A Dichotomy for Real Boolean Holant Problems

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***Abstract*—We prove a complexity dichotomy for Holant problems on the boolean domain with arbitrary sets of real-valued constraint functions. These constraint functions need not be symmetric nor do we assume any auxiliary functions. It is proved that for every set $\mathcal{F}$ of real-valued constraint functions, $\operatorname{Holant}(\mathcal{F})$ is either P-time computable or #P-hard. The classification has an explicit criterion. This is a culmination of much research on this problem, and it uses many previous results and techniques. Dealing with some concrete functions plays an important role in this proof. In particular, two functions, called f_6 and f_8, and their associated families exhibit intriguing and extraordinary closure properties related to Bell states in quantum information theory.**

***Keywords*-Holant problems; complexity dichotomy; quantum information theory; Bell states.**

I. Introduction

Counting problems arise in many different fields, e.g., statistical physics, economics and machine learning. In order to study the complexity of counting problems, several natural frameworks have been proposed. Two well-studied frameworks are counting constraint satisfaction problems (#CSP) [6], [24], [7], [11], [9] and counting graph homomorphisms (#GH) [23], [8], [26], [10] which is a special case of #CSP. These frameworks are expressive enough so that they can express many natural counting problems but also specific enough so that complete complexity classifications can be established.

Holant problems are a more expressive framework which generalizes #CSP and #GH. It is a broad class of sum-of-products computation. It was proposed in the context of holographic algorithms [18], [19]. These are also called edge-coloring models studied in [31] which are essentially Holant problems with symmetric constraint functions. It is known that some prototypical Holant problems, such as counting perfect matchings, cannot be expressed in #GH [25], [16], which are vertex-coloring models. Unlike #CSP and #GH for which full complexity dichotomies have been established, the understanding of Holant problems, even restricted to the Boolean domain, is still limited. In this paper, we establish the first Holant dichotomy on the Boolean domain with arbitrary real-valued constraint functions. These constraint functions need not be symmetric nor do we assume any auxiliary functions.

We briefly recap the definition of Holant problems on the Boolean domain. It is parameterized by a set of constraint functions, also called signatures; such a signature maps $\{0,1\}^n \to \mathbb{C}$ for some $n > 0$. Let $\mathcal{F}$ be any fixed set of signatures. A signature grid $\Omega = (G, \pi)$ over $\mathcal{F}$ is a tuple, where $G = (V, E)$ is a graph without isolated vertices, π labels each $v \in V$ with a signature $f_v \in \mathcal{F}$ of arity $\deg(v)$, and labels the incident edges $E(v)$ at v with input variables of f_v. We consider all 0-1 edge assignments σ, and each gives an evaluation $\prod_{v \in V} f_v(\sigma|_{E(v)})$, where $\sigma|_{E(v)}$ denotes the restriction of σ to $E(v)$.

Definition I.1 (Holant problems)**.** *The input to the problem* $\operatorname{Holant}(\mathcal{F})$ *is a signature grid* $\Omega = (G, \pi)$ *over* $\mathcal{F}$*. The output is the partition function*

$$\operatorname{Holant}(\Omega) = \sum_{\sigma: E(G) \to \{0,1\}} \prod_{v \in V(G)} f_v(\sigma|_{E_{(v)}}).$$

Bipartite Holant problems $\operatorname{Holant}(\mathcal{F} \mid \mathcal{G})$ *are Holant problems over bipartite graphs* $H = (U, V, E)$*, where each vertex in* U *or* V *is labeled by a signature in* $\mathcal{F}$ *or* $\mathcal{G}$ *respectively. When* $\{f\}$ *is a singleton set, we write* $\operatorname{Holant}(\{f\})$ *as* $\operatorname{Holant}(f)$ *and* $\operatorname{Holant}(\{f\} \cup \mathcal{F})$ *as* $\operatorname{Holant}(f, \mathcal{F})$.

Weighted #CSP is a special class of Holant problems. So are all weighted #GH. Other problems expressible as Holant problems include counting matchings and perfect matchings [32], counting weighted Eulerian orientations (#EO problems) [28], [13], computing the partition functions of six-vertex models [29], [15] and eight-vertex models [3], [12], and a host of other so-called *vertex models* from statistical physics [4]. Many Holant problems such as counting perfect matchings, all

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matchings, vertex disjoint cycle covers, etc. cannot be expressed by #GH [25], [16]. Thus, Holant problems are provably more expressive.

Significant progress has been made in the complexity classification of Holant problems, mainly over the Boolean domain. When all signatures are restricted to be *symmetric*, a full dichotomy is proved [17]. When asymmetric signatures are allowed, some dichotomies are proved for special families of Holant problems by assuming that certain auxiliary signatures are available, e.g., Holant^*, Holant^+ and Holant^c [20], [1], [22], [2]. Without assuming auxiliary signatures a Holant dichotomy is established for non-negative real-valued signatures [27], and for all real-valued signatures where a signature of odd arity is present [14]. In this paper, we prove a full complexity dichotomy for Holant problems with real values.

Theorem I.2. *Let $\mathcal{F}$ be a set of real-valued signatures. If $\mathcal{F}$ satisfies the tractability condition* (T) *in Theorem II.3, then* $\operatorname{Holant}(\mathcal{F})$ *is polynomial-time computable; otherwise,* $\operatorname{Holant}(\mathcal{F})$ *is #P-hard.*

This theorem is the culmination of a large part of previous research on dichotomy theorems on Holant problems, and it uses many previously established results and techniques. However, as it turned out, the journey to this theorem has been arduous; interested readers are referred to our EPILOGUE. The overall plan of the proof is by induction on arities of signatures in $\mathcal{F}$. Since a dichotomy has been proved when $\mathcal{F}$ contains a signature of odd arity, we only need to consider signatures of even arity. For signatures of small arity 2 or 4 (base cases) and large arity at least 10, we give an induction proof based on classification results of #CSP, #EO problems and the eight-vertex model. However, two signatures f_6 and f_8 of arity 6 and 8 (and their associated families) are discovered in this journey, which have extraordinary closure properties. We call these Bell properties [14]. These amazing signatures are wholly unexpected, and their existence presented a formidable obstacle to the induction proof.

All four binary Bell signatures (related to Bell states [5] in quantum information theory) are realizable from f_6 by gadget construction. As a technical tool, we introduce Holant^b problems where the four binary Bell signatures are available. This is specifically to handle the signature f_6. We prove a #P-hardness result for $\operatorname{Holant}^b(f_6, \mathcal{F})$. In this proof, we find other miraculous signatures with special structures such that all signatures realizable from them by merging gadgets are affine signatures, while themselves are not affine signatures.

In order to handle the signature f_8, we introduce Holant problems with limited appearance, where some signatures are only allowed to appear a limited number of times in all instances. We turn the obstacle of the closure property of f_8 in our favor to prove non-constructively a P-time reduction from $\operatorname{Holant}^b(f_8, \mathcal{F})$ to $\operatorname{Holant}(f_8, \mathcal{F})$. In fact, it is provable that except $=_2$, the other three binary Bell signatures are *not* realizable from f_8 by gadget construction. However, we show that we can realize, in the sense of a non-constructive complexity reduction, the desired binary Bell signatures which appear an unlimited number of times. This utilizes the framework where these signatures occur only a limited number of times. Then, we give a #P-hardness result for $\operatorname{Holant}^b(f_8, \mathcal{F})$ similar to $\operatorname{Holant}^b(f_6, \mathcal{F})$.

II. PRELIMINARIES

A. Definitions and notations

Let f be a complex-valued signature. If $\overline{f(\alpha)} = f(\overline{\alpha})$ for all α, we say f satisfies *arrow reversal symmetry* (ARS). Here overlines denote complex conjugation and bit-wise complement respectively. The support $\mathscr{S}(f)$ of f is $\{\alpha \in \mathbb{Z}_2^n \mid f(\alpha) \neq 0\}$. If $\mathscr{S}(f) = \emptyset$, then f is a zero signature, denoted by $f \equiv 0$. We use $\operatorname{wt}(\alpha)$ to denote the Hamming weight of α. Let $\mathscr{E}_n = \{\alpha \in \mathbb{Z}_2^n \mid \operatorname{wt}(\alpha) \text{ is even}\}$, and $\mathscr{O}_n = \{\alpha \in \mathbb{Z}_2^n \mid \operatorname{wt}(\alpha) \text{ is odd}\}$. A signature f of arity n has even or odd parity if $\mathscr{S}(f) \subseteq \mathscr{E}_n$ or $\mathscr{S}(f) \subseteq \mathscr{O}_n$ respectively. In both cases, we say that f has parity.

The EQUALITY signature $(=_n)$ of arity n takes value 1 on all-0 and all-1 inputs, and 0 elsewhere. Let $\mathcal{EQ}_k = \{=_k, =_{2k}, \ldots, =_{nk}, \ldots\}$ and $\mathcal{EQ} = \mathcal{EQ}_1$. Then, #CSP($\mathcal{F}$) is exactly $\operatorname{Holant}(\mathcal{EQ} \mid \mathcal{F})$. Also, we define $\#\text{CSP}_k(\mathcal{F})$ to be $\operatorname{Holant}(\mathcal{EQ}_k \mid \mathcal{F})$. The binary DISEQUALITY $(\neq_2)$ has truth table $(0,1,1,0)$ indexed by $\{0,1\}^2$. If f is a 4-ary signature with even parity, then $\operatorname{Holant}(\neq_2 \mid f)$ is the problem of computing the partition function of an eight-vertex model. If $\mathcal{F}$ is a set of signatures all having support on half-weighted inputs, then $\operatorname{Holant}(\neq_2 \mid \mathcal{F})$ is a #EO problem.

A signature f of arity $2n$ is called a DISEQUALITY signature $\neq_{2n}$ if $f = 1$ when $x_1 = \ldots = x_n \neq x_{n+1} = \ldots = x_{2n}$, and 0 otherwise. Functions obtained by permuting variables of $\neq_{2n}$ are also called DISEQUALITY signatures. Let $\mathcal{DEQ} = \{\neq_2, \neq_4, \ldots, \neq_{2n}, \ldots\}$. The four binary Bell signatures are

$$=_2, \neq_2, (=_2^-) = (1,0,0,-1), \text{ and } (\neq_2^-) = (0,1,-1,0).$$

Let $\mathcal{B} = \{=_2, \neq_2, =_2^-, \neq_2^-\}$. A signature f of arity $n \geqslant 2$ has a $2^k \times 2^{n-k}$ signature matrix $M_{[k],[n-k]}(f)$, with

2^k assignments of some k variables as row index. In particular,

$$M(=_2) = I_2 = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right] \text{ and } M(\neq_2) = N_2 = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right].$$

Also, let $H = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}\right]$.

B. Holographic transformation

Any $T \in \mathbf{GL}_2(\mathbb{C})$ defines a transformation $f \mapsto Tf = T^{\otimes n} f$, for f of arity n, written as a column vector $f \in \mathbb{C}^{2^n}$. For a set $\mathcal{F}$, define $T\mathcal{F} = \{Tf \mid f \in \mathcal{F}\}$. For signatures written as row vectors we define fT^{-1} and $\mathcal{F}T^{-1}$ similarly. The holographic transformation defined by T is the following operation: any signature grid $\Omega = (H, \pi)$ of $\operatorname{Holant}(\mathcal{F} \mid \mathcal{G})$ is transformed to $\Omega' = (H, \pi')$ of $\operatorname{Holant}(\mathcal{F}T^{-1} \mid T\mathcal{G})$, on the same bipartite graph H, by replacing each signature $f \in \mathcal{F}$ and $g \in \mathcal{G}$ with fT^{-1} and Tg, respectively.

Theorem II.1 (Valiant [33]). *For every $T \in \mathbf{GL}_2(\mathbb{C})$, $\operatorname{Holant}(\mathcal{F} \mid \mathcal{G}) \equiv_T \operatorname{Holant}(\mathcal{F}T^{-1} \mid T\mathcal{G})$.*

$\operatorname{Holant}(\mathcal{F})$ is the same as its bipartite form $\operatorname{Holant}(=_2 \mid \mathcal{F})$. Let $\mathbf{O}_2 \subseteq \mathbb{R}^{2\times 2}$ be the set of all 2×2 real orthogonal matrices. For all $Q \in \mathbf{O}_2$, $(=_2)Q^{-1} = (=_2)$, and $\operatorname{Holant}(=_2 \mid \mathcal{F}) \equiv_T \operatorname{Holant}(=_2 \mid Q\mathcal{F})$.

Let $Z = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ \mathfrak{i} & -\mathfrak{i}\end{smallmatrix}\right]$. Then $(=_2)Z = (\neq_2)$, and

$$\operatorname{Holant}(=_2 \mid \mathcal{F}) \equiv_T \operatorname{Holant}(\neq_2 \mid Z^{-1}\mathcal{F}).$$

Let $\widehat{\mathcal{F}} = Z^{-1}\mathcal{F}$, and $\widehat{f} = Z^{-1}f$.

Lemma II.2 ([13]). *A (complex-valued) signature f is a real-valued signature iff $\widehat{f}$ satisfies* ARS.

We say a real-valued binary signature f is orthogonal if $M(f)M^{\mathtt{T}}(f) = \lambda I_2$ for some real $\lambda > 0$. Let $\mathcal{O}$ denote the set of all binary orthogonal signatures and the binary zero signature. For any binary f, $f \in \mathcal{O}$ iff $\widehat{f}$ has ARS and parity. Then, $\widehat{\mathcal{O}} = Z^{-1}\mathcal{O}$ is the set of all binary signatures with ARS and parity. Note that $\mathcal{B} \subseteq \mathcal{O}$ and $\widehat{\mathcal{B}} \subseteq \widehat{\mathcal{O}}$. Let $\widehat{\mathbf{O}_2} = \{\widehat{Q} = Z^{-1}QZ \mid Q \in \mathbf{O}_2\}$. Then, for $\widehat{Q} \in \widehat{\mathbf{O}_2}$,

$$\widehat{Q}\widehat{\mathcal{F}} = (Z^{-1}QZ)(Z^{-1}\mathcal{F}) = Z^{-1}(Q\mathcal{F}) = \widehat{Q\mathcal{F}}.$$

We use $\mathcal{F}^{\otimes k}$ $(k \geqslant 1)$ to denote the set $\{\lambda \cdot \bigotimes_{i=1}^{k} f_i \mid \lambda \in \mathbb{R}\backslash\{0\}, f_i \in \mathcal{F}\}$, and $\mathcal{F}^{\otimes}$ to denote $\bigcup_{k=1}^{\infty} \mathcal{F}^{\otimes k}$.

C. Gadget construction

We say f is *realizable* from $\mathcal{F}$ by gadget construction if there is a graph $G = (V, E, D)$ with internal edges E and dangling edges D such that by labeling $v \in V$ with $f_v \in \mathcal{F}$, G defines f by its sum-of-product, with inputs on D. If so, then $\operatorname{Holant}(f, \mathcal{F}) \equiv_T \operatorname{Holant}(\mathcal{F})$. We use $\partial_{ij} f = f_{ij}^{00} + f_{ij}^{11}$ to denote the signature realized by the *merging* gadget: connect variables x_i and x_j of f using $(=_2)$. Here f_{ij}^{ab} denotes the signature by setting $(x_i, x_j) = (a, b) \in \{0,1\}^2$. In $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$, the merging operation is connecting x_i and x_j of $\widehat{f}$ using $\neq_2$. This is denoted by $\widehat{\partial}_{ij}\widehat{f}$, and we have $\widehat{\partial_{ij} f} = \widehat{\partial}_{ij}\widehat{f} = \widehat{f}_{ij}^{01} + \widehat{f}_{ij}^{10}$. Also, if $=_2^-$ and $\neq_2^-$ are realizable, then we construct $\partial_{ij}^- f$ and $\widehat{\partial}_{ij}^- f$ by connecting x_i and x_j by $=_2^-$ and $\neq_2^-$. For a signature f of arity $n > m \geqslant 1$, the *mating* gadget connects $n-m$ variables of f with the corresponding variables of another copy of f. For $m = 1$, there is one dangling variable x_i of each copy of f and mating realizes a signature of arity 2, denoted by $\mathfrak{m}_i f$, and $M(\mathfrak{m}_i f) = M_i(f)M_i^{\mathtt{T}}(f)$. In $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$, the conections are by $(\neq_2)$, and we denote it by $\widehat{\mathfrak{m}}_i\widehat{f} = \widehat{\mathfrak{m}_i f}$. For $m = 2$, we use $\mathfrak{m}_{ij} f$ and $\widehat{\mathfrak{m}}_{ij}\widehat{f}$ to denote the signatures realized by mating f using $=_2$ and mating $\widehat{f}$ using $\neq_2$ respectively. In $\operatorname{Holant}(=_2 \mid \mathcal{F})$ and $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ we identify the left-hand side (LHS) vertices by an edge labeled with $=_2$ or $\neq_2$.

D. Some known results

There are three known and most interesting tractable signature sets that define P-time computable counting problems: product-type signatures $\mathscr{P}$, affine signatures $\mathscr{A}$, and local affine signatures $\mathscr{L}$. Please see the full paper [30] or [22] for definitions and more details. Let $\mathscr{T}$ be the set of tensor products of unary and binary signatures. Problems defined by $\mathscr{T}$ are also tractable.

A signature set $\mathcal{F}$ is $\mathscr{C}$-transformable if there exists a $T \in \mathrm{GL}_2(\mathbb{C})$ such that $(=_2)(T^{-1})^{\otimes 2} \in \mathscr{C}$ and $T\mathcal{F} \subseteq \mathscr{C}$. By Theorem II.1, if $\operatorname{Holant}(\mathscr{C})$ is tractable, then $\operatorname{Holant}(\mathcal{F})$ is tractable for any $\mathscr{C}$-transformable set $\mathcal{F}$. The following tractable results are known.

Theorem II.3. *Let $\mathcal{F}$ be a set of complex-valued signatures. Then $\operatorname{Holant}(\mathcal{F})$ is tractable if*

$$\begin{aligned} &\mathcal{F} \subseteq \mathscr{T}, \\ &\textit{or } \mathcal{F} \textit{ is } \mathscr{P}\textit{-transformable}, \\ &\textit{or } \mathcal{F} \textit{ is } \mathscr{A}\textit{-transformable} \\ &\textit{or } \mathcal{F} \textit{ is } \mathscr{L}\textit{-transformable}. \end{aligned} \qquad \text{(T)}$$

Remark: As a special case, the above tractability condition applies to real-valued signatures.

If $\mathcal{F}$ does not satisfy condition (T), then for all $Q \in \mathbf{O}_2$, $Q\mathcal{F}$ also does not satisfy condition (T), and based on dichotomy results of $\#\mathrm{CSP}_2(\mathcal{F})$ and $\#\mathrm{CSP}_k(\neq_k, \mathcal{F})$ [21], [22], [2], we have $\#\mathrm{CSP}_2(Q\mathcal{F})$ and $\#\mathrm{CSP}_k(\neq_2, \widehat{Q\mathcal{F}})$ are #P-hard. If $\mathcal{F}$ does not satisfy condition (T) and $\mathcal{F}$ contains a nonzero signature of odd arity, then

$\operatorname{Holant}(\mathcal{F})$ is #P-hard [14]. We will also use dichotomy results of eight-vertex models [12] and #EO problems [13]. Please see the full paper [30] for more details.

A nonzero signature f is irreducible if f cannot be written as $g \otimes h$ (with possible variable permutation) and it is reducible otherwise. For real-valued reducible signatures, Lin and Wang proved the following result.

Lemma II.4 ([27])**.** *If a nonzero real-valued signature f has a real factorization $g \otimes h$, then*

$$\operatorname{Holant}(g, h, \mathcal{F}) \equiv_T \operatorname{Holant}(f, \mathcal{F}),$$

$$\textit{and } \operatorname{Holant}(\neq_2 \mid \widehat{g}, \widehat{h}, \widehat{F}) \equiv_T \operatorname{Holant}(\neq_2 \mid \widehat{f}, \widehat{F})$$

for any signature set $\mathcal{F}$ ($\widehat{\mathcal{F}}$). We say g ($\widehat{g}$) and h ($\widehat{h}$) are realizable from f ($\widehat{f}$) by factorization.

In the following, without other specification, f is a real-valued signature and $\mathcal{F}$ is a set of real-valued signatures, and $\widehat{f}$ denotes a signature satisfying ARS and $\widehat{\mathcal{F}}$ denotes a set of such signatures. We use Q to denote a matrix in $\mathbf{O}_2$, and $\widehat{Q} = Z^{-1}QZ$ to denote a matrix in $\widehat{\mathbf{O}_2}$.

III. Proof Organization

The proof of Theorem I.2 is organized as follows.

By Theorem II.3, if $\mathcal{F}$ satisfies condition (T), then $\operatorname{Holant}(\mathcal{F})$ is P-time computable. So, we only need to prove the #P-hardness when $\mathcal{F}$ does not satisfy condition (T). If $\mathcal{F}$ contains a nonzero signature of odd arity, then we get #P-hardness. In the following without other specifications, when refer to a real-valued signature set $\mathcal{F}$ or a corresponding signature set $\widehat{\mathcal{F}} = Z^{-1}\mathcal{F}$ satisfying ARS, we assume that they consist of signatures of even arity, and $\mathcal{F}$ does not satisfy condition (T).

In Section IV, we introduce the second order orthogonality (2nd-Orth) as a key tool in our proof. We show that all irreducible signatures in $\mathcal{F}$ satisfy 2nd-Orth, or else, we get #P-hardness based on results of #CSP problems, #EO problems and eight-vertex models (Lemma IV.3).

In Section V, we give the induction framework of the proof. Since $\mathcal{F}$ does not satisfy condition (T), $\mathcal{F} \nsubseteq \mathscr{T}$. Also since $\mathcal{O}^{\otimes} \subseteq \mathscr{T}$, $\mathcal{F} \nsubseteq \mathcal{O}^{\otimes}$. $\mathcal{F}$ contains a signature $f \notin \mathcal{O}^{\otimes}$ of arity $2n$. We want to achieve a proof of #P-hardness by induction on $2n$. The case of $2n = 2$ is proved in Lemma IV.1. When $2n = 4$, we show that $\operatorname{Holant}(\mathcal{F})$ is #P-hard (Lemma V.1). When $2n \geqslant 10$, we show that $\operatorname{Holant}(\mathcal{F})$ is #P-hard or a signature $g \notin \mathcal{O}^{\otimes}$ of even arity at most 8 is realizable (Lemma V.2).

In Section VI, we handle the case of arity 6. We show that either $\operatorname{Holant}(\mathcal{F})$ is #P-hard, or the extraordinary signature f_6 with the Bell property is realizable after a holographic transformation by some $Q \in \mathbf{O}_2$, i.e., $\operatorname{Holant}(f_6, Q\mathcal{F}) \leqslant_T \operatorname{Holant}(\mathcal{F})$ (Lemma VI.1). By gadget construction, all four Bell signatures in $\mathcal{B}$ are realizable from f_6, i.e., $\operatorname{Holant}(\mathcal{B}, f_6, Q\mathcal{F}) \leqslant_T \operatorname{Holant}(f_6, Q\mathcal{F})$. Note that $Q\mathcal{F}$ does not satisfy condition (T) iff $\mathcal{F}$ does not satisfy it. Then we prove the #P-hardness of $\operatorname{Holant}(\mathcal{B}, f_6, Q\mathcal{F})$ when $Q\mathcal{F}$ does not satisfy condition (T) (Lemma VI.5).

We denote $\operatorname{Holant}^b(\mathcal{F}) = \operatorname{Holant}(\mathcal{B}, \mathcal{F})$. This notion plays an important role in the proof for arity 8 as well.

In Section VII, we handle the case of arity 8. We show that $\operatorname{Holant}(\mathcal{F})$ is #P-hard or an even more amazing signature called f_8 with the strong Bell property is realizable after a holographic transformation by some $Q \in \mathbf{O}_2$, i.e., $\operatorname{Holant}(f_8, Q\mathcal{F}) \leqslant_T \operatorname{Holant}(\mathcal{F})$ (Lemma VII.1). One can prove that $\mathcal{B}$ cannot be realized from f_8 by gadget construction. However, by introducing Holant problems with limited appearance and using the strong Bell property of f_8, we show $\operatorname{Holant}^b(f_8, Q\mathcal{F}) \leqslant_T \operatorname{Holant}(f_8, Q\mathcal{F})$ (Lemmas VII.7). Then, we prove the #P-hardness of $\operatorname{Holant}^b(f_8, Q\mathcal{F})$ when $Q\mathcal{F}$ does not satisfy condition (T).

In the actual proof, for convenience, many results are proved in the setting of $\operatorname{Holant}(\neq \mid \widehat{\mathcal{F}})$ which is equivalent to $\operatorname{Holant}(\mathcal{F})$ under the Z^{-1} transformation. Because Theorem I.2 is a culmination of previous dichotomies on Holant problems, our proof is built on many layers of previous results and techniques. Due to space limit, many proof steps described below are compressed. Indeed many single sentences require substantial justifications. Please refer to the full paper [30] for more details.

IV. Second Order Orthogonality

For convenience, we will consider the problem $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ which is equivalent to $\operatorname{Holant}(\mathcal{F})$ via a holographic transformation by Z^{-1}. Remember that we assume $\mathcal{F}$ does not satisfy condition (T). Consider signatures $\widehat{\mathfrak{m}}_i \widehat{f}$ realized by mating using $\neq_2$.

Lemma IV.1. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T)*. Let $\widehat{\mathcal{F}} = Z^{-1}\mathcal{F}$. If $\widehat{\mathcal{F}}$ contains a nonzero signature $\widehat{f}$ of arity $2n \geqslant 2$, then either $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ is #P-hard or there exists some $\mu \neq 0$ such that for all i, $\widehat{\mathfrak{m}}_i \widehat{f} = \mu N_2$. In particular, if $\widehat{\mathcal{F}}$ contains a binary signature $\widehat{b} \notin \widehat{\mathcal{O}}$, then $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ is #P-hard.*

If for all i, $\widehat{\mathfrak{m}}_i \widehat{f} = \mu N_2$ ($\mu \neq 0$), then $\widehat{f}$ is said to satisfy first order orthogonality (1st-Orth) [14]. As a generalization, we introduce the following second order orthogonality (2nd-Orth).

Definition IV.2. *Let f be a complex-valued signature of arity $n \geqslant 4$. It satisfies* second order orthogonality *if there exists some $\lambda \neq 0$ such that for all $\{i,j\} \subset [n]$, the entries of f satisfy*

$$|\mathbf{f}_{ij}^{00}|^2 = |\mathbf{f}_{ij}^{01}|^2 = |\mathbf{f}_{ij}^{10}|^2 = |\mathbf{f}_{ij}^{11}|^2 = \lambda,$$

$$\text{and} \qquad \langle \mathbf{f}_{ij}^{ab}, \mathbf{f}_{ij}^{cd} \rangle = 0 \qquad \text{for all } (a,b) \neq (c,d).$$

Here $\langle \cdot, \cdot \rangle$ denotes the (complex) inner product (with conjugation), $\mathbf{f}_{ij}^{ab} \in \mathbb{C}^{2^{n-2}}$ denotes the vector by setting $x_i = a$ and $x_j = b$ in f, and $|\mathbf{f}_{ij}^{ab}|$ denotes its 2-norm defined by this inner product.

Remark: When f is a real-valued signature, the complex inner product is just the usual dot product which can be represented by mating using $(=_2)$. Thus, f satisfies 2ND-ORTH iff there is $\lambda \neq 0$ such that for all (i,j), $M(\mathfrak{m}_{ij} f) = \lambda I_4 = \lambda I_2^{\otimes 2}$. On the other hand, when $\widehat{f}$ is a signature with ARS, the complex inner product can be represented by mating using $\neq_2$, due to ARS. Thus, $\widehat{f}$ satisfies 2ND-ORTH iff there is some $\lambda \neq 0$ such that for all (i,j), $M(\widehat{\mathfrak{m}}_{ij} \widehat{f}) = \lambda N_4 = \lambda N_2^{\otimes 2}$. Moreover, f satisfies 2ND-ORTH iff $\widehat{f}$ satisfies it.

Lemma IV.3. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *Let $\widehat{\mathcal{F}} = Z^{-1}\mathcal{F}$. If $\widehat{\mathcal{F}}$ contains an irreducible signature $\widehat{f}$ of arity $\geqslant 4$, then either* $\operatorname{Holant}(\neq \mid \widehat{\mathcal{F}})$ *is #P-hard, or $\widehat{f}$ satisfies* 2ND-ORTH.

Proof Sketch. We may assume that $\widehat{f}$ satisfies 1ST-ORTH, i.e., there exists $\mu \neq 0$ such that for all i, $M(\widehat{\mathfrak{m}}_i \widehat{f}) = \mu N_2$. Then, we can show that for all (i,j), $\widehat{\mathfrak{m}}_{ij} \widehat{f}$ has even parity and thus it represents an eight-vertex model. Then, by the dichotomy for eight-vertex models [12], we may assume all $\widehat{\mathfrak{m}}_{ij} \widehat{f}$ belong to the tractable family for that. Except for the case that $\widehat{\mathfrak{m}}_{ij} \widehat{f} = \lambda_{ij} N_4$ for all $\{i,j\}$, in all other tractable cases, we can give a reduction from $\#\text{CSP}(Q\mathcal{F})$, $\#\text{CSP}_k(\neq_2, \widehat{Q}\widehat{\mathcal{F}})$ or $\#\text{EO}(\widehat{\mathcal{F}})$ to $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ for some orthogonal matrix $Q \in \mathbf{O}_2$ and some even k. These problems are all #P-hard when $\mathcal{F}$ does not satisfy condition (T). Thus, we may assume that $\widehat{\mathfrak{m}}_{ij} \widehat{f} = \lambda_{ij} N_4$ for all $\{i,j\}$. Then, we can show that all λ_{ij} have the same value $\lambda \neq 0$. ∎

V. The Induction Framework

Since $\mathcal{F}$ does not satisfy condition (T), we have $\widehat{\mathcal{F}} \nsubseteq \mathscr{T}$. Also, since $\widehat{\mathcal{O}}^{\otimes} \subseteq \mathscr{T}$, there is a signature $\widehat{f} \in \widehat{\mathcal{F}}$ of arity $2n \geqslant 4$ such that $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$. We want to achieve a proof of #P-hardness by induction on $2n$. The general framework is that we start with a signature $\widehat{f}$ of arity $2n \geqslant 4$ that is not in $\widehat{\mathcal{O}}^{\otimes}$, and realize a signature $\widehat{g}$ of arity $2k \leqslant 2n-2$ that is also not in $\widehat{\mathcal{O}}^{\otimes}$, or otherwise we can directly show $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ is #P-hard. If we can reduce the arity down to 2 (by a sequence of reductions of length independent of the problem instance size), then we have a binary signature $\widehat{b} \notin \widehat{\mathcal{O}}$. By Lemma IV.1 and recall that $\mathcal{F}$ does not satisfy condition (T), we are done.

For the inductive step, we first consider the case that $\widehat{f}$ is reducible. Suppose that $\widehat{f} = \widehat{f_1} \otimes \widehat{f_2}$. If $\widehat{f_1}$ or $\widehat{f_2}$ have odd arity, then we can realize a signature of odd arity by factorization and we are done. Otherwise, $\widehat{f_1}$ and $\widehat{f_2}$ have even arity. Since $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$, we know $\widehat{f_1}$ and $\widehat{f_2}$ cannot both in $\widehat{\mathcal{O}}^{\otimes}$. Then, we can realize a signature of lower arity that is not in $\widehat{\mathcal{O}}^{\otimes}$ by factorization, and we are done. Thus, in the following we may assume that $\widehat{f}$ is irreducible. Then, we may further assume that $\widehat{f}$ satisfies 2ND-ORTH. Otherwise, we get #P-hardness by Lemma IV.3. We use merging to realize signatures of arity $2n-2$ from $\widehat{f}$. Consider $\widehat{\partial}_{ij} \widehat{f}$ for all pairs of indices $\{i,j\}$. If there exists a pair $\{i,j\}$ such that $\widehat{\partial}_{ij} \widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$, then we can realize $\widehat{g} = \widehat{\partial}_{ij} \widehat{f}$ which has arity $2n-2$, and we are done. Thus, we may assume $\widehat{\partial}_{ij} \widehat{f} \in \widehat{\mathcal{O}}^{\otimes}$ for all $\{i,j\}$. We denote this property by $\widehat{f} \in \widehat{\int} \widehat{\mathcal{O}}^{\otimes}$. We want to achieve our induction proof based on these two properties, namely $\widehat{f} \in \widehat{\int} \widehat{\mathcal{O}}^{\otimes}$ and $\widehat{f}$ satisfies 2ND-ORTH. We first consider the case that $2n = 4$.

Lemma V.1. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *Let $\widehat{\mathcal{F}} = Z^{-1}\mathcal{F}$. If $\widehat{\mathcal{F}}$ contains a 4-ary signature $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$, then* $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ *is #P-hard.*

Proof Sketch. We show that there is no irreducible signature $\widehat{f}$ of arity 4 that satisfies both 2ND-ORTH and $\widehat{f} \in \widehat{\int} \widehat{\mathcal{O}}^{\otimes}$. In this proof, 2ND-ORTH plays a key role. It essentially gives a set of equations on the entries of $\widehat{f}$. Suppose that $\widehat{f}$ satisfies 2ND-ORTH. By analyzing these equations on the 16 entries of $\widehat{f}$ given by 2ND-ORTH, we can show that all entries of $\widehat{f}$ have the same (nonzero) norm. Then we show that such a signature $\widehat{f}$ does not satisfy $\widehat{f} \in \widehat{\int} \widehat{\mathcal{O}}^{\otimes}$. ∎

Then, we show that the above induction framework works for signatures of arity $2n \geqslant 10$.

Lemma V.2. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *Let $\widehat{\mathcal{F}} = Z^{-1}\mathcal{F}$. If $\widehat{\mathcal{F}}$ contains a signature $\widehat{f}$ of arity $2n \geqslant 10$ and $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$, then*

- $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ *is #P-hard, or*
- *there is a signature $\widehat{g} \notin \widehat{\mathcal{O}}^{\otimes}$ of arity $2k \leqslant 2n-2$ such that* $\operatorname{Holant}(\neq \mid \widehat{g}, \widehat{\mathcal{F}}) \leqslant_T \operatorname{Holant}(\neq \mid \widehat{\mathcal{F}})$.

Proof Sketch. Again, we show that there is no signature of arity $2n \geqslant 10$ that satisfies both 2ND-ORTH and $\widehat{f} \in \int\widehat{\mathcal{O}}^{\otimes}$. Here, the property $\widehat{f} \in \int\widehat{\mathcal{O}}^{\otimes}$ plays a key role. For signatures of arity $2n \geqslant 10$, the property $\widehat{f} \in \int\widehat{\mathcal{O}}^{\otimes}$ implies that $\widehat{f}$ is a sum of a signature in $\widehat{\mathcal{O}}^{\otimes}$ and a signature whose support is on the all-0 and all-1 inputs. Then, we show such a signature does not satisfy 2ND-ORTH. ■

If Lemma V.2 were to hold for signatures of arity 6 and 8, i.e., there is no irreducible signature $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$ of arity 6 or 8 such that $\widehat{f}$ satisfies both 2ND-ORTH and $\widehat{f} \in \int\widehat{\mathcal{O}}^{\otimes}$, then the induction proof holds and we are done. However, there are extraordinary signatures of arity 6 and 8 with special closure properties (Bell properties) such that they satisfy both 2ND-ORTH and $\widehat{f} \in \int\widehat{\mathcal{O}}^{\otimes}$.

VI. First Major Obstacle: 6-ary Signatures with the Bell Property

Consider the 6-ary signature

$$\widehat{f_6} = \chi_S \cdot (-1)^{x_1x_2+x_2x_3+x_1x_3+x_1x_4+x_2x_5+x_3x_6}$$

where χ_S is the 0-1 indicator function on the set

$$S = \mathscr{S}(\widehat{f_6}) = \{\alpha \in \mathbb{Z}_2^6 \mid \mathrm{wt}(\alpha) \equiv 0 \bmod 2\}.$$

It has the following matrix form,

$$M_{123,456}(\widehat{f_6}) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

One can check that $\widehat{f_6} \notin \widehat{\mathcal{O}}^{\otimes}$, $\widehat{f_6}$ satisfies 2ND-ORTH and $\widehat{f_6} \in \int\widehat{\mathcal{O}}^{\otimes}$. The following lemma shows how this extraordinary signature $\widehat{f_6}$ was discovered.

Lemma VI.1. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *Let $\widehat{\mathcal{F}} = Z^{-1}\mathcal{F}$. If $\widehat{\mathcal{F}}$ contains a* 6*-ary signature $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$, then*

- $\mathrm{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ *is #P-hard, or*
- $\mathrm{Holant}(\neq_2 \mid \widehat{f_6}, \widehat{Q}\widehat{\mathcal{F}}) \leqslant_T \mathrm{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ *for some* $\widehat{Q} \in \widehat{\mathbf{O}_2}$.

Proof Sketch. The general strategy of this proof is to show that we can realize signatures with special properties from $\widehat{f}$ step by step and finally we can realize $\widehat{f_6}$. Otherwise, we can realize signatures that lead to #P-hardness. Again, we may assume that $\widehat{f}$ is irreducible.

1) Firstly, if $\widehat{f} \notin \int\widehat{\mathcal{O}}^{\otimes}$, then we get #P-hardness by Lemma V.1. Now assume $\widehat{f} \in \int\widehat{\mathcal{O}}^{\otimes}$, we can realize an irreducible 6-ary signature f' where $\widehat{f'}(\alpha) = 0$ for all α with $\mathrm{wt}(\alpha) = 2$ or 4.
2) Secondly, if f' does not satisfy 2ND-ORTH, then we get #P-hardness by Lemma IV.3. Now assume that f' satisfies 2ND-ORTH, we show that $\mathcal{S}(f') = \{\alpha \in \mathbb{Z}_2^6 \mid \mathrm{wt}(\alpha) = 1 \bmod 2\}$, and all nonzero entries of f' have the same norm.
3) Thirdly, by ARS, after a holographic transformation by some $\widehat{Q} \in \widehat{\mathbf{O}_2}$ and normalization, we show that we can realize an irreducible 6-ary signature $\widehat{f''}$ where $\mathscr{S}(\widehat{f''}) = \{\alpha \in \mathbb{Z}_2^6 \mid \mathrm{wt}(\alpha) = 1 \bmod 2\}$ and there exists $\lambda = 1$ or $\mathfrak{i}$ such that for all $\alpha \in \mathscr{S}(\widehat{f''})$, $\widehat{f''}(\alpha) = \pm\lambda$.
4) Finally, we show that $\widehat{f_6}$ is realizable from $\widehat{f''}$. ■

Remark: Recall that for all $\widehat{Q} \in \widehat{\mathbf{O}_2}$, $\widehat{Q}\widehat{\mathcal{F}} = \widehat{Q\mathcal{F}}$ where $Q \in \mathbf{O}_2$. Moreover, for any real-valued $\mathcal{F}$ not satisfying condition (T), $Q\mathcal{F}$ is also a real-valued signature set not satisfying condition (T).

Now, we want to show that $\mathrm{Holant}(\neq_2 \mid \widehat{f_6}, \widehat{Q}\widehat{\mathcal{F}})$ is #P-hard for any real-valued $\mathcal{F}$ that does not satisfy condition (T). If so, then $\mathrm{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ is #P-hard given that $\widehat{\mathcal{F}}$ contains a 6-ary signature $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$ and $\mathcal{F}$ does not satisfy condition (T). By the remark above, it suffices to show that $\mathrm{Holant}(\neq_2 \mid \widehat{f_6}, \widehat{\mathcal{F}})$ is #P-hard for any real-valued $\mathcal{F}$ that does not satisfy condition (T). We go back to real-valued Holant problems under the Z-transformation. Consider $\mathrm{Holant}(f_6, \mathcal{F})$. We have

$$f_6 = \chi_S(-1)^{x_1+x_2+x_3+x_1x_2+x_2x_3+x_1x_3+x_1x_4+x_2x_5+x_3x_6}$$

where χ_S is the 0-1 indicator function on the set

$$S = \mathscr{S}(f_6) = \{\alpha \in \mathbb{Z}_2^6 \mid \mathrm{wt}(\alpha) \equiv 0 \bmod 2\}.$$

f_6 has quite a similar matrix form as that of $\widehat{f_6}$,

$$M_{123,456}(f_6) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}.$$

By merging two variables of f_6 using $=_2$, we can realize all four Bell binary signatures, i.e., $\mathrm{Holant}(\mathcal{B}, f_6, \mathcal{F}) \leqslant_T \mathrm{Holant}(f_6, \mathcal{F})$. We define the problem $\mathrm{Holant}^b(\mathcal{F})$ to be $\mathrm{Holant}(\mathcal{B} \cup \mathcal{F})$. For all $\{i, j\}$, consider 4-ary signatures $\partial^-_{ij} f_6$, $\widehat{\partial}_{ij} f_6$ and $\widehat{\partial}^-_{ij} f_6$ realized by connecting variables x_i and x_j of f using $=^-_2$, $\neq_2$ and $\neq^-_2$ respectively. If there is one that is not in $\mathcal{O}^{\otimes}$, then by Lemma V.1, we get #P-hardness. However, it is observed in [14] that f_6 satisfies the following Bell property. Note that $\mathcal{B} \subseteq \mathcal{O}$, we have $\mathcal{B}^{\otimes} \subseteq \mathcal{O}^{\otimes}$.

Definition VI.2 (Bell property)**.** *A signature f satisfies the Bell property if for all pairs of indices $\{i, j\}$, $\partial_{ij} f$, $\partial^-_{ij} f$, $\widehat{\partial}_{ij} f$ and $\widehat{\partial}^-_{ij} f$ are in $\mathcal{B}^{\otimes}$.*

Let $f_6^H = Hf_6$, where $H = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}\right]$. Let $\mathcal{F}_6$ and $\mathcal{F}_6^H$ be the sets of 6-ary signatures realized by connecting each variable x_i of f_6 and f_6^H with one variable of a binary signature $b_i \in \mathcal{B}$ respectively. All signatures in $\mathcal{F}_6 \cup \mathcal{F}_6^H$ satisfy the Bell property. Together with $\mathcal{B}^{\otimes}$, they capture all 6-ary signatures that are a serious obstacle to our induction proof.

Lemma VI.3. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *If $\mathcal{F}$ contains a nonzero signature f of arity at most* 6 *where $f \notin \mathcal{B}^{\otimes} \cup \mathcal{F}_6 \cup \mathcal{F}_6^H$, then* $\text{Holant}(f_6, \mathcal{F})$ *is #P-hard.*

By Lemma VI.3, if we could realize a signature of arity at most 6 that is not in $\mathcal{B}^{\otimes} \cup \mathcal{F}_6 \cup \mathcal{F}_6^H$ using $\mathcal{B}$ and f_6, then there would be a somewhat more straightforward proof. However, after many failed attempts, we believe there is a more intrinsic reason why this approach cannot work.

Conjecture VI.4. *All signatures of arity at most* 6 *realizable from $\mathcal{B} \cup \{f_6\}$ are in $\mathcal{B}^{\otimes} \cup \mathcal{F}_6$, and all signatures of arity at most* 6 *realizable from $\mathcal{B} \cup \{f_6^H\}$ are in $\mathcal{B}^{\otimes} \cup \mathcal{F}_6^H$.*

So to prove the #P-hardness of $\text{Holant}^b(f_6, \mathcal{F})$, we have to make additional use of $\mathcal{F}$. In particular, we need to use a non-affine signature in $\mathcal{F}$.

Lemma VI.5. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *Then,* $\text{Holant}^b(f_6, \mathcal{F})$ *is #P-hard.*

Proof Sketch. We first show that every signature in $\mathcal{F}$ has parity. Otherwise, by induction we can realize a binary signature with no parity, which is not in $\mathcal{B}$. By Lemma VI.3, we get #P-hardness.

Since $\mathcal{F}$ does not satisfy condition (T), $\mathcal{F} \nsubseteq \mathscr{A}$. Thus, $\mathcal{F}$ contains a signature $f \notin \mathscr{A}$. Suppose that it has arity $2n$. We prove #P-hardness by induction on $2n$. Note that $\mathcal{B}^{\otimes} \cup \mathcal{F}_6 \cup \mathcal{F}_6^H \subseteq \mathscr{A}$. When $2n = 2, 4$ or 6, $f \notin \mathscr{A}$ implies that $f \notin \mathcal{B}^{\otimes} \cup \mathcal{F}_6 \cup \mathcal{F}_6^H$. Then, by Lemma VI.3, we get #P-hardness.

Suppose that $2n \geqslant 8$. We want to show that we can realize a signature $g \notin \mathscr{A}$ of arity $2k \leqslant 2n-2$ from f. Again, we may assume that f is irreducible and hence f satisfies 2ND-ORTH. For all pairs of indices $\{i, j\}$, consider signatures $\partial_{ij} f$, $\partial_{ij}^{-} f$, $\widehat{\partial}_{ij} f$ and $\widehat{\partial}_{ij}^{-} f$. If there is one signature that is not in $\mathscr{A}$, then we are done by induction. Otherwise, $\partial_{ij} f, \partial_{ij}^{-} f, \widehat{\partial}_{ij} f, \widehat{\partial}_{ij}^{-} f \in \mathscr{A}$ for all $\{i, j\}$. We denote this property by $f \in \int_{\mathcal{B}} \mathscr{A}$. Assuming that f has parity, f satisfies 2ND-ORTH and $f \in \int_{\mathcal{B}} \mathscr{A}$, we want to reach a contradiction by showing that this would force f itself belong to $\mathscr{A}$.

1) First, under the above assumptions, we ask whether all nonzero entries of f have the same norm. We can show that the answer is yes, but only for signatures of arity $2n \geqslant 10$. For a signature f of arity $2n = 8$, we show that either all nonzero entries of f have the same norm, or the following signatures g_8 or g_8' are realizable. These two signatures are defined by $g_8 = \chi_S - 4 \cdot f_8$ and $g_8' = q_8 - 4 \cdot f_8$, where
$$S = \mathscr{S}(q_8) = \{\alpha \in \mathbb{Z}_2^8 \mid \text{wt}(\alpha) \equiv 0 \bmod 2\},$$
$$q_8 = \chi_S (-1)^{\sum_{1 \leqslant i < j \leqslant 8} x_i x_j},$$
and $f_8 = \chi_T$ is the indicator function on the set
$$\begin{aligned} T = \mathscr{S}(f_8) = \{(x_1, x_2, \ldots, x_8) \in \mathbb{Z}_2^8 \mid \\ x_1 + x_2 + x_3 + x_4 = 0, \\ x_5 + x_6 + x_7 + x_8 = 0, \\ x_1 + x_2 + x_5 + x_6 = 0, \\ x_1 + x_3 + x_5 + x_7 = 0\}. \end{aligned}$$
It is here the function f_8 makes its first appearance. It is discovered by analyzing the independence number of a family of spacial graphs (with vertex set on even-weighted hypercube nodes), which should be of independent interest. Later we will give some more extraordinary properties about f_8. Clearly, $g_8, g_8' \notin \mathscr{A}$ since their nonzero entries have two different norms 1 and 3. One can check that g_8 and g_8' have parity, g_8 and g_8' satisfy 2ND-ORTH and $g_8, g_8' \in \int_{\mathcal{B}} \mathscr{A}$. Thus, one cannot get a non-affine signature by connecting two variables of g_8 or g_8' using signatures in $\mathcal{B}$. However, fortunately by merging two arbitrary variables of g_8 using $=_2$ and two arbitrary variables of g_8' using $=_2^-$, we can get 6-ary irreducible signatures that do not satisfy 2ND-ORTH. Thus, we get #P-hardness.
2) Then, by further assuming that nonzero entries of f have the same norm, we show that f has affine support, or we get #P-hardness directly by Lemma VI.3.
3) Finally, further assuming that f has affine support, we ask whether f itself is an affine signature. Again the answer is yes only for signature of arity $2n \geqslant 10$. For a signature f of arity $2n = 8$, we show that either $f \in \mathscr{A}$ or the following signature is realizable.
$$h_8 = \chi_T (-1)^{x_1x_2x_3 + x_1x_2x_5 + x_1x_3x_5 + x_2x_3x_5},$$
where $T = \mathscr{S}(h_8) = \mathscr{S}(f_8)$. Note that in the support $\mathscr{S}(h_8)$, by taking x_1, x_2, x_3, x_5 as free

variables, the remaining 4 variables are mod 2 sums of $\binom{4}{3}$ subsets of $\{x_1, x_2, x_3, x_5\}$. Clearly, h_8 is not affine, but it has affine support and all its nonzero entries have the same norm. One can check that h_8 satisfies 2ND-ORTH and $h_8 \in \int_{\mathcal{B}} \mathscr{A}$. But still fortunately, by merging h_8 using $=_2$, we can realize a 6-ary signature that is not in $\mathcal{B}^{\otimes} \cup \mathcal{F}_6 \cup \mathcal{F}_6^H$. By Lemma VI.3, we get #P-hardness.

Thus, starting with a signature $f \notin \mathscr{A}$ of arity $2n \geqslant 10$, by merging f using $\mathcal{B}$ step by step, we can finally realize a signature $g \notin \mathscr{A}$ of arity at most 8, or we get #P-hardness directly. Then, from the 8-ary signature g, we can realize a signature h of arity 6 which is not in $\mathcal{B}^{\otimes} \cup \mathcal{F}_6 \cup \mathcal{F}_6^H$. Thus, by Lemma VI.3, we are done. ■

Combining Lemmas VI.1 and VI.5, we have the following result.

Lemma VI.6. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *Let $\widehat{\mathcal{F}} = Z^{-1}\mathcal{F}$. If $\widehat{\mathcal{F}}$ contains a 6-ary signature $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$, then* $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ *is #P-hard.*

VII. FINAL OBSTACLE: AN 8-ARY SIGNATURE WITH THE STRONG BELL PROPERTY

Now, we analyze the signature f_8. One can check that f_8 also satisfies both 2ND-ORTH and $f_8 \in \int \mathcal{O}^{\otimes}$. Also, f_8 is unchanged under the holographic transformation by $\mathbb{Z}^{-1}$, i.e., $\widehat{f_8} = Z^{-1} f_8 = f_8$. The following theorem shows how f_8 was discovered.

Lemma VII.1. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *Let $\widehat{\mathcal{F}} = Z^{-1}\mathcal{F}$. If $\widehat{\mathcal{F}}$ contains an 8-ary signature $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$, then*

- $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ *is #P-hard, or*
- $\operatorname{Holant}(\neq_2 \mid \widehat{f_8}, \widehat{Q}\widehat{\mathcal{F}}) \leqslant_T \operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ *for some* $\widehat{Q} \in \widehat{\mathbf{O}_2}$.

Now, we want to show that $\operatorname{Holant}(\neq_2 \mid \widehat{f_8}, \widehat{Q}\widehat{\mathcal{F}})$ is #P-hard for any real-valued $\mathcal{F}$ that does not satisfy condition (T). If so, then $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ is #P-hard given $\widehat{\mathcal{F}}$ contains an 8-ary signature $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$ and $\mathcal{F}$ does not satisfy condition (T). By the remark after Lemma VI.1, it suffices to show that $\operatorname{Holant}(\neq_2 \mid \widehat{f_8}, \widehat{\mathcal{F}})$ is #P-hard for any real-valued $\mathcal{F}$ that does not satisfy condition (T). Again, we go back to real-valued Holant problems under the Z-transformation. Consider $\operatorname{Holant}(f_8, \mathcal{F})$. Remember that $f_8 = \widehat{f_8}$.

Lemma VII.2. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *If $\mathcal{F}$ contains a nonzero binary signature $b \notin \mathcal{B}^{\otimes}$, then* $\operatorname{Holant}(b, f_8, \mathcal{F})$ *is #P-hard.*

Similar to the proof of Lemma VI.5, we can show the following result.

Lemma VII.3. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *Then,* $\operatorname{Holant}^b(f_8, \mathcal{F})$ *is #P-hard.*

So, if we could realize $\mathcal{B}$ from f_8 by gadget construction as we did for f_6, then we would be done. However, since f_8 has even parity and all its entries are non-negative, all gadgets realizable from f_8 have even parity and have non-negative entries. Thus, $=_2^-$, $\neq_2$ and $\neq_2^-$ *cannot* be realized from f_8 by gadget construction. In fact, f_8 satisfies the following strong Bell property [14] .

Definition VII.4. *A signature f satisfies the strong Bell property if for all pairs of indices $\{i,j\}$, $\partial_{ij} f \in \{=_2\}^{\otimes}$, $\partial_{ij}^- f \in \{=_2^-\}^{\otimes}$, $\widehat{\partial}_{ij} f \in \{\neq_2\}^{\otimes}$ and $\widehat{\partial_{ij}^-} f \in \{\neq_2^-\}^{\otimes}$.*

However, critically based on the strong Bell property of f_8, we prove that $\operatorname{Holant}^b(f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(f_8, \mathcal{F})$ in a novel way. We define the following Holant problems with limited appearance.

Definition VII.5. *Let $\mathcal{F}$ be a signature set containing a signature f. The problem* $\operatorname{Holant}(f^{\leqslant k}, \mathcal{F})$ *contains all instances of* $\operatorname{Holant}(\mathcal{F})$ *where the signature f appears at most k times.*

Lemma VII.6. *For any $b \in \mathcal{B}$ and any $\mathcal{F}$,* $\operatorname{Holant}(b, f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(b^{\leqslant 2}, f_8, \mathcal{F})$.

Proof: Consider an instance Ω of $\operatorname{Holant}(b, f_8, \mathcal{F})$. Suppose that b appears n times in Ω. If $n \leqslant 2$, then Ω is already an instance of $\operatorname{Holant}(b^{\leqslant 2}, f_8, \mathcal{F})$. Otherwise, $n \geqslant 3$. Consider the gadget realized by connecting two variables x_i and x_j of f_8 using b. We denote it by $\partial_{ij}^b f$. Since f_8 satisfies the strong Bell property, $\partial_{ij}^b f = b^{\otimes 3}$. Thus, by replacing three occurrences of b in Ω by the gadget $\partial_{ij}^b f$, we can reduce the number of occurrences of b by 2. We can carry out this replacement a linear number of times to obtain an equivalent instance of $\operatorname{Holant}(b^{\leqslant 2}, f_8, \mathcal{F})$, of size linear in Ω. ■

Now, we are ready to prove the reduction $\operatorname{Holant}^b(f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(f_8, \mathcal{F})$. Note that if $\operatorname{Holant}(f_8, \mathcal{F})$ is #P-hard, then the reduction holds trivially. For any $b \in \mathcal{B}$, if we connect a variable of b with a variable of another copy of b using $=_2$, we get $\pm(=_2)$. Also, for any $b_1, b_2 \in \mathcal{B}$ where $b_1 \neq b_2$ if we connect the two variables of b_1 with the two variables of b_2, we get a value 0.

Lemma VII.7. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *Then,* $\operatorname{Holant}^b(f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(f_8, \mathcal{F})$.

Proof: We prove this reduction in two steps.

Step 1. There exists a signature $b_1 \in \mathcal{B}\backslash\{=_2\}$ such that $\operatorname{Holant}(b_1, f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(f_8, \mathcal{F})$.

We consider all binary signatures and 4-ary signatures realizable by gadget constructions from $\{f_8\} \cup \mathcal{F}$. If a binary signature $g \notin \mathcal{B}$ is realizable from $\{f_8\} \cup \mathcal{F}$, then by Lemma VII.2, $\operatorname{Holant}(f_8, \mathcal{F})$ is #P-hard, and we are done. If a binary signature $g \in \mathcal{B}\backslash\{=_2\}$ is realizable from $\{f_8\} \cup \mathcal{F}$, then we are done with $b_1 = g$. So we may assume that all binary signatures g realizable from $\{f_8\} \cup \mathcal{F}$ are $=_2$ (up to a scalar) or the binary zero signature, i.e., $g = \mu \cdot (=_2)$ for some $\mu \in \mathbb{R}$. Similarly, if a nonzero 4-ary signature $h \notin \mathcal{B}^{\otimes 2}$ is realizable, then we have $\operatorname{Holant}(f_8, \mathcal{F})$ is #P-hard. If a nonzero 4-ary signature $h \in \mathcal{B}^{\otimes 2}\backslash\{=_2\}^{\otimes 2}$ is realizable, then we can realize a binary signature $b_1 \in \mathcal{B}\backslash\{=_2\}$ by factorization, and we are done. Thus, we may assume that all 4-ary signatures h realizable from $\{f_8\} \cup \mathcal{F}$ are scalar multiples of $(=_2)^{\otimes 2}$, i.e., $h = \lambda \cdot (=_2)^{\otimes 2}$ for some $\lambda \in \mathbb{R}$.

Now, let b_1 be a signature in $\mathcal{B}\backslash\{=_2\}$. We show that $\operatorname{Holant}(b_1^{\leqslant 2}, f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(f_8, \mathcal{F})$. Consider an instance Ω of $\operatorname{Holant}(b_1^{\leqslant 2}, f_8, \mathcal{F})$.

- If b_1 does not appear in Ω, then Ω is already an instance of $\operatorname{Holant}(f_8, \mathcal{F})$.
- If b_1 appears exactly once in Ω (we may assume it does connect to itself), then we may consider *the rest of* Ω *that connects to* b_1 as a gadget realized from $\{f_8\} \cup \mathcal{F}$, which must have signature $\lambda \cdot (=_2)$, for some $\lambda \in \mathbb{R}$. Connecting the two variables of b_1 by $(=_2)$ for every $b_1 \in \mathcal{B}\backslash\{=_2\}$ will always gives 0. Thus, $\operatorname{Holant}(\Omega) = 0$.
- Suppose b_1 appears exactly twice in Ω. It is easy to handle when the two copies of b_1 form a gadget of arity 0 or 2 to the rest of Ω. So we may assume they are connected to the rest of Ω in such a way that the rest of Ω forms a 4-ary gadget h realized from $\{f_8\} \cup \mathcal{F}$. We can name the four dangling edges of h in any specific ordering as (x_1, x_2, x_3, x_4). Then

  $$h(x_1, x_2, x_3, x_4) = \lambda \cdot (=_2)(x_1, x_j) \otimes (=_2)(x_k, x_\ell)$$

  for some partition $\{1, 2, 3, 4\} = \{1, j\} \sqcup \{k, \ell\}$, and some $\lambda \in \mathbb{R}$. (Note that while we have named four specific dangling edges as (x_1, x_2, x_3, x_4), the specific partition $\{1, 2, 3, 4\} = \{1, j\} \sqcup \{k, \ell\}$ and the value λ are unknown at this point.) We consider the following three instances Ω_{12}, Ω_{13}, and Ω_{14}, where Ω_{1s} $(s \in \{2, 3, 4\})$ is the instance formed by merging variables x_1 and x_s of h using $=_2$, and merging the other two variables of h using $=_2$. Since h is a gadget realized from $\{f_8\} \cup \mathcal{F}$, Ω_{12}, Ω_{13}, and Ω_{14} are instances of $\operatorname{Holant}(f_8, \mathcal{F})$. Note that $\operatorname{Holant}(\Omega_{1s}) = 4\lambda$ when $s = j$ and $\operatorname{Holant}(\Omega_{1s}) = 2\lambda$ otherwise. Thus, by computing $\operatorname{Holant}(\Omega_{1s})$ for $s \in \{2, 3, 4\}$, we can get λ, and if $\lambda \neq 0$ the partition $\{1, j\} \sqcup \{k, \ell\}$ of the four variables. Thus we can get the exact structure of the 4-ary gadget h. In either case (whether $\lambda = 0$ or not), we can compute the value of $\operatorname{Holant}(\Omega)$.

Thus, $\operatorname{Holant}(b_1^{\leqslant 2}, f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(f_8, \mathcal{F})$. By Lemma VII.6, $\operatorname{Holant}(b_1, f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(f_8, \mathcal{F})$.

Step 2. For any $b_1 \in \mathcal{B}\backslash\{=_2\}$, we have $\operatorname{Holant}^b(f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(b_1, f_8, \mathcal{F})$.

We show that for some binary signature $b_2 \in \mathcal{B}\backslash\{=_2, b_1\}$ we have the reduction $\operatorname{Holant}(b_2, b_1, f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(b_1, f_8, \mathcal{F})$. In other words, we can get another $b_2 \in \mathcal{B}\backslash\{=_2, b_1\}$. Then, (using properties of $\mathcal{B}$) by connecting one variable of b_1 and one variable of b_2 using $=_2$, we get the remaining signature in $\mathcal{B}\backslash\{=_2, b_1, b_2\}$. Then, the lemma is proved. The proof is similar to the proof in Step 1. We consider *all* binary and 4-ary gadgets realizable from $\{b_1, f_8\} \cup \mathcal{F}$. Still, we may assume that all realizable binary signatures are of the form $\mu \cdot (=_2)$ or $\mu \cdot b_1$ for some $\mu \in \mathbb{R}$, and all realizable 4-ary signatures are of form $\lambda \cdot (=_2)^{\otimes 2}$, $\lambda \cdot b_1^{\otimes 2}$ or $\lambda \cdot (=_2) \otimes b_1$ for some $\lambda \in \mathbb{R}$. Otherwise, we can show that $\operatorname{Holant}(b_1, f_8, \mathcal{F})$ is #P-hard or we realize a signature $b_2 \in \mathcal{B}\backslash\{=_2, b_1\}$ directly by gadget construction.

Then, let b_2 be an arbitrary signature in $\mathcal{B}\backslash\{=_2, b_1\}$. We show that

$$\operatorname{Holant}(b_2^{\leqslant 2}, b_1, f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(b_1, f_8, \mathcal{F}).$$

Consider an instance Ω of $\operatorname{Holant}(b_2^{\leqslant 2}, b_1, f_8, \mathcal{F})$. If b_2 does not appear in Ω, then Ω is already an instance of $\operatorname{Holant}(b_1, f_8, \mathcal{F})$. If b_2 appears exactly once in Ω, then it is connected with a binary gadget g where $g = \mu \cdot (=_2)$ or $g = \mu \cdot b_1$. In both cases, the evaluation is 0. Thus, $\operatorname{Holant}(\Omega) = 0$. Suppose b_2 appears exactly twice in Ω. Again it is easy to handle the case if the rest of Ω forms a gadget of arity 0 or 2 to the two occurrences of b_2. So we may assume the two occurrences of b_2 are connected to a 4-ary gadget $h = \lambda \cdot (=_2)^{\otimes 2}$, $\lambda \cdot b_1^{\otimes 2}$ or $\lambda \cdot (=_2) \otimes b_1$. We denote the four variables of h by (x_1, x_2, x_3, x_4), by an arbitrary ordering of the four dangling edges. Then $h(x_1, x_2, x_3, x_4) = \lambda \cdot h_1(x_1, x_j) \otimes h_2(x_k, x_\ell)$ where $h_1, h_2 \in \{=_2, b_1\}$, for some λ and $\{j, k, \ell\} = \{2, 3, 4\}$. (Note that at the moment the values λ and j, k, ℓ are unknown.) We consider the following three instances Ω_{12}, Ω_{13} and Ω_{14}, where Ω_{1s} $(s \in \{2, 3, 4\})$ is the instance formed by connecting variables x_1 and x_s of h using $=_2$, and connecting the other two variables of

h using $=_2$. Clearly, Ω_{12}, Ω_{13} and Ω_{14} are instances of $\operatorname{Holant}(b_1, f_8, \mathcal{F})$. Consider the evaluations of these instances. We have three cases.

- If $h_1 = h_2 = (=_2)$, then $\operatorname{Holant}(\Omega_{1s}) = 4\lambda$ when $s = j$ and $\operatorname{Holant}(\Omega_{1s}) = 2\lambda$ when $s \neq j$.
- If $h_1 = h_2 = b_1$, then $\operatorname{Holant}(\Omega_{1s}) = 0$ when $s = j$. If $M(b_1)$ is the 2 by 2 matrix form for the binary signature b_1 where we list its first variable as row index and second variable as column index, then we have $\operatorname{Holant}(\Omega_{1k}) = \lambda \cdot \operatorname{tr}(M(b_1)M(b_1)^{\mathrm{T}})$, and $\operatorname{Holant}(\Omega_{1\ell}) = \lambda \cdot \operatorname{tr}(M(b_1)^2)$, where tr denotes trace. For $b_1 = (=_2^-)$ or $(\neq_2^+)$, the matrix $M(b_1)$ is symmetric, and the value $\operatorname{Holant}(\Omega_{1s}) = 2\lambda$ in both cases $s = k$ or $s = \ell$. For $b_1 = (\neq_2^-)$, $M(b_1)^{\mathrm{T}} = -M(b_1)$, and we have $\operatorname{Holant}(\Omega_{1k}) = 2\lambda$, and $\operatorname{Holant}(\Omega_{1\ell}) = -2\lambda$.
- If one of h_1 and h_2 is $=_2$ and the other is b_1, then $\operatorname{Holant}(\Omega_{1s}) = 0$ for all $s \in \{j, k, \ell\}$.

Thus, if the values of $\operatorname{Holant}(\Omega_{1s})$ for $s \in \{2, 3, 4\}$ are not all zero, then $\lambda \neq 0$ and the third case is impossible, and we can tell whether h is in the form $\lambda \cdot (=_2)^{\otimes 2}$ or $\lambda \cdot (b_1)^{\otimes 2}$. Moreover we can get the exact structure of h, i.e., the value λ and the decomposition form of h_1 and h_2. Otherwise, the values of $\operatorname{Holant}(\Omega_{1s})$ for $s \in \{2, 3, 4\}$ are all zero. Then we can write $h = \lambda \cdot (=_2)(x_1, x_j) \otimes b_1(x_k, x_\ell)$ or $h = \lambda \cdot b_1(x_1, x_j) \otimes (=_2)(x_k, x_\ell)$, including possibly $\lambda = 0$, which means $h \equiv 0$. (Note that if $\lambda \neq 0$, this uniquely identifies that we are in the third case; if $\lambda = 0$ then this form is still formally valid, even though we cannot say this uniquely identifies the third case. But when $\lambda = 0$ all three cases are the same, i.e., $h \equiv 0$.) At this point we still do not know the exact value of λ and the decomposition form of h.

We further consider the following three instances Ω'_{12}, Ω'_{13} and Ω'_{14}, where Ω'_{1s} $(s \in \{2, 3, 4\})$ is the instance formed by connecting variables x_1 and x_s of h using b_1, and connecting the other two variables of h using $=_2$. It is easy to see that Ω'_{12}, Ω'_{13} and Ω'_{14} are instances of $\operatorname{Holant}(b_1, f_8, \mathcal{F})$. Consider the evaluations of these instances.

- If $h_1 = (=_2)(x_1, x_j)$, then $\operatorname{Holant}(\Omega'_{1s}) = 0$ when $s = j$. Also we have $\operatorname{Holant}(\Omega_{1k}) = \lambda \cdot \operatorname{tr}(M(b_1)^2)$, and $\operatorname{Holant}(\Omega_{1\ell}) = \lambda \cdot \operatorname{tr}(M(b_1)M(b_1)^{\mathrm{T}})$. For $b_1 = (=_2^-)$ or $\neq_2^+$, the matrix $M(b_1)$ is symmetric, and the value $\operatorname{Holant}(\Omega_{1s}) = 2\lambda$ in both cases $s = k$ or $s = \ell$. For $b_1 = (\neq_2^-)$, $M(b_1)^{\mathrm{T}} = -M(b_1)$, and we have $\operatorname{Holant}(\Omega_{1k}) = -2\lambda$, and $\operatorname{Holant}(\Omega_{1\ell}) = 2\lambda$.
- If $h_1 = b_1(x_1, x_j)$, then $\operatorname{Holant}(\Omega'_{1s}) = 4\lambda$ when $s = j$ and $\operatorname{Holant}(\Omega'_{1s}) = 2\lambda$ when $s \neq j$.

Thus, by further computing $\operatorname{Holant}(\Omega'_{1s})$ for $s \in \{2, 3, 4\}$, we can get the exact structure of h.

Therefore, by querying $\operatorname{Holant}(b_1, f_8, \mathcal{F})$ at most 6 times, we can compute h exactly. Then, we can compute $\operatorname{Holant}(\Omega)$ easily. Thus,

$$\operatorname{Holant}(b_2^{\leqslant 2}, b_1, f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(b_1, f_8, \mathcal{F}).$$

By Lemma VII.6,

$$\operatorname{Holant}(b_2, b_1, f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(b_1, f_8, \mathcal{F}).$$

The other signature in $\mathcal{B} \backslash \{=_2, b_1, b_2\}$ can be realized by connecting b_1 and b_2. Thus, $\operatorname{Holant}^b(f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(b_1, f_8, \mathcal{F})$.

Therefore, $\operatorname{Holant}^b(f_8, \mathcal{F}) \leqslant_T \operatorname{Holant}(f_8, \mathcal{F})$. ∎

Combining with Lemmas VII.1, VII.3 and VII.7, we have the following result.

Lemma VII.8. *Suppose that $\mathcal{F}$ is a set of real-valued signatures of even arity and $\mathcal{F}$ does not satisfy condition* (T). *Let $\widehat{\mathcal{F}} = Z^{-1}\mathcal{F}$. If $\widehat{\mathcal{F}}$ contains an 8-ary signature $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$, then $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ is #P-hard.*

Finally, we give a proof of Theorem I.2.

Proof: First, we may assume that $\mathcal{F}$ is a set of signatures of even arities and it does not satisfy condition (T). We show that $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}}) \equiv_T \operatorname{Holant}(\mathcal{F})$ is #P-hard. Since $\mathcal{F}$ does not satisfy condition (T), $\widehat{\mathcal{F}} \not\subseteq \mathscr{T}$. Since $\widehat{\mathcal{O}}^{\otimes} \subseteq \mathscr{T}$, there is a signature $\widehat{f} \in \widehat{\mathcal{F}}$ of arity $2n$ such that $\widehat{f} \notin \widehat{\mathcal{O}}^{\otimes}$. We prove this theorem by induction on $2n$. When $2n \leqslant 8$, by Lemmas IV.1, V.1, VI.6 and VII.8, $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ is #P-hard.

Inductively, suppose for some $2k \geqslant 8$, if $2n \leqslant 2k$, then $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ is #P-hard. We consider $2n = 2k + 2 \geqslant 10$. By Lemma V.2, $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ is #P-hard, or $\operatorname{Holant}(\neq \mid \widehat{g}, \widehat{\mathcal{F}}) \leqslant_T \operatorname{Holant}(\neq \mid \widehat{\mathcal{F}})$ for some $\widehat{g} \notin \widehat{\mathcal{O}}^{\otimes}$ of arity $\leqslant 2k$. By the induction hypothesis, $\operatorname{Holant}(\neq \mid \widehat{g}, \widehat{\mathcal{F}})$ is #P-hard. Thus, $\operatorname{Holant}(\neq_2 \mid \widehat{\mathcal{F}})$ is #P-hard. ∎

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Epilogue

The young knight Siegfried and his tutor Jeyoda set out for their life journey together. Their aim is to pacify the real land of Holandica, to bring order and unity.

In the past decade, brave knights have battled innumerable demons and creatures, and have conquered the more familiar portion of Holandica, called Holandica Symmetrica. Along the way they have also been victorious by channeling the power of various deities known as Unary Oracles. In the past few years this brotherhood of the intrepid have also gained great power from the beneficent god Orieneuler and enhanced their skills in a more protected world ruled by Count SeaEspie.

"But prepared we must be," Jeyoda reminds Siegfried, "arduous, our journey will become." As the real land of Holandica is teeming with unknowns, who knows what wild beasts and creatures they may encounter. Siegfried nods, but in his heart he is confident that his valor and power will be equal to the challenge.

They have recently discovered a treasure sword. This is their second gift from the Cathedral Orthogonia, more splendid and more powerful than the first. In their initial encounters with the minion creatures in their journey, the second sword from Cathedral Orthogonia proved to be invincible.

These initial victories laid the foundation for their journey, but also a cautious optimism sets in. Perhaps with their new powerful sword in hand, final victory will not be that far away.

Just as they savor these initial victories, things start to change. As they enter the Kingdom of Degree-Six everything becomes strange. Subliminally they feel the presence of a cunning enemy hiding in the darkness. Gradually they are convinced that this enemy possesses a special power that eludes the ongoing campaign, and in particular their magic sword. After a series of difficult and protracted battles with many twists and turns, their nemesis, the Lord of Intransigence slowly reveals his face. The Lord of Intransigence has a suit of magic armor, called the Bell Spell, that hides and protects him so well that the sword of Cathedral Orthogonia cannot touch him.

Siegfried and Jeyoda know that in order to conquer the Lord of Intransigence, they need all the skills and wisdom they have. Although the Lord of Intransigence has a strong armor, he has a weakness. The armor is maintained by four little elfs called the Bell Binaries. The next battle is tough and long. Siegfried and Jeyoda hit upon the idea of convincing the Bell Binaries to stage a mutiny. With his four little elfs turning against him, his armor loses its magic, and the Kingdom of Degree-Six is conquered. In the aftermath of this victory, Siegfried and Jeyoda also collect some valuable treasures that will come in handy in their next campaign.

After defeating the Lord of Intransigence, Siegfried and Jeyoda enter the Land of Degree-Eight. Now they are very careful. After meticulous reconnaissance, they finally identify the most fearsome enemy, the Queen of the Night. Taking a page from their battle with the Lord of Intransigence they look for opportunities to gain help from within the enemy camp. However, the Queen of the Night has the strongest protective coat called the Strong Bell Spell. This time there is no way to summon help from within the Queen's own camp. In fact, her protective armor is so strong that any encounter with Siegfried and Jeyoda's sword makes her magically disappear in a puff of white smoke.

But, everyone has a weakness. For the Queen, her vanishing act also brings the downfall. After plotting the strategy for a long time, Siegfried and Jeyoda use a magical potion to create from nothing the helpers needed to defeat the Queen.

Buoyed by their victory, they summon their last strength to secure the Land of Degree-Eight and beyond. Finally they bring complete order to the entire real land of Holandica. At their celebratory banquet, they want to share the laurels with Knight Ming and Knight Fu who provided invaluable assistance in their journey; but the two brave and generous knights have retreated to their Philosopher's Temple and are nowhere to be found.

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