lemma 3 (∗). *Let (G, T, k) be an instance of* MULTIWAY CUT WITH DELETABLE TERMINALS(k). *An equivalent instance (G', T', k') with $k' \leq k$ and $|T'| \leq 2k'$ can be computed in polynomial time.*

For the representative set, we form a graph G' by adding to every non-terminal vertex v two sink-only copies v' and v'', and form the gammoid on G' with T as source set. We form the set S of triples $\{v, v', v''\}$ for $v \in V \setminus T$, and let S^* be a k-representative set of S. Further, we let $V^* \subseteq V$ be the set of vertices v with $\{v, v', v''\} \in S^*$. We know that $|V^*| = \mathcal{O}(k^3)$; we argue (again) that there is an optimal solution X such that $(X \setminus T) \subseteq V^*$.

Lemma 4 (∗). *Let (G, T, k) be an instance of* MULTIWAY CUT WITH DELETABLE TERMINALS(k), *and let X be a minimum size multiway cut (of size k) which as a secondary criterion has a maximum size intersection with T. Let V^* be a representative set of V, as constructed above. Then for every $x \in (X \setminus T)$, the set $X + x' + x''$ is linked to T, and $X \subseteq T \cup V^*$.*

Proof (sketch): The proof is given by matching arguments on an auxiliary bipartite graph $H = (X' \cup T', E_H)$ where $X' = X \setminus T$ and $T' = T \setminus X$ (skipping deleted terminals $t \in T \cap X$), and with an edge $\{x, t\}$ if x can reach t in $G - (X - x)$. There are two crucial facts. First, any pseudomatching $M \subseteq E_H$ such that edges in M share no endpoint in T' corresponds to $|M|$ paths from T' to X' which overlap only on vertices in X'; this follows from the fact that each component of $G - X$ contains at most one terminal. Second, it follows from the maximum overlap of X with T that any subset $S \subseteq X$ has at least $|S| + 2$ neighbors in H, since instead of S we could delete all but one terminal reachable from S. An application of Hall's Theorem completes the proof. ■

By this lemma, we may set all vertices of $V \setminus (V^* \cup T)$ as undeletable, without changing the existence of a solution of size at most k. The easiest way to achieve this is by adding shortcut edges $\{u, v\}$ between any two vertices $u, v \in V^* \cup T$ which are connected by a path with internal vertices from V', and subsequently deleting V' from the graph. Thus we get a polynomial kernel.

Theorem 8. MULTIWAY CUT WITH DELETABLE TERMINALS(k) *has a randomized kernel of $\mathcal{O}(k^3)$ vertices. The error probability can be made exponentially small in n; all errors are false negatives.*

B. Bounded terminals Multiway Cut

We will now focus on s-MULTIWAY CUT(k), i.e., the variant of MULTIWAY CUT(k) where the number of terminals is bounded by some fixed constant s; we show a randomized polynomial kernel with $\mathcal{O}(k^{s+1})$ vertices.

Unlike for MULTIWAY CUT WITH DELETABLE TERMINALS(k) we are not able to directly form a set of representative vertices which is guaranteed to contain an optimal solution. Instead we show how to identify irrelevant vertices via representative sets techniques, and get the kernel by iterating computation of representative vertices and deletion of a single irrelevant vertex.

We begin by identifying a condition under which at most $\mathcal{O}(k^{s+1})$ vertices are representative, and such that for any single non-representative vertex v of G there is an optimal solution not containing v. Let $T = \{t_1, \ldots, t_s\}$ be the set of terminals, arbitrarily ordered. Assume that the LP-based reductions of Guillemot [27] have been applied, so that the terminals have disjoint neighborhoods and $|N(T)| \leq 2k$ (see Lemma 3). Our condition of representativeness, called *high reachability*, is defined as follows.

Definition 2. *Let (G, T, k) be an instance of s-*MULTIWAY CUT(k) *with $G = (V, E)$ and a set $T \subseteq V$ of at most s terminals. Let $X \subseteq V \setminus T$ be a multiway cut of at most k vertices. Furthermore, let G' be the directed graph obtained from G by adding one sink-only copy v' for each $v \in V \setminus T$. We say that $v \in X$ is* highly reachable *under X if $X + v' + N(t)$ is independent for every $t \in T$, in the gammoid $(G', N(T))$. We say that $v \in V$ is* highly reachable *if there exists some multiway cut X of at most k vertices such that v is highly reachable under X.*

Intuitively, after removing any one terminal from T, the solution X should still be independent with two vertex-disjoint paths to v. Though there are other ways to express this, the exact wording of Definition 2 will be useful for the group feedback vertex set problems later in this section.

We may identify the highly reachable vertices in polynomial time.

Lemma 5. *We can identify a set of $\mathcal{O}(k^{s+1})$ vertices, in randomized polynomial time with error probability $\mathcal{O}(2^{-n})$, which contains all highly reachable vertices.*

Proof: Create a gammoid which is the disjoint union of $s + 1$ layers $0, 1, \ldots, s$, where layers $1, \ldots, s$ are copies

[2]Given a graph G, create G' by attaching a terminal v' to each vertex v. Multiway cuts in G' correspond to vertex covers of G and vice versa.

of the gammoid $(G', N(T))$, where G' is as in Definition 2. For layer 0 we take the uniform matroid of rank k on base set V, i.e., we take $(V, \binom{V}{\leq k})$. Then, create a set of tuples $(v(0), v'(1), v'(2), \ldots, v'(s))$ for every $v \in V(G)$, where $v(0)$ is the copy of v in layer 0 and each $v'(i)$, with $1 \leq i \leq s$ is the sink-only copy of v in the i:th matroid layer. Use Lemma 1 to identify a $\mathcal{O}(k^{s+1})$-sized set of representative tuples (note that the rank can be bounded by $\mathcal{O}(ks)$).

We show that the tuples $(v(0), v'(1), v'(2), \ldots, v'(s))$ corresponding to highly reachable vertices v (with respect to some solution X) will be the unique choice for extending some particular independent sets. Hence they must be contained in the representative subset of the tuples.

Let X be a solution to the instance, and let $v \in X$ be highly reachable under X. Consider the set X^* which contains the following elements in the $s+1$ layers:

1) In layer 0 it contains all vertices $u \in X \setminus \{v\}$; these are at most $k-1$.
2) In each other layer $i \in \{1, \ldots, s\}$ it contains the vertices $X + N(t_i)$.

By assumption the tuple $(v(0), v'(1), v'(2), \ldots, v'(s))$ extends this set (to an independent set). Note that $v(0)$ is not among the vertices chosen for layer 0, since those are only the copies of $X \setminus \{v\}$.

Now assume that there is some vertex $u \neq v$ such that $(u(0), u'(1), u'(2), \ldots, u'(s))$ extends X^*. If $u \in X$, then its tuple intersects $X \setminus \{v\}$ in layer 0. If $u \notin X$, then it is reachable from only one terminal, say t_i, in $G - X$. Hence, in layer i, the vertex $u'(i)$ cannot be added to X^* since we already request paths which saturate (the layer i copies of) X and $N(t_i)$.

Thus v is the unique vertex whose tuple can extend the independent set X^*, implying that its tuple will be among the representative tuples computed via Lemma 2. Thus the corresponding $\mathcal{O}(k^{s+1})$ vertices contain all highly reachable vertices. ■

Now, we need to show that any single vertex which is not highly reachable can be removed without harm, i.e., that such a vertex is irrelevant. It suffices to show that any optimal solution X containing v can be converted into a solution of the same size and avoiding v.

Lemma 6 $(*)$. *Let v be a vertex which is not highly reachable under any multiway cut of size at most k and is not contained in $N(T)$. Then there is an optimal multiway cut which does not contain v.*

Proof (sketch): Let X be an optimal multiway cut for G with $v \in X$. Let $t \in T$ such that $X + v' + N(t)$ is not independent in $(G', N(T))$, according to Definition 2. It follows that there is a minimal cut C of size at most $|X| + |N(t)|$ separating $N(T)$ and $X + v' + N(t)$. The proof goes by constructing a new multiway cut X' from C and X, such that $|X'| \leq |X|$ and $v \notin X'$. ■

To conclude our result, we only need one simple reduction rule.

Reduction Rule 1. *Let $(G = (V, E), T, k)$ be a multiway cut instance, and let $v \in V \setminus (T \cup N(T))$ be a vertex which is not identified as potentially highly reachable by Lemma 5. Create G' from G by replacing $N(v)$ by a clique and removing v from the graph. Return (G', T, k).*

Theorem 9 $(*)$. *s-MULTIWAY CUT(k) has a randomized kernel of $\mathcal{O}(k^{s+1})$ vertices. The error probability can be made exponentially small in n, and errors are limited to false negatives.*

The kernel is, arguably, *near-combinatorial*, in the sense that it consists of a reduction rule which performs simple direct modifications to the input graph, while the condition of applicability for the rule may be seen as non-combinatorial, as it at the moment requires a representation of a gammoid and the multilinear algebra of Lemma 1.

C. Covering Graph Cuts

In this section, we will adapt the above approach to prove the more general Theorems 3 and 4 about covering terminal min-cuts and minimum multiway cuts. In essence, we find that the representative sets condition of Lemma 5 gives us much more power than we need for kernelization. We begin with the statement for cuts in directed graphs.

Proof of Theorem 3: Let G, S, T be as given; we may assume w.l.o.g. that S are sources and T sinks in the graph (by adding source vertices before S and sink vertices after T; this does not modify any cuts). Let a vertex $v \in V$ be *essential* if there are sets $A \subseteq S$, $B \subseteq T$ such that every minimum (A, B)-vertex cut contains v, and *irrelevant* otherwise. Let $A \subseteq S$ and $B \subseteq T$, and let C_A resp. C_B be the minimum (A, B)-vertex cut closest to A resp. to B. The idea of the proof is that vertices essential for an (A, B)-cut are exactly those in $C_A \cap C_B$, and, in turn, by Proposition 1, we can use representative sets to find such vertices.

Claim 1 $(*)$. *Let r denote the size of a minimum (S, T)-vertex cut (which may intersect S and T). We can find a set $Z \subseteq V$ of $\mathcal{O}(|S| \cdot |T| \cdot r)$ vertices which includes all essential vertices.*

Now, any vertex in $V \setminus (Z \cup S \cup T)$ may safely be made undeletable (as in Reduction Rule 1) without changing the size of any minimum (A, B)-vertex cut. By induction, this gives a set of $\mathcal{O}(|S| \cdot |T| \cdot r)$ vertices which covers a minimum (A, B)-vertex cut in G for every $A \subseteq S$, $B \subseteq T$, as requested. ■

By a construction in [10], we can transfer this result to preserving minimum cuts through a set of terminals, allowing for deleting terminals.

Corollary 3 $(*)$. *Let $G = (V, E)$ be a directed graph, and $X \subseteq V$ a set of terminals. We can identify, in* *polynomial time, a set Z of $\mathcal{O}(|X|^3)$ vertices such that for any $S, T, R \subseteq X$, a minimum (S,T)-vertex cut in $G - R$ is contained in Z.*

Regarding tightness, it is easy to show that $\Omega(|S| \cdot |T|)$ vertices are necessary, even in a very simple setup: Let S and T be disjoint sets of vertices of weight 2, and for every $u \in S, w \in T$ create a connecting vertex $v_{u,w}$ of weight 1, with $N(v_{u,w}) = \{u, w\}$. Then, the sets $A = \{u\}$ and $B = \{w\}$ show that we must include $v_{u,w}$ to preserve the unique minimum (A, B)-cut. This is easily converted into a setting without weights, by copying vertices in S and T into pairs of twins. We do not know whether the further factor of r in our upper bound is necessary.

The above gives us *direct* polynomial kernels for a number of problems (as opposed to the implicit polynomial kernels resulting from polynomial compression due to NP-hardness reductions, e.g. as in [10]).

Corollary 4 (∗). DIGRAPH PAIR CUT *has a kernel of* $\mathcal{O}(k^4)$ *vertices;* ALMOST 2-SAT *has a kernel of* $\mathcal{O}(k^6)$ *vertices. The* signed graph *generalizations of* ODD CYCLE TRANSVERSAL *and* EDGE BIPARTIZATION*, where edges are marked as even or odd and the goal is to kill all odd-parity cycles, have kernels with* $\mathcal{O}(k^{4.5})$ *respectively* $\tilde{\mathcal{O}}(k^3)$ *vertices. All the above kernels are randomized, and can be derandomized with polynomial advice.*

Covering multiway cuts: We now turn to a multiway cut variant of the above. To state our results, we define a multiway cut of a partition as follows: Let $\mathcal{X} = (X_1, \ldots, X_s)$ be a tuple of pairwise disjoint sets of vertices, $X := \bigcup X_i$. A *multiway cut of* $\mathcal{X}$ is a set of vertices $C \subseteq V$, which may intersect X, such that for $i \neq j$, there is no path between $(X_i \setminus C)$ and $(X_j \setminus C)$ in $G - C$. In other words, it is a multiway cut of the instance produced by adding terminals $t_1, \ldots, t_s$, with $N(t_i) = X_i$. We show that we can find a set which covers a minimum multiway cut for *every* partition $\mathcal{X} = (X_1, \ldots, X_s)$ with $\bigcup_i X_i \subseteq X$.

Proof of Theorem 4: The proof will again be an irrelevant vertex construction, using the representative set condition of Lemma 5 as guide. Call a vertex $v \in (V \setminus X)$ *essential* if there is a partition $\mathcal{X} = (X_1, \ldots, X_s)$ of $X' \subseteq X$ such that every minimum multiway cut of $\mathcal{X}$ contains v. As in Section V-B, we find that such an essential vertex must be highly reachable under the partition $\mathcal{X}$. Essentially, what remains to be proven is that Lemma 5 does not really depend on knowing the partition $\mathcal{X}$ in advance.

Claim 2 (∗). *We can identify (in polynomial time) a set of* $\mathcal{O}(|X|^{s+1})$ *vertices which includes all essential vertices.*

The rest of the proof now proceeds as for Theorem 3. ■

D. Further kernelization implications

We now briefly mention two further related kernelization results for cut problems, providing the last parts of Theorem 2. First, we extend the result for s-MULTIWAY CUT(k) to also give a polynomial kernel for s-MULTICUT(k), i.e., MULTICUT(k) restricted to instances having at most s terminal pairs. The key fact is that an optimal multicut X for some instance (G, T, k), where $T \subseteq \binom{V}{2}$ of size at most s, is also a multiwaycut for some partition of the terminal vertices $V(T) = \{v \mid \exists u : \{u, v\} \in T\}$ in which no set contains both vertices of some terminal pair. By considering partitions that are maximally coarse, i.e., merging any two sets would violate this condition, we get a relation to multiway cut instances with up to (roughly) $\sqrt{2s}$ terminals.

Theorem 10 (∗). *s-MULTICUT(k) admits a randomized polynomial kernel with $f(s)k^{\lceil\sqrt{2s}\rceil+1}$ vertices. The error probability can be made exponentially small in n, and errors are limited to false negatives.*

Second, we obtain a polynomial kernel for the GROUP FEEDBACK VERTEX SET(k) problem, for groups of bounded size. FPT algorithms for this problem have been given by Guillemot [27] and, in a very general form, Cygan et al. [42]. We focus on the following version, which can be easily seen to generalize OCT, and furthermore encompasses s-MULTIWAY CUT(k) for any group Γ with at least s elements (see [35]).

Γ-FEEDBACK VERTEX SET(k)	**Parameter:** k.

Input: A directed graph $D = (V, A)$, an edge labeling $\phi : A \to \Gamma$, and an integer k.
Question: Is there a set X of at most k vertices such that $D - X$ has a consistent labeling, i.e., a function $\pi : V \to \Gamma$ such that $\pi(u) \otimes \phi((u, v)) = \pi(v)$ for all arcs $(u, v) \in A \setminus F$.

Our proof essentially has three parts. First, we show that we can use known approximation results for VERTEX MULTICUT [43] to get a solution of $\mathcal{O}(s^2k^2)$ elements; second, we show that the iterative compression form of GROUP FEEDBACK VERTEX SET(k) reduces to MULTIWAY CUT, albeit with a superpolynomial number of calls; finally, we adapt the above to give an irrelevant vertex reduction rule for GROUP FEEDBACK VERTEX SET(k). We get the following kernelization result.

Theorem 11 (∗). *Let Γ be a fixed group with s elements. Γ-FEEDBACK VERTEX SET(k) admits a randomized polynomial kernel with $\mathcal{O}(k^{2s+2})$ vertices. The error probability can be made exponentially small in n, and errors are limited to false negatives.*

VI. CONCLUSION

We give powerful new techniques for polynomial kernelization, centered around applications of a lemma from matroid theory due to Lovász [17] and Marx [18]. The resulting tools significantly advance the field of kernelization, and imply polynomial kernels for a range of problems, including ALMOST 2-SAT(k), s-terminal MULTIWAY CUT(k), and MULTICUT(k) with s terminal pairs, for constant s, among other results. In particular, we show how the lemma can be applied to find irrelevant vertices for graph cut problems. In addition to the aforementioned kernels, this lets us find a form of cut-covering sets of small size: given a graph G and terminal set T, we can find a set Z, of size polynomial in $|T|$, such that for every $A, B \subseteq T$, a minimum (A, B)-vertex cut is contained in Z. Similarly, for a constant s, we can find a set Z of polynomial size such that for every partition of T (or a subset of T) into at most s partitions, a minimum multiway cut of the partition is contained in Z. We foresee further applications of these results. Similarly to in [10], our kernels are randomized; unlike in [10], they can all be made reduction rule-based. Furthermore, the failure probability can be reduced to be exponentially small in the input length, implying non-uniform polynomial-time kernelization.

Despite being randomized, we note that all our kernels are compatible with the lower bound framework of Bodlaender et al. [6] and Fortnow and Santhanam [7]; see the discussion in [10] regarding coRP-kernels. Similar arguments can be made regarding non-uniform kernels; see [7, Corollary 3.4]. Hence, concrete polynomial upper and lower bounds for the current problems is a relevant path of research (see Dell and van Melkebeek [8], Dell and Marx [44], and Hermelin and Wu [45]). Research on non-uniform kernels in general is also left as future work.

Further significant open questions include the existence of polynomial kernels for the general form of MULTIWAY CUT(k) and MULTICUT $(s + k)$, in edge- and vertex-deletion variants, and for the GROUP FEEDBACK ARC SET(k) and GROUP FEEDBACK VERTEX SET(k) problems with arbitrary groups. Additionally, a polynomial kernel for DIRECTED FEEDBACK VERTEX SET remains an open problem.

ACKNOWLEDGMENT

The authors are grateful to Dániel Marx for comments on an early draft of the paper, and to Saket Saurabh for an early version of [15]. The second author thanks Madhusudan Manjunath for enlightening discussions. The first author acknowledges support by the Netherlands Organization for Scientific Research (NWO), project "KERNELS: Combinatorial Analysis of Data Reduction".

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