Explicit near-fully X-Ramanujan graphs

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Abstract—**Let p(Y1, ..., Yd, Z1, ..., Ze) be a self-adjoint noncommutative polynomial, with coefficients from $\mathbf{C}^{r \times r}$, in the indeterminates Y1,... , Yd (considered to be self-adjoint), the indeterminates Z1, ..., Ze, and their adjoints Z1*, ..., Ze*. Suppose Y1, ..., Yd are replaced by independent random n x n matching matrices, and Z1, ..., Ze are replaced by independent random n x n permutation matrices. Assuming for simplicity that p's coefficients are 0-1 matrices, the result can be thought of as a kind of random rn-vertex graph G. As n goes to infinity, there will be a natural limiting infinite graph X that covers any finite outcome for G. A recent landmark result of Bordenave and Collins shows that for any eps > 0, with high probability the spectrum of a random G will be eps-close in Hausdorff distance to the spectrum of X (once the suitably defined "trivial" eigenvalues are excluded). We say that G is "eps-near fully X-Ramanujan".**

Our work has two contributions: First we study and clarify the class of infinite graphs X that can arise in this way. Second, we derandomize the Bordenave–Collins result: for any X, we provide explicit, arbitrarily large graphs G that are covered by X and that have (nontrivial) spectrum at Hausdorff distance at most eps from that of X. This significantly generalizes the recent work of Mohanty et al., which provided explicit near-Ramanujan graphs for every degree d (meaning d-regular graphs with all nontrivial eigenvalues bounded in magnitude by 2sqrt(d-1) + eps).

As an application of our main technical theorem, we are also able to determine the "eigenvalue relaxation value" for a wide class of average-case degree-2 constraint satisfaction problems.

Keywords-spectral graph theory, random matrices, constraint satisfaction problems

I. INTRODUCTION

Let G be an n-vertex, d-regular graph. Its adjacency matrix A will always have a "trivial" eigenvalue of d corresponding to the eigenvector $\frac{1}{n}(1, 1, \ldots, 1)$, the stationary probability distribution for the standard random walk on G. Excluding this eigenvalue, a bound on the magnitude λ of the remaining nontrivial eigenvalues can be very useful; for example, λ can be used to control the mixing time of the random walk on G [1], the maximum cut in G [2], and the error in the Expander Mixing Lemma for G [3].

The Alon–Boppana theorem [4] gives a lower bound on how small λ can be, namely $2\sqrt{d-1} - o_{n\to\infty}(1)$. This number $2\sqrt{d-1}$ arises from the spectral radius $\rho(\mathbb{T}_d)$ of the infinite d-regular tree $\mathbb{T}_d$, which is the universal cover for all d-regular graphs ($d \geq 3$). Celebrated work of Lubotzky–Phillips–Sarnak [5] and Margulis [6] (see also [7]) shows

that for infinitely many d, there exists an explicit infinite family of d-regular graphs satisfying $\lambda \leq 2\sqrt{d-1}$. Graphs meeting this bound were dubbed *d-regular Ramanujan graphs*, and subsequent constructions [8], [9] gave explicit families of d-regular Ramanujan graphs whenever $d-1$ is a prime power. The fact that these graphs are optimal (spectral) expanders, together with the fact that they are *explicit* (constructible deterministically and efficiently), has made them useful in a variety of application areas in computer science, including coding theory [10], cryptography [11], and derandomization [12].

The analysis of LPS/Margulis Ramanujan graphs famously relies on deep results in number theory, and it is still unknown whether infinitely many d-regular Ramanujan exist when $d-1$ is not a prime power. On the other hand, if one is willing to settle for *nearly*-Ramanujan graphs, there is a simple though inexplicit way to construct them for any d and n: Friedman's landmark resolution [13] of Alon's conjecture shows that for any $\varepsilon > 0$, a *random* n-vertex d-regular graph has $\lambda \leq 2\sqrt{d-1} + \varepsilon$ with high probability (meaning probability $1 - o_{n\to\infty}(1)$). The proof of Friedman's theorem is also very difficult, although it was notably simplified by Bordenave [14]. The distinction between Ramanujan and nearly-Ramanujan does not seem to pose any problem for applications, but the lack of explicitness does, particularly (of course) for applications to derandomization.

There are several directions in which Friedman's theorem could conjecturally be generalized. One major such direction was conjectured by Friedman himself [15]: that for any fixed base graph K with universal cover tree $\mathfrak{X}$, a random n-lift $\mathbf{G}$ of K is nearly "$\mathfrak{X}$-Ramanujan" with high probability. Here the term "$\mathfrak{X}$-Ramanujan" refers to two properties: first, $\mathfrak{X}$ *covers* $\mathbf{G}$ in the graph theory sense; second, the "nontrivial" eigenvalues of $\mathbf{G}$, namely those not in $\mathrm{spec}(K)$, are bounded in magnitude by the spectral radius $\rho(\mathfrak{X})$ of $\mathfrak{X}$. The modifier "nearly" again refers to relaxing $\rho(\mathfrak{X})$ to $\rho(\mathfrak{X}) + \varepsilon$, here. (We remark that for bipartite K, Marcus, Spielman, and Srivastava [16] showed the existence of an exactly $\mathfrak{X}$-Ramanujan n-lift for every n.) An even stronger version of this conjecture would hold that $\mathbf{G}$ is near-*fully* $\mathfrak{X}$-Ramanujan with high probability; by this we mean that for every $\varepsilon > 0$, the nontrivial spectrum of $\mathbf{G}$ is ε-close in Hausdorff distance to the spectrum of $\mathfrak{X}$ (i.e., every nontrivial eigenvalue of $\mathbf{G}$ is within ε of a point in $\mathfrak{X}$'s spectrum, and vice versa).

This stronger conjecture — and in fact much more — was recently proven by Bordenave and Collins [17]. Indeed their work implies that for a wide variety of *non-tree* infinite graphs $\mathfrak{X}$, there is a random-lift method for generating arbitrarily large finite graphs, covered by $\mathfrak{X}$, whose nontrivial spectrum is near-fully $\mathfrak{X}$-Ramanujan. However besides universal cover trees, it is not made clear in [17] precisely to which $\mathfrak{X}$'s their results apply.

Our work has two contributions. First, we significantly clarify and partially characterize the class of infinite graphs $\mathfrak{X}$ for which the Bordenave–Collins result can be used; we term these *MPL graphs*. We establish that all free products of finite vertex-transitive graphs [18] (including Cayley graphs of free products of finite groups), free products of finite rooted graphs [19], additive products [20], and amalgamated free products [21], inter alia, are MPL graphs — but also, that MPL graphs must be unimodular, hyperbolic, and of finite treewidth. The second contribution of our work is to derandomize the Bordenave–Collins result: for every MPL graph $\mathfrak{X}$ and every $\varepsilon > 0$, we give a poly(n)-time deterministic algorithm that outputs a graph on $n' \sim n$ vertices that is covered by $\mathfrak{X}$ and whose nontrivial spectrum is ε-close in Hausdorff distance to that of $\mathfrak{X}$.

A. Bordenave and Collins's work

Rather than diving straight into the statement of Bordenave and Collins's main theorem, we will find it helpful to build up to it in stages.

d-regular graphs.: Let us return to the most basic case of random n-vertex, d-regular graphs. A natural way to obtain such a graph $\mathcal{G}_n$ (provided n is even) is to independently choose d uniformly random matchings $\mathcal{M}_1, \ldots \mathcal{M}_d$ on the same vertex set $V_n = [n] = \{1, 2, \ldots, n\}$ and to superimpose them. It will be important for us to remember which edge in $\mathcal{G}_n$ came from which matching, so let us think $\mathcal{M}_1, \ldots, \mathcal{M}_d$ as being *colored* with colors $1, \ldots, d$. Then $\mathcal{G}_n$ may be thought of as a "color-regular graph"; each vertex is adjacent to a single edge of each color.

Moving to linear algebra, the adjacency matrix $\mathbf{A}_n$ for $\mathcal{G}_n$ may be thought of as follows: First, we take the formal polynomial $p(Y_1, \ldots, Y_d) = Y_1 + \cdots + Y_d$. Next, we obtain $\mathbf{A}_n$ by substituting $Y_j = P_{\sigma_j}$ for each $j \in [d]$, where the σ_j's are independent uniformly random matchings on $[n]$ (i.e., permutations in $\mathfrak{S}(n)$ with all cycles of length 2) and where P_σ denotes the permutation matrix associated to σ.

If we fix a vertex $o \in V_n$ and a number $\ell \in \mathbb{N}$, with high probability the radius-ℓ neighborhood of o in $\mathcal{G}_n$ will look like the radius-ℓ neighborhood of the root of an infinite d-color-regular tree (i.e., the infinite d-regular tree in which each vertex is adjacent to one edge of each color). This tree may be identified with the Cayley graph of the free group $V_\infty = \mathbb{Z}_2 \star Z_2 \star \cdots \mathbb{Z}_2$ with generators $g_1, \ldots, g_d$. These generators act as permutations on V_∞ by left-multiplication. Indeed, if one writes P_{g_j} for the associated permutation

operator on $\ell_2(V_\infty)$, then the adjacency operator for the Cayley graph is $A_\infty = p(P_{g_1}, \ldots, P_{g_d})$.

Bordenave and Collins's generalization of Friedman's theorem may thus be viewed as follows: for $p(Y_1, \ldots, Y_d) = Y_1 + \cdots + Y_d$ we have that for any $\varepsilon > 0$, if $\mathbf{A}_n = p(P_{\sigma_1}, \ldots, P_{\sigma_d})$ and $A_\infty = p(P_{g_1}, \ldots, P_{g_d})$, then with high probability the "nontrivial" spectrum of $\mathbf{A}_n$ is ε-close in Hausdorff distance to the spectrum of A_∞. Here "nontrivial" refers to excluding $p(1, \ldots, 1) = d$.

Weighted color-regular graphs.: The Bordenave–Collins theorem is more general than this, however. It also applies to *(edge-)weighted* color-regular graphs. Let $a_1, \ldots, a_d \in \mathbb{R}$ be real weights associated with the d colors, and consider the more general linear polynomial $p(Y_1, \ldots, Y_d) = a_1 Y_1 + \cdots + a_d Y_d$. Then $\mathbf{A}_n = p(P_{\sigma_1}, \ldots, P_{\sigma_d})$ is the (weighted) adjacency matrix of a random "color-regular" graph in which each vertex is adjacent to one edge each of colors $1, \ldots, d$, with edge-weights $a_1, \ldots, a_d$ respectively. Similarly, $A_\infty = p(P_{g_1}, \ldots, P_{g_d})$ is the adjacency operator on $\ell_2(V_\infty)$ for the version of the d-color-regular infinite tree in which the edges of color j are weighted by a_j. Again, the Bordenave–Collins result implies that for all $\varepsilon > 0$, with high probability the nontrivial spectrum of $\mathbf{A}_n$ (meaning, when $p(1, \ldots, 1) = \sum_j a_j$ is excluded) is ε-close in Hausdorff distance to the spectrum of A_∞.

There are several examples where this may be of interest. The first is non-standard random walks on color-regular graphs; for example, taking $a_1 = 1/2$, $a_2 = 1/3$, $a_3 = 1/6$ models random walks where one always "takes the red edge with probability $1/2$, the blue edge with probability $1/3$, and the green edges with probability $1/6$". Another example is the case of $a_1 = \cdots = a_{d/2} = +1$, $a_{d/2+1} = \cdots = a_d = -1$. Here $\mathbf{A}_n$ is a d-regular random graphs in which each vertex is adjacent to $d/2$ edges of weight $+1$ and $d/2$ edges of weight -1. This is a natural model for random d-regular instances of the *2XOR* constraint satisfaction problem. Studying the maximum-magnitude eigenvalue of $\mathbf{A}_n$ is interesting because it commonly used to efficiently compute an upper bound on the optimal CSP solution (which is NP-hard to find in the worst case); see Section I-D for further discussion. Conveniently, the "trivial eigenvalue" of $\mathbf{A}_n$ is 0, and the spectrum of A_∞ is easily seen to be identical to that of the d-regular infinite tree, $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. Thus this setting is very similar to that of unweighted random d-regular graphs, but without the annoyance of the eigenvalue of d.

Self-loops and general permutations.: The Bordenave–Collins theorem is more general than this, however. Here are two more modest generalizations it allows for. First, one can allow "self-loops" in our template polynomials. In other words, one can generalize to polynomials $p(Y_1, \ldots, Y_d) = a_0 \mathbb{1} + a_1 Y_1 + \cdots + a_d Y_d$, where $a_0 \in \mathbb{R}$ and $\mathbb{1}$ can be thought of as a new "indeterminate" which is always substituted with

the identity operator (both in the finite case of producing $\mathbf{A}_n$ and in the infinite case of A_∞). Second, in addition to having Y_i indeterminates that are substituted with random $n \times n$ matching matrices, one may also allow new indeterminates that are substituted with uniformly random $n \times n$ general permutation matrices. One should be careful to create self-adjoint matrices, i.e. undirected (weighted) graphs, though. To this end, Bordenave and Collins consider polynomials of the form

$$p(Y_1, \ldots, Y_d, Z_1, \ldots, Z_e) = a_0 \mathbb{1} + a_1 Y_1 \tag{1}$$
$$+ \cdots + a_d Y_d + a_{d+1} Z_1$$
$$+ \cdots + a_{d+e} Z_e + a_{d+1}^* Z_1^*$$
$$+ \cdots + a_{d+e}^* Z_e^*.$$

Here $a_0, \ldots, a_d \in \mathbb{R}$, $a_{d+1}, \ldots, a_{d+e} \in \mathbb{C}$, and $Z_1, \ldots, Z_e$ are new indeterminates that in the finite case are always substituted with random $n \times n$ general permutation matrices. We say that the above *polynomial* is "self-adjoint", with the indeterminates $\mathbb{1}, Y_1, \ldots, Y_d$ being treated as self-adjoint. Note that the finite adjacency matrix $\mathbf{A}_n = p(P_{\sigma_1}, \ldots, P_{\sigma_d}, P_{\sigma_{d+1}}, \ldots, P_{\sigma_{d+e}})$ that is self-adjoint and hence that represents a (weighted) undirected n-vertex graph. (As a reminder, here $\sigma_1, \ldots, \sigma_d$ are random matching permutations and $\sigma_{d+1}, \ldots, \sigma_{d+e}$ are random general permutations.) As for the infinite case, we extend the notation V_∞ to denote $\mathbb{Z}_2^{\star d} \star \mathbb{Z}^{\star e}$, the free product of d copies of $\mathbb{Z}_2$ and e copies of $\mathbb{Z}$. Then $A_\infty = p(P_{g_1}, \ldots, P_{g_d}, P_{g_{d+1}}, \ldots, P_{g_{d+e}})$, where $g_{d+1}, \ldots, g_{d+e}$ denote the generators of the $\mathbb{Z}$ factors (and note that $P_{g_{d+i}}^* = P_{g_{d+i}}^{-1} = P_{g_{d+i}^{-1}}$).

Matrix coefficients.: Now comes one of the more dramatic generalizations: the Bordenave–Collins result also allows for *matrix edge-weights/coefficients*. One motivation for this generalization is that it is needed for the "linearization" trick, discussed below. But another motivation is that it allows the theory to apply to non-regular graphs. The setup now is that for a fixed dimension $r \in \mathbb{N}^+$, we will consider color-regular graphs where each color j is now associated with an edge-weight that may be a *matrix* $a_j \in \mathbb{C}^{r \times r}$. The adjacency matrix of an n-vertex graph with $r \times r$ matrix edge-weights is, naturally, the $n \times n$ block matrix whose (u, v) block is the $r \times r$ weight matrix for edge (u, v). (To be careful here, an undirected edge should be thought of as two opposing directed edges; we insist these directed edges get matrix weights that are adjoints of one another, so as to overall preserve self-adjointness.) In case all edges have the same matrix weight, the resulting adjacency matrix is just the Kronecker product of the original adjacency matrix and the weight. For example, if P_σ is the adjacency matrix of a matching on $[n]$, and each edge in the matching is assigned

the weight

$$a = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(the adjacency matrix of } \rhd\text{—)},$$

then the resulting matrix-weighted graph has adjacency matrix $P_\sigma \otimes a$, an operator on $\mathbb{C}^n \otimes \mathbb{C}^r$.

Note that a matrix weighted graph's adjacency operator on $\mathbb{C}^n \otimes \mathbb{C}^r$ can simultaneously be viewed as an operator on $\mathbb{C}^{nr}$. In this viewpoint, it is the adjacency matrix of an (uncolored) scalar-weighted nr-vertex graph, which we call the *extension* of the underlying matrix-weighted graph. The situation is particularly simple when the matrix edge-weights are 0-1 matrices; in this case, the extension is an ordinary unweighted graph. In our above example, $P_\sigma \otimes a$ is the adjacency matrix of $n/2$ disjoint copies of the graph formed from C_6 by taking two opposing vertices and hanging a pendant edge on each. Notice that this is a non-regular graph, even though the original matching is regular.

The Bordenave–Collins theorem shows that for any self-adjoint polynomial as in Equation (1), where the coefficients a_j are from $\mathbb{C}^{r \times r}$, we again have that for all $\varepsilon > 0$, the resulting random adjacency operator $\mathbf{A}_n$ (on $\mathbb{C}^{nr}$) has its nontrivial spectrum ε-close in Hausdorff distance to that of the operator A_∞ (on $\ell_2(V_\infty \times [r])$). Here the "nontrivial spectrum" refers to the $nr - r$ eigenvalues obtained by removing the eigenvalues of $p(1, \ldots, 1) = \sum_{j=0}^{d} a_j + \sum_{j=1}^{e}(a_j + a_j^*)$ from the spectrum of $\mathbf{A}_n$.

The most notable application of this result is the generalized Friedman conjecture about the spectrum of random lifts of a base graph $K = (R, E)$. That result is obtained by taking $r = |R|$, $e = |E|$, $d = 0$, $a_0 = 0$, and $p = \sum_{(u,v) \in E} a_{uv} Z_{uv}$, where (u, v) denotes a directed edge, a_{uv} is the $r \times r$ matrix that has a single 1 in the (u, v) entry (0's elsewhere), and where Z_{vu} denotes Z_{uv}^* when $v > u$. In this case, the random matrices $\mathbf{A}_n$ are adjacency matrices of (extension) graphs that are random n-lifts of K, and the operator $\mathbf{A}_\infty$ is the adjacency operator for the universal cover tree $\mathfrak{X}$ of K.

Nonlinear polynomials.: We now come to Bordenave and Collins's other dramatic generalization: the polynomials p that serve as "recipes" for producing random finite graphs and their infinite covers need not be linear. Restricting to 0-1 matrix weights but allowing for nonlinear polynomials leads to a wealth of possible infinite graphs $\mathfrak{X}$ (not necessarily trees), which we term *MPL (matrix polynomial lift) graphs*. (Note that even if one ultimately only cares about matrix weights in $\{0, 1\}^{r \times r}$, the "linearization" reduction produces linear polynomials with general matrix weights.) An example MPL graph is depicted on our title

page; it arises from the polynomial

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix} \cdot \mathbb{1} + a_{14}Z_1 + a_{41}Z_1^* + a_{36}Z_2
$$
$$
+ a_{63}Z_2^* + a_{25}Z_2 Z_1 + a_{52}Z_1^* Z_2^*,
$$

where again a_{uv} denotes the 6×6 matrix with a 1 in the (u, v) entry.

We may now finally state Bordenave and Collins's main theorem:

Theorem I.1. *Let p be a self-adjoint noncommutative polynomial with coefficients from $\mathbb{C}^{r \times r}$ in the self-adjoint indeterminates $\mathbb{1}, Y_1, \ldots, Y_d$ and the indeterminates $Z_1, \ldots, Z_e, Z_1^*, \ldots, Z_e^*$. Then for all $\varepsilon, \beta > 0$ and sufficiently large n, the following holds:*

Let $\mathbf{A}_n$ be the operator on $\mathbb{C}^n \otimes \mathbb{C}^r$ obtained by substituting the $n \times n$ identity matrix for $\mathbb{1}$, independent random $n \times n$ matching matrices for $Y_1, \ldots, Y_d$, and independent random $n \times n$ permutation matrices for $Z_1, \ldots, Z_e$. Write $\mathbf{A}_{n,\perp}$ for the restriction of $\mathbf{A}_n$ to the codimension-r subspace orthogonal to $\mathrm{span}\{(1, \ldots, 1)\} \otimes \mathbb{C}^r$. Then except with probability at most β, the spectra $\sigma(\mathbf{A}_{n,\perp})$ and $\sigma(A_\infty)$ are at Hausdorff distance at most ε.

Here A_∞ is the operator acting $\ell_2(V_\infty) \otimes \mathbb{C}^r$, where $V_\infty = \mathbb{Z}_2^{\star d} \star \mathbb{Z}^{\star e}$ is the free product of d copies of the group $\mathbb{Z}_2$ and e copies of the group $\mathbb{Z}$, obtained by substituting for $Y_1, \ldots, Y_d$ and $Z_1, \ldots, Z_e$ the left-regular representations of the generators of V_∞.

As discussed further in Section I-C, our work derandomizes this theorem by providing explicit (deterministically $\mathrm{poly}(n)$-time computable) $n \times n$ permutation matrices $P_{\sigma_1}, \ldots, P_{\sigma_d}$ (matchings), $P_{\sigma_{d+1}}, \ldots, P_{\sigma_{d+e}}$ (general), for which the conclusion holds. In fact, our result has the stronger property that for fixed constants $d, e, r, k, R,$ and ε, we construct in deterministic $\mathrm{poly}(n)$ time $P_{\sigma_1}, \ldots, P_{\sigma_{d+e}}$ that have the desired ε-Hausdorff closeness *simultaneously for all polynomials p* (with degree bounded by k and coefficient matrices bounded in norm by R). A very simple but amusing consequence of this is that for every constant $D \in \mathbb{N}^+$ and $\varepsilon > 0$ we get explicit n-vertex matchings $\mathcal{M}_1, \ldots, \mathcal{M}_D$ such that $\mathcal{M}_1 + \mathcal{M}_2 + \cdots + \mathcal{M}_d$ is ε-nearly d-regular Ramanujan for each $d \leq D$.

B. X-Ramanujan graphs

We would like to now rephrase the Bordenave–Collins result in terms of a new definition of "X-Ramanujan" graphs. Over the years, a number of works have raised the question of how to generalize the classic notion of a d-regular Ramanujan graph to the case of non-regular graphs G; see, e.g., [20, Sec. 2.2] for an extended discussion. A

natural first possibility is simply to compare the "nontrivial" spectral of G to that of its universal cover tree $\mathfrak{X}$. However this idea is rather limited in scope, particularly because it only pertains to locally tree-like graphs. To illustrate the deficiency, consider the sorts of graph G arising from average-case analysis of constraint satisfaction problems. For example, random regular instances of the 3Sat or NAE-3Sat problem lead one to study graphs G composed of triangles, arranged such that every vertex participates in a fixed number of triangles — say, 4, for a concrete example. Such graphs are 8-regular, so in analyzing G's second-largest eigenvalue one might be tempted to compare it to the Alon–Boppana bound $2\sqrt{7}$, inherited from the 8-regular infinite tree. However as the below theorem of Grigorchuk and Żuk shows, the graph G will in fact always have second-largest eigenvalue at least $1 + 2\sqrt{6} - o(1) > 2\sqrt{7}$. The reason is that the finite triangle graph G is *covered* (in the graph-theoretic sense) by the (non-tree) infinite free product graph $\mathfrak{X} = C_3 \star C_3 \star C_3 \star C_3$, which is known to have spectral radius $\rho(\mathfrak{X}) = 1 + 2\sqrt{6}$.

Theorem I.2. *([22]'s generalization of the Alon–Boppana bound.) Let $\mathfrak{X}$ be an infinite graph and let $\varepsilon > 0$. Then there exists $c > 0$ such that any n-vertex graph G covered by $\mathfrak{X}$ has at least cn eigenvalues at least $\rho(\mathfrak{X}) - \varepsilon$. (In particular, for large enough n the second-largest eigenvalue of G is at least $\rho(\mathfrak{X}) - \varepsilon$.)*

In light of this, and following [22], [23], [20], we instead take the perspective that the property of "Ramanujan-ness" should derive from the nature of the *infinite graph* $\mathfrak{X}$, rather than that of the finite graph G:

Definition I.3 (X-Ramanujan, slightly informal[1])**.** Given an infinite graph $\mathfrak{X}$, we say that finite graph G is $\mathfrak{X}$-*Ramanujan* if:

- $\mathfrak{X}$ covers G;
- the "nontrivial eigenvalues" of G are bounded in magnitude by $\rho(\mathfrak{X})$.

If the bound is relaxed to $\rho(\mathfrak{X}) + \varepsilon$, we say that G is ϵ-*nearly* $\mathfrak{X}$-Ramanujan.

Thus the classic definition of G being a "d-regular Ramanujan graph" is equivalent to being $\mathbb{T}_d$-Ramanujan for $\mathbb{T}_d$ the infinite d-regular tree. It was shown in [20] (via non-explicit methods) that for a fairly wide variety of $\mathfrak{X}$, infinitely many $\mathfrak{X}$-Ramanujan graphs exist. This wide variety includes all free products of Cayley graphs, and all "additive products". Friedman's generalized conjecture (proven by Bordenave–Collins) holds that whenever $\mathfrak{X}$ is the universal cover tree of a base graph K, random lifts of K are ε-nearly $\mathfrak{X}$ with high probability (for any fixed $\varepsilon > 0$).

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[1]Both of the two bullet points in this definition require a caveat: (i) does "covering" allow for disconnected G? (ii) what exactly counts as a "nontrivial eigenvalue"? These points are addressed at the end of this section.

With this perspective in hand, one can be much more ambitious. Take the earlier example of graphs G where every vertex participates in 4 triangles; i.e., graphs covered by $\mathfrak{X} = C_3 \star C_3 \star C_3 \star C_3$, which is known to have spectrum $\sigma(\mathfrak{X}) = [1 - 2\sqrt{6}, 1 + 2\sqrt{6}]$. The above definition of $\mathfrak{X}$-Ramanujan asks for G's nontrivial eigenvalues to be upper-bounded in magnitude by $1 + 2\sqrt{6}$. But it seems natural to ask if G can also have these eigenvalues bounded *below* by $1 - 2\sqrt{6}$. As another example, it well known that the spectrum of the (c, d)-biregular infinite tree $\mathbb{T}_{c,d}$ with $d > c$ is $[-(\sqrt{d-1} + \sqrt{c-1}), -(\sqrt{d-1} - \sqrt{c-1})] \cup \{0\} \cup [(\sqrt{d-1} - \sqrt{c-1}), (\sqrt{d-1} + \sqrt{c-1})]$, which (when 0 is excluded) contains a notable *gap* between $\pm(\sqrt{d-1} - \sqrt{c-1})$. Are there infinitely many (c, d)-biregular finite graphs G with nontrivial spectrum inside these two intervals? (As far as we are aware, the answer to this question is unknown, but if an ε-tolerance is allowed, the Bordenave–Collins theorem gives a positive answer.) Even further we might ask for (c, d)-biregular G's whose nontrivial spectrum does not have any *other* gaps besides the one between $\pm(\sqrt{d-1} - \sqrt{c-1})$. Of course, since G's spectrum is a finite set this is not strictly possible, but we might ask for it to hold up to an ϵ. Taking these questions to their limit leads to following definition:

Definition I.4 (Near-fully X-Ramanujan, slightly informal). Given an infinite graph $\mathfrak{X}$ and $\varepsilon > 0$, we say that finite graph G is *ε-near fully $\mathfrak{X}$-Ramanujan* if:

- $\mathfrak{X}$ covers G;
- every nontrivial eigenvalue of G is within ϵ of a point $\mathfrak{X}$'s spectrum and vice versa — i.e., the Hausdorff distance of G's nontrivial spectrum to that of $\mathfrak{X}$ is at most ε.

Example I.5. It is known that for every $\varepsilon > 0$, a sufficiently large d-regular LPS/Ramanujan graph G is ε-near fully $\mathbb{T}_d$-Ramanujan; i.e., every point in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ is within distance ε of $\mathrm{spec}(G)$.[2] Rather remarkably, is has also been shown this is true of *every* sufficiently large d-regular Ramanujan graph [25].

Bordenave and Collins's Theorem I.1 precisely implies that for any "MPL graph" — i.e., any infinite graph $\mathfrak{X}$ arising from the "infinite lift" of a $\{0, 1\}^r$-edge-weighted polynomial — we can use finite random lifts to (inexplicitly) produce arbitrarily large ε-near fully $\mathfrak{X}$-Ramanujan graphs. Our work, describes in the next section, makes this construction explicit, and significantly characterizes *which* graphs are MPL.

Technical definitional matters: "nontrivial" spectrum and connectedness.: As soon as we move away from d-regular graphs, it's no longer particularly clear what the correct definition of "nontrivial spectrum" should be. For

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example, in Friedman's generalized conjecture about random lifts G of a base graph K with universal covering tree $\mathfrak{X}$, then "nontrivial spectrum" of G is taken to be $\mathrm{spec}(G) \setminus \mathrm{spec}(K)$. But note that this definition is not a function just of $\mathfrak{X}$, since many different base graphs K can have $\mathfrak{X}$ as their universal cover tree. Rather, it depends on the "recipe" by which $\mathfrak{X}$ is realized, namely as the "infinite lift" of K. Taking this as our guide, we will pragmatically define "nontrivial spectrum" only in the context of a specific matrix polynomial p whose infinite lift generates $\mathfrak{X}$; as in Bordenave–Collins's Theorem I.1, the trivial spectrum is precisely $\mathrm{spec}(p(1, \dots, 1))$, the spectrum of the "1-lift of p".

We also need to add a word about connectedness in the context of graph covering. Traditionally, to say that "$\mathfrak{X}$ covers G" one requires that both $\mathfrak{X}$ and G be connected. In the context of the classic "d-regular Ramanujan graph" definition, there are no difficulties because $d \leq 2$ is typically excluded; note that for $d = 1$ or 2 we have the random d-regular graphs are surely or almost surely disconnected. However when we move away from trees it does not seem to be a good idea to insist on connectedness. For one, there are many MPL graphs consist of multiple disjoint copies of some infinite graph $\mathfrak{X}$; it seems best to admit $\mathfrak{X}$ as an MPL graph in this case. For two, it's a remarkably delicate question as to when (the extension of) a random n-lift of a matrix polynomial is connected. Fortunately, in most cases $\mathfrak{X}$ is "non-amenable" and this implies that the infinite explicit families of near fully $\mathfrak{X}$-Ramanujan graphs we produce are connected. Nevertheless, for convenience in this work we will make say that "$\mathfrak{X}$ covers G" provided each connected component of G is covered by some connected component of $\mathfrak{X}$.

C. Our results, and comparison with prior work

The first part of our paper is devoted understanding the class of "MPL graphs". Recall these are defined as follows: Suppose the Bordenave–Collins Theorem I.1 is applied with matrix coefficients $a_j \in \{0, 1\}^{r \times r}$. Then the resulting operator A_∞ on $\ell_2(V_\infty) \otimes \mathbb{C}^r$ can be viewed as the adjacency operator of an infinite graph on vertex set $V_\infty \times [r]$. We say that $\mathfrak{X}$ is an MPL graph if (one or more disjoint isomorphic copies of) $\mathfrak{X}$ can be realized in this way.

Our main results concerning MPL graphs are as follows:

- All free products of Cayley graphs of finite groups are MPL graphs. More generally, all "additive products" (as defined in [20]) are MPL graphs. Additionally, all "amalgamated free products" (as defined in [21]) are MPL graphs. (Furthermore, there are still more MPL graphs that do not appear to fit either category; e.g., the graph depicted on the title page.)
- Some zig-zag products and replacement products (as defined in [26]) of finite graphs may be viewed as lifts of matrix-coefficient noncommutative polynomials.

- All MPL graphs have finite treewidth. (So, e.g., an infinite grid is not an MPL graph.)
- Each connected component of an MPL graph is hyperbolic. (So, e.g., an MPL graph's simple cycles are of bounded length.)
- All MPL graphs are unimodular. (So, e.g., Trofimov [27]'s "grandparent graph" is not an MPL graph.)
- Given an MPL graph (by its generating polynomial), as well as two vertices, it is efficiently decidable whether or not these vertices are connected.

The remainder of our paper is devoted to derandomizing the Bordenave–Collins Theorem I.1; i.e., obtaining *explicit* (deterministically polynomial-time computable) arbitrarily large ε-near fully $\mathfrak{X}$-Ramanujan graphs. Thanks to the linearization trick utilized in [17], it eventually suffice to derandomize Theorem I.1 in the case of *linear* polynomials with matrix coefficients. This means that one is effectively seeking ε-near fully $\mathfrak{X}$-Ramanujan graphs for $\mathfrak{X}$ being a (matrix-weighted) color-regular infinite tree.

Our technique is directly inspired by the recent work of Mohanty et al. [28], which obtained an analogous derandomization of Friedman's theorem, based on Bordenave's proof [14]. Although the underlying idea (dating further back to [29]) is the same, the technical details are significantly more complex, in the same way that [17] is significantly more complex than [14]. (See the discussion toward the end of [17, Sec. 4.1] for more on this comparison.) Some distinctions include the fact that the edge-weights no longer commute, one needs the spectral radius of the nonbacktracking operator to directly arise in the trace method calculations (as opposed to its square-root arising as a proxy for the graph growth rate), and one needs to simultaneously handle a net of all possible matrix edge-weights.

Similar to [28], our key technical theorem concerns random edge-signings (essentially equivalent to random 2-lifts) of sufficiently "bicycle-free" color-regular graphs. Here bicycle-freeness (also referred to as "tangle-freeness") refers to the following:

Definition I.6 (Bicycle-free). An undirected multigraph is said to be λ-*bicycle free* provided that the distance-λ neighborhood of every vertex has at most one cycle.

We also use this terminology for an "n-lift" — i.e., a sequence of permutations $\sigma_1, \ldots, \sigma_d$ (matchings), $\sigma_{d+1}, \ldots, \sigma_{d+e}$ (general permutations) on $V_n = [n]$ — when the multigraph with adjacency matrix $P_{\sigma_1} + \cdots + P_{\sigma_d} + P_{\sigma_{d+1}} + P_{\sigma_{d+1}}^* + \cdots + P_{\sigma_{d+e}} + P_{\sigma_{d+e}}^*$ is λ-bicycle free.

Let us state our key technical theorem in an informal way (see the full version for the full statement):

Theorem I.7. *(Informal statement.) Let G be an n-vertex color-regular graph with matrix weights $a_1, \ldots, a_d, a_{d+1}, \ldots, a_{d+e}, a_{d+1}^*, \ldots, a_{d+e}^* \in \mathbb{C}^{r \times r}$, and assume G is λ-bicycle free for $\lambda \gg (\log \log n)^2$. Consider*

a uniformly random edge-signing of G, and let $\mathbf{B}_n$ denote the nonbacktracking operator of the result. Then for any $\varepsilon > 0$, with high probability we have $\rho(\mathbf{B}_n) \leq \rho(B_\infty) + \varepsilon$, where B_∞ denotes the nonbacktracking operator of the color-regular infinite tree with matrix weights $a_1, \ldots, a_{d+e}^*$.

As in [28], although our proof of Theorem I.7 is similar to, and inspired by, the proof of the key technical theorem of Bordenave–Collins [17, Thm. 17], it does not follow from it in a black-box way; we needed to fashion our own variant of it. Incidentally, this (non-derandomized) theorem on random edge-signings is also needed for our applications to random CSPs; see Section I-D.

After proving Theorem I.7, and carefully upgrading it so that the conclusion holds simultaneously for *all* weight sets (a_j) (of bounded norm), the overall derandomization task *is* a straightforward recapitulation of the method from [29], [28]. Namely, we first run through the proof of the Bordenave–Collins theorem to establish that $O(\log n)$-wise uniform random permutations are sufficient to derandomize it. Constructing these requires $n^{O(\log n)}$ deterministic time, but we apply them with $n = N_0 = 2^{\Theta(\sqrt{\log N})}$, where N is (roughly) the size of the final graph we wish to construct. Thus in $N_0^{O(\log N_0)} = \text{poly}(N)$ deterministic time we obtain a "good" N_0-lift. We also show that this derandomized N_0-lift will preserve the property of a truly random lift, that its associated graph is (with high probability) λ-bicycle free for $\lambda = \Theta(\log N_0) = \Theta(\sqrt{\log N}) \gg (\log \log N)^2$. Next we show that $1/\text{poly}(N)$-almost $O(\log N)$-wise uniform random *bit-strings* are sufficient to derandomize Theorem I.7, and recall that these can be constructed in $\text{poly}(N)$ deterministic time. It then remains to repeatedly apply this 2-lifts arising from the derandomized Theorem I.7 to obtain an explicit "good" N'-lift, for $N' \sim N_0$. We remark that, as in [29], [28], this final N'-lift is not "strongly explicit", although it does have the intermediate property of being "probabilistically strongly explicit". In the end we obtain the following theorem (informally stated; see the full version for the full statement):

Theorem I.8. *(Informal statement.) For fixed constants d, e, r, R, and $\varepsilon > 0$, there is a deterministic algorithm that, on input n, runs in $\text{poly}(n)$ time and outputs an n'-vertex unweighted color-regular graph ($n' \sim n$) such that the following holds: For all ways of choosing edge-weights $a_1, \ldots, a_{d+e}^* \in \mathbb{C}^{r \times r}$ for the colors with Frobenius-norm bounds $\|a_j\|_F, \|a_j^{-1}\|_F \leq R$, the resulting color-regular graph's nonbacktracking operator $B_{N'}$ has its nontrivial eigenvalues bounded in magnitude by $\rho(B_\infty) + \varepsilon$, where B_∞ denotes the nonbacktracking operator for the analogously weighted color-regular infinite tree.*

At this point, it would seem that we are essentially done, and we need only apply the (non-random) results from [17] that let them go from nonbacktracking operator spectral

radius bounds for linear polynomials (as in Theorem I.8) to adjacency operator Hausdorff-closeness for general polynomials (as in Theorem I.1), taking a little care to make sure the parameters in these reductions only depend on d, e, r, R, ε, and k (the degree of the polynomial), and not on the polynomial coefficients a_j themselves. The tools needed for these reductions include: (i) a version of the Ihara–Bass formula for matrix-weighted (possibly infinite) color-regular graphs, to pass from nonbacktracking operators to adjacency operators ([17, Prop. 9, Prop. 10]); (ii) a reduction from bounding the spectral radius of nonbacktracking operators to obtaining Hausdorff closeness for linear polynomials ([17, Thm. 12, relying on Prop. 10]); (iii) a way to ensure that this reduction does not blow up the norm of the coefficients a_j involved; (iv) the linearization trick to reduce Hausdorff closeness for general polynomials to that for linear polynomials. Unfortunately, and not to put too fine a point on it, there are bugs in the proof of each of (i), (ii), (iii) in [17]. Correcting these is why the remainder of our paper (Chapters 9 and 10 in the full version) still requires new material.[3] In brief, the bug in (i) involves a missing case for the spectrum of non-self-adjoint operators B; the bug in (ii) arises because their reduction converts self-adjoint linear polynomials to non-self-adjoint ones, to which their Theorem 17 does not apply. We fill in the former gap, and derive an alternative reduction for (ii) preserving self-adjointness. Finally, the bug in (iii) seems to require more serious changes. We patched it by first establishing only norm bounds for linear polynomials, and then appealing to an alternative version of Anderson's linearization [30], namely Pisier's linearization [31], the quantitative ineffectiveness of which required some additional work on our part.

D. Implications for degree-2 constraint satisfaction problems

In this section we discuss applications of our main technical theorem on random edge-signings, Theorem I.7, to the study of constraint satisfaction problems (CSPs). In fact, putting this theorem together with the finished Bordenave–Collins theorem implies the following variant (cf. [28, Thm. 1.15]):

Theorem I.9. *(Informal statement.) In the setting of Theorem I.1, if $\mathbf{A}_n$ is the operator produced by substituting random ± 1-signed permutations into the matrix polynomial p, then the Hausdorff distance conclusion holds for the full spectrum $\sigma(\mathbf{A}_n)$ vis-a-vis $\sigma(A_\infty)$; i.e., we do not have to remove any "trivial eigenvalues" from $\mathbf{A}_n$.*

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[3]After consultation with the authors, we are hopeful they will soon be able to published amended proofs. The bug in (i) is not too serious with multiple ways to fix it. The bug in (ii) is fixed satisfactorily in our work, and the authors of [17] outlined to alternate fix involving generalizing [17, Thm. 17] to non-self-adjoint polynomials. The bug in (iii) is perhaps the most serious, but the authors may have an alternative patch in mind.

This theorem will allow us to determine the "eigenvalue relaxation value" for a wide class of average-case CSPs. Roughly speaking, we consider random regular instances of Boolean valued CSPs where the constraints are expressible as degree-2 polynomials (with no linear term). Our work determines the typical eigenvalue relaxation bound for these CSPs; recall that this is a natural, efficiently-computable upper bound on the optimum value of Boolean quadratic programs (and on the SDP/quantum relaxation value). This generalizes previous work [32], [33], [34] on random Max-Cut, NAE3-Sat, and "2-eigenvalue 2XOR-like CSPs", respectively. We remark again that our results here do not require the derandomization aspect of our work, but they *do* rely on Theorem I.9 concerning random *signed* lifts, which is not derivable in a black-box fashion from the work of Bordenave–Collins.

We will be concerned throughout this section with Boolean CSPs: optimization problems over a Boolean domain, which we take to be $\{\pm 1\}$ (equivalent to $\{0, 1\}$ or $\{\mathsf{True}, \mathsf{False}\}$). The hallmark of a CSP is that it is defined by a collection of *local* constraints of similar type. Our work is also general enough to handle certain *valued CSPs*, meaning ones where the constraints are not simply predicates (which are satisfied/unsatisfied) but are real-valued functions. These may be thought of giving as "score" for each assignment to the variables in the constraint's scope.

Definition I.10 (Degree-2 valued CSPs). A *Boolean valued CSP* is defined by a set Ψ of *constraint types* ψ. Each ψ is a function $\psi : \{\pm 1\}^r \to \mathbb{R}$, where r is the *arity*. We will say such a CSP is *degree-2* if each ψ can be represented as a degree-2 polynomial, with no linear terms, in its inputs.

An *instance* $\mathcal{I}$ of such a CSP is defined by a set of n variables V and a list of m constraints $C = (\psi, S)$, where each $\psi \in \Psi$ and each S is an r-tuple of distinct variables from V. More generally, in an instance with *literals allowed*, each constraint is of the form $C = (\psi, S, \ell)$, where $\ell \in \{\pm 1\}^r$.

The computational task associated with $\mathcal{I}$ is to determine the value of the optimal *assignment* $x : V \to \{\pm 1\}$; i.e.,

$$\mathrm{Opt}(\mathcal{I}) = \max_{x : V \to \{\pm 1\}} \mathrm{obj}(x) \qquad (2)$$

where

$$\mathrm{obj}(x) = \sum_{C = (\psi, S, \ell) \in \mathcal{I}} \psi(\ell_1 x(S_1), \ldots, \ell_r x(S_r)).$$

Note that for a degree-2 CSP, the objective $\mathrm{obj}(x)$ may be considered as a degree-2 homogeneous polynomial (plus a constant term) over the Boolean cube $\{\pm 1\}^n$.

Example I.11. Let us give several examples where Ψ contains just a single constraint ψ.

When $r = 2$ and $\psi(x_1, x_2) = \frac{1}{2} - \frac{1}{2} x_1 x_2$ we obtain the Max-Cut CSP. If we furthermore allow literals here, we

obtain the 2XOR CSP. If literals are disallowed, and ψ is changed to $\psi(x_1, x_2, x_3, x_4) = \frac{1}{2} - \frac{1}{2}x_1x_2 + \frac{1}{2} - \frac{1}{2}x_3x_4$, we get a version of the 2XOR CSP in which instances must have an equal number of "equality" and "unequality" constraints.

When $r = 3$ and $\psi(x_1, x_2, x_3) = \frac{3}{4} - \frac{1}{4}(x_1x_2 + x_1x_3 + x_2x_3)$, the CSP becomes 2-coloring a 3-uniform hypergraph; if literals are allowed here, the CSP is known as NAE-3Sat.

The case of the predicate $\psi(x_1, x_2, x_3, x_4) = \frac{1}{2} + \frac{1}{4}(x_1x_2 + x_2x_3 + x_3x_4 - x_1x_4)$ with literals allowed yields the Sort$_4$ CSP; the predicate here is equivalent to "CHSH game" from quantum mechanics [35].

Remark I.12. As additive constants are irrelevant for the task of optimization, we will henceforth assume without loss of generality that degree-2 CSPs involve *homogeneous* degree-2 polynomial constraints.

Definition I.13 (Instance graph). Given an instance $\mathcal{I}$ of a degree-2 CSP as above, we may associate an *instance graph G*, with adjacency matrix A. This G is the undirected, edge-weighted graph with vertex set $V = [n]$ where, for each nonzero monomial $w_{ij}x_ix_j$ appearing in $\mathrm{obj}(x)$ from Equation (2), G contains edge $\{i, j\}$ with weight $\frac{1}{2}w_{ij}$. The factor of $\frac{1}{2}$ is included so that given an assignment $x \in \{\pm 1\}^n$, we have $\mathrm{obj}(x) = x^\top A x = \langle A, xx^\top \rangle$, where $\langle A, B \rangle$ denotes the matrix inner product.

For almost all Boolean CSPs, exactly solving the quadratic program to determine Opt is NP-hard; hence computationally tractable relaxations of the problem are interesting. Perhaps the simplest and most natural such relaxation is the *eigenvalue bound* — the generalization of the Fiedler bound [2] for Max-Cut:

Definition I.14 (Eigenvalue bound). Let $\mathcal{I}$ be a degree-2 CSP instance and let A be the adjacency matrix of its instance graph. The *eigenvalue bound* is

$$\mathrm{Eig}(\mathcal{I}) = \max_{\substack{x \in \ell_2(V) \\ \|x\|_2^2 = n}} \langle A, xx^\top \rangle = \max_{\substack{X \in \mathbb{R}^{n \times n} \text{ PSD} \\ \mathrm{tr}(X) = n}} \langle A, X \rangle = n \cdot \lambda_{\max}(A).$$

It is clear that

$$\mathrm{Opt}(\mathcal{I}) \leq \mathrm{Eig}(\mathcal{I})$$

always, and $\mathrm{Eig}(\mathcal{I})$ can be computed (to arbitrary precision) in polynomial time. For many average-case optimization instances, this kind of spectral certificate provides the best known efficiently-computable bound on the instance's optimal value; it is therefore of great interest to characterize $\mathrm{Eig}(\mathcal{I})$ for random instances. See, e.g., [34] for further discussion. As we will describe below, for a natural model of random instances $\mathcal{I}$ of a degree-2 CSP, our Theorem I.9 allows us to determine the typical value of $\mathrm{Eig}(\mathcal{I})$ for large instances. Often one can show this exceeds the typical value of $\mathrm{Opt}(\mathcal{I})$ for these random instances, thus leading to a potential *information-computation gap* for the certification task.

We should also mention another efficiently-computable upper bound on $\mathrm{Opt}(\mathcal{I})$:

Definition I.15 (SDP bound). Let $\mathcal{I}$ be a degree-2 CSP instance and let A be the adjacency matrix of its instance graph. The *SDP bound* is

$$\mathrm{SDP}(\mathcal{I}) = \max_{\substack{X \in \mathbb{R}^{n \times n} \text{ PSD} \\ X_{ii} = 1 \ \forall i}} \langle A, X \rangle.$$

This quantity can also be computed (to arbitrary precision) in polynomial time, and it is easy to see that $\mathrm{Opt}(\mathcal{I}) \leq \mathrm{SDP}(\mathcal{I}) \leq \mathrm{Eig}(\mathcal{I})$ always. Thus the SDP value can only be a better efficiently-computable upper bound on $\mathrm{Opt}(\mathcal{I})$, and it would be of interest also to characterize its typical value for random degree-2 CSPs, as was done in [32], [33], [34]. Proving that $\mathrm{SDP}(\mathcal{I}) \sim \mathrm{Opt}(\mathcal{I})$ with high probability (as happened in those previous works) seems to require that the CSP has certain symmetry properties that do not hold in our present very general setting. We leave investigation of this to future work.

We now describe a model of random degree-2 CSPs for which Bordenave–Collins's Theorem I.1 and our Theorem I.9 lets one determine the (high probability) value of the eigenvalue relaxation bound. It is based on the "additive lifts" construction from [34].

Suppose we have a degree-2 Boolean CSP with constraints $\Psi = \{\psi_1, \ldots, \psi_t\}$ of arity r. We wish to create random "constraint-regular" instances defined by numbers c_{jk} ($1 \leq j \leq r$, $1 \leq k \leq t$) in which each variable appears in the jth position of c_{jk} constraints of type ψ_k. Let $c = \sum_{jk} c_{jk}$, and for notational simplicity let $\mathcal{I}_0 = (\phi_1, \ldots, \phi_c)$ denote a minimal such CSP instance on a set $[r]$ of variables, where each ϕ_i stands for some ψ_k with permuted variables, and every scope is considered to be $(1, \ldots, r)$. For each constraint ϕ_i, we associate an *atom* graph A_i, which is the instance graph on vertex set $[r]$ defined by the single constraint ϕ_i applied to the r variables.

Now given $n \in \mathbb{N}^+$, we will construct a random CSP on nr variables with nc constraints as a "random lift". We begin with a *base graph*: the complete bipartite graph $K_{r,c}$, with the r vertices in one part representing the variables, and the c vertices in the other part representing the constraints. We call any n-lift of this base graph a *constraint graph*; we can view it as an instance of the CSP where the edges encode which variables participate in which constraints for the random CSP. When we do a random n-lift of this $K_{r,c}$, we create r groups of n variables each, and c groups of n constraints, with a random matching being placed between every group of variables and every group of constraints. In this random bipartite graph, each variable participates in exactly c constraints, one in each group, while each constraint gets exactly one variable from each group.

To obtain the instance graph from the constraint graph,

we start with an empty graph with nr vertices corresponding to the variables, and then, for each constraint vertex in the constraint graph from some group j, we place a copy of the atom A_j on the r vertices to which the constraint vertex is adjacent.

With this lift-based model for generating random constraint-regular CSP instances, it is important to allow for *random literals* in the lifted instance. (Otherwise one may easily generate trivially satisfiable CSPs. Consider, for example, random NAE-3Sat instances., as in [33]. Without literals, all lifted instances will be trivially satisfiable due to the partitioned structure of the $3n$ variables: one can just assigning two of the three n-variable groups the label $+1$ and the other n-variable group the label -1. Hence the need for random literals.) Thus instead of doing a random n-lift of the base $K_{r,c}$ graph, we do a random *signed n-lift*, placing a random ± 1 sign on each edge in the lifted random graph. Then, if u and v are variables and they are connected to a constraint vertex y in the constraint graph, then, in the instance graph the weight of the edge $\{u, v\}$ (if it exists in the atom) will pick up an additional sign of $\mathrm{sign}(\{u, y\}) \cdot \mathrm{sign}(\{v, y\})$. This has the effect of uniformly randomly negating a variable when a predicate is applied to it.

Next, we will see how this model of constructing a random regular CSP is equivalent to a random lift of a particular matrix polynomial. The coefficients of the polynomial will be in $\mathbb{C}^{r \times r}$, indexed by the r variables which the atoms act on. The rc indeterminates in the polynomial are indexed by the edges of the base constraint graph; i.e., we have one (non-self-adjoint) indeterminate $X_{u,j}$ for each variable u and each constraint j. We construct the polynomial p iteratively: for each constraint j, and every pair of variables u and v, if $\{u, v\}$ is an edge in in A_j then we add the terms

$$\frac{1}{2} w_{vu} \, |v\rangle\langle u| \, X_{v,j} X_{u,j}^* + \frac{1}{2} w_{uv} \, |u\rangle\langle v| \, X_{u,j} X_{v,j}^*$$

to p. Then, a random signed lift fits precisely into the model of our Theorem I.9, and we conclude that for this model of random regular general degree-2 CSPs, the eigenvalue relaxation bound is, with high probability, $n \cdot (\lambda_{\max(A_\infty)} \pm \varepsilon)$ for any $\varepsilon > 0$.

NOTES

A full version of the paper may be found at https://arxiv.org/abs/2009.02595.

ACKNOWLEDGMENT

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