Framework for ER-Completeness of Two-Dimensional Packing Problems

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***Abstract*—We show that many natural two-dimensional packing problems are algorithmically equivalent to finding real roots of multivariate polynomials. A two-dimensional packing problem is defined by the type of pieces, containers, and motions that are allowed. The aim is to decide if a given set of pieces can be placed inside a given container. The pieces must be placed so that in the resulting placement, they are pairwise interior-disjoint, and only motions of the allowed type can be used to move them there. We establish a framework which enables us to show that for many combinations of allowed pieces, containers, and motions, the resulting problem is ER-complete. This means that the problem is equivalent (under polynomial time reductions) to deciding whether a given system of polynomial equations and inequalities with integer coefficients has a real solution.**

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***Keywords*-Existential Theory of the Reals, Geometric Packing,**

I. Introduction

Packing problems are everywhere in our daily lives. To give a few examples, you solve packing problems when deciding where to put your furniture in your home, the manufacturer of your clothing arranges cutting patterns on a large piece of fabric in order to minimize waste, and at Christmas time you are trying to cut out as many cookies from a dough as you can. In a large number of industries, it is crucial to solve complicated instances of packing problems efficiently. In addition to clothing manufacturing, we mention leather, glass, wood, and sheet metal cutting, selective laser sintering, shipping (packing goods in containers), and 3D printing (arranging the parts to be printed in the printing volume); see Figure 1. (https://www.mirisys.com/)

Packing problems can be easily and precisely defined in a mathematical manner, but many important questions are still completely elusive. In this work, we uncover a fundamental aspect of many versions of geometric packing by settling their computational difficulty.

We denote **Pack**($\mathcal{P} \to \mathcal{C}, \mathcal{M}$) as the packing problem with pieces of the type $\mathcal{P}$, containers of type $\mathcal{C}$, and motions of type $\mathcal{M}$. In an instance of **Pack**($\mathcal{P} \to \mathcal{C}, \mathcal{M}$), we are given pieces $p_1, \ldots, p_n$ of type $\mathcal{P}$ and a container C of type $\mathcal{C}$. We want to decide if there is a motion of type $\mathcal{M}$ for each piece such that after moving the pieces by these motions, each piece is in C and the pieces are pairwise interior-disjoint. Such a placement of the pieces is called a *valid* placement.

As the allowed motions, we consider *translations* (✣) and *rigid motions* (⟲✣), where a rigid motion is a combination of a translation and a rotation. As containers and pieces, we consider squares (□), convex polygons (⬠), simple polygons (⊵), convex curved polygons (⌓), and curved polygons (⊾), where a *curved polygon* is a region enclosed by a simple closed curve consisting of a finite number of line segments and arcs contained in hyperbolae

(such as the graph of $y = 1/x$).

The problems with only translations allowed are relevant to some industries; for instance when arranging cutting patterns on a roll of fabric for clothing production, where the orientation of each piece has to follow the orientation of the weaving or a pattern printed on the fabric. In other contexts such as leather, glass, or sheet metal cutting, there are usually no such restrictions, so rotations can be used to minimize waste. As can be seen from Figure 1, it is relevant to study packing problems where the pieces as well as the containers may be non-convex and have boundaries consisting of many types of curves (not just straight line segments).

We show that many of the above mentioned variants of packing are $\exists\mathbb{R}$-complete. The complexity class $\exists\mathbb{R}$ will be defined below. We call our developed techniques a *framework*, since the same techniques turn out to be applicable to prove hardness for many versions of packing. With adjustments or additions, they can likely be used for other versions or proofs of other types of hardness as well.

The Existential Theory of the Reals.: The term *Existential Theory of the Reals* refers ambiguously to a formal language, a corresponding algorithmic problem (ETR), and a complexity class ($\exists\mathbb{R}$). Let us start with the formal logic. Let

$$\Sigma := \{\forall, \exists, 0, 1, x_1, \ldots, x_n, +, \cdot, =, \leq, <, \wedge, \vee, \neg\}$$

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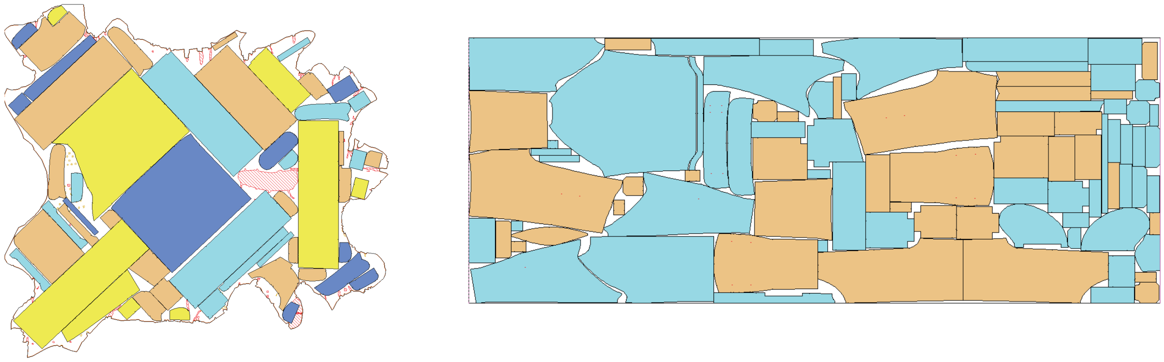

Figure 1. Real examples of nesting on a leather hide (left) and a piece of fabric (right), kindly provided by MIRISYS and produced by their software for automatic nesting.

be an alphabet for some $n \geq 1$. A *sentence* over Σ is a well-formed formula with no free variables, i.e., so that every variable is bound to a quantifier. The *Existential Theory of the Reals* is the true sentences of the form

$$\exists x_1, \ldots, x_n \; \Phi(x_1, \ldots, x_n),$$

where Φ is a quantifier-free formula. The algorithmic problem ETR is to decide whether a sentence of this form is true or not. At last, this leads us to the complexity class *Existential Theory of the Reals* ($\exists\mathbb{R}$), which consists of all those languages that are many-one reducible to ETR in polynomial time. Given a quantifier-free formula Φ, we define the solution space of Φ as $V(\Phi) := \{\mathbf{x} \in \mathbb{R}^n : \Phi(\mathbf{x})\}$. Thus in other words, ETR is to decide if $V(\Phi)$ is empty or not. It is currently known that

$$\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}. \tag{1}$$

To show the first inclusion is an easy exercise, whereas the second is non-trivial and was first proven by Canny [13]. A problem P is $\exists\mathbb{R}$-hard if ETR is many-one reducible to P in polynomial time, and P is $\exists\mathbb{R}$-complete if P is $\exists\mathbb{R}$-hard and in $\exists\mathbb{R}$. [1] None of the inclusions (1) are known to be strict, but the first is widely believed to be [11], implying that the $\exists\mathbb{R}$-hard problems are not in NP. As examples of $\exists\mathbb{R}$-complete problems, we mention problems related to realization of order-types [49], [47], [55], graph drawing [12], [20], [40], [43], recognition of geometric graphs [15], [16], [38], [45], straightening of curves [23], the art gallery problem [2], Nash-equilibria [10], [28], linkages [1], [51], [52], matrix-decompositions [19], [53], [54], and polytope theory [49]. See also the surveys [14], [44], [50].

$\exists\mathbb{R}$-membership.: Showing that the packing problems we are dealing with in this paper are contained in $\exists\mathbb{R}$ is easy using the following recent result.

[1]There seems to be no general convention about how to pronounce terms such as $\exists\mathbb{R}$-complete. We propose to use the pronunciation *E-T-R*-complete.

Theorem (Erickson, Hoog, Miltzow [24])**.** *For any decision problem P, there is a real verification algorithm for P if and only if $P \in \exists\mathbb{R}$.*

A *real verification algorithm* is like a verification algorithm for a problem in NP with the additional feature that it accepts real inputs for the witness and runs on the real RAM. (We refer to [24] for the full definition, as it is too long to include here.)

Thus in order to show that our packing problems lie in $\exists\mathbb{R}$, we have to specify a witness and a real verification algorithm. The witness is simply the motions that move the pieces to a valid placement. The verification algorithm checks that the pieces are pairwise interior-disjoint and contained in the container. Note that without the theorem above, we would need to describe an ETR-formula equivalent to a given packing instance in order to show $\exists\mathbb{R}$-membership. Although this is not difficult for packing, it would still require some work.

Results.: We show that various two-dimensional packing problems are $\exists\mathbb{R}$-complete. A compact overview of our results is displayed in Table 2. In the table, the second row (with problems **Pack**($\mathcal{P} \to$ ⊵, $\mathcal{M}$)) is in some sense redundant, since the $\exists\mathbb{R}$-completeness results can be deduced from the more restricted third row (the problems **Pack**($\mathcal{P} \to \square, \mathcal{M}$)). We anyway include the row since a majority of our reduction is to establish hardness of problems with polygonal containers, and only later we reduce these problems to the case where the container is a square. For the green cells, we have constructions that work both with and without rotations allowed; for instance, we can use the same reduction to **Pack**(⬠ → ⊾, ⟲✢) as to **Pack**(⬠ → ⊾, ✢). For the orange cells, the reductions only work when rotations are allowed.

A strength of our reductions is that in the resulting constructions, all corners can be described with rational coordinates that require a number of bits only logarithmic in the total number of bits used to represent the instance. Therefore, we show that the problems are *strongly* $\exists\mathbb{R}$-

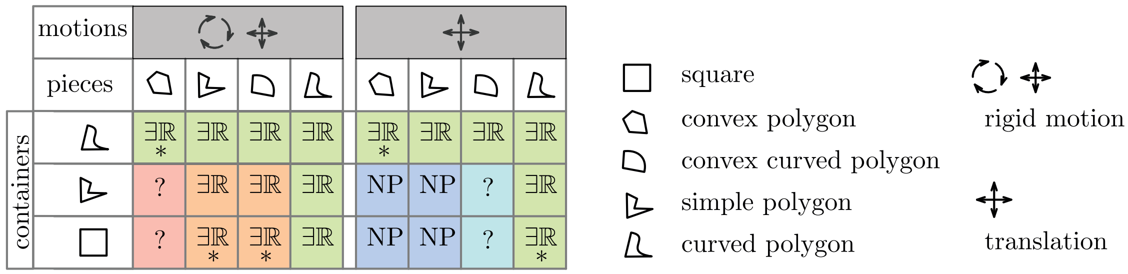

Figure 2. This table displays 12 variants of the packing problem with rotations and translations, and 12 with translations only. ∃ℝ means ∃ℝ-complete and NP means NP-complete. We show that 16 of the problems are ∃ℝ-complete. The problems marked with * are the *basic* problems. The ∃ℝ-completeness of the remaining problems follow since there is a basic problem in the table which is more restricted. The complexities of the four variants marked with ? remain elusive.

hard. Another strength is that all the pieces have constant complexity, i.e., each piece can be described by its boundary as a union of $O(1)$ straight line segments and arcs contained in hyperbolae.

It might be folklore that the problems in the blue cells, i.e., the problems with polygonal pieces and containers and only translations allowed, are in NP, and in the following we sketch an argument why, which Günter Rote told the the second author. We show that a valid placement can be specified as the translations of the pieces represented by a number of bits polynomial in the input size. Consider a valid placement of the pieces. For each pair of a segment s and a corner c (each of a piece or the container), we consider the line $\ell(s)$ containing s and note which of the closed half-planes bounded by $\ell(s)$ contains c. Then we build a linear program (LP) using that information in the natural way. Here, the translation of each piece is described by two variables and for each pair (s, c), we have one constraint involving at most four variables, enforcing c to be on the correct side of $\ell(s)$. It is easy to verify that the constraint is linear. The solution of the LP gives a valid placement of every piece and as the LP is polynomial in the input, so is the number of the bits of the solution to the LP. Note that if rotations are included, the corresponding constraints become non-linear.

To sum up, we conclude that two distinct features are key for ∃ℝ-hardness of packing problems: Rotations and non-polygonal shapes. The results provide evidence that many packing problems are likely harder than problems in NP. This gives a confirmation to the operations research community that they cannot employ standard algorithm techniques (solvers for IP and SAT, etc.) that work well for many NP-complete problems like scheduling, TSP, and SAT. The main message for the theory community is that dealing with rotations or non-polygonal shapes can probably only be done if we relax the problem considerably.

An important step in the proof of ∃ℝ-hardness of the art gallery problem [2] was the introduction of ETR-INV formulae.

Definition 1. *An* ETR-INV formula $\Phi = \Phi(x_1, \ldots, x_n)$ *is a conjunction*

$$\left(\bigwedge_{i=1}^{n} 1/2 \leq x_i \leq 2\right) \wedge \left(\bigwedge_{i=1}^{m} C_i\right),$$

where $m \geq 0$ and each C_i is of one of the forms

$$x + y = z, \quad x \cdot y = 1$$

for $x, y, z \in \{x_1, \ldots, x_n\}$.

It was proven to be ∃ℝ-complete to decide if a given ETR-INV formula Φ has a solution [2]. This has since then been used to prove ∃ℝ-completeness of a geometric graph drawing problem with prescribed face areas [20], completing a partially (straight-line) drawn graph [43], and the polytope nesting problem [19]. In this paper, we introduce the following promise problem which we prove is ∃ℝ-complete.

Definition 2. *An instance $\mathcal{I} = [\Phi, \delta, (I(x_1), \ldots, I(x_n))]$ of the* RANGE-ETR-INV problem *consists of an ETR-INV formula Φ, a number $\delta := 2^{-l}$ for a positive integer l, and, for each variable $x \in \{x_1, \ldots, x_n\}$, an interval $I(x) \subseteq [1/2, 2]$ such that $|I(x)| \leq 2\delta$. For every* inversion constraint *$x \cdot y = 1$, we have either $I(x) = I(y) = [1-\delta, 1+\delta]$ or $I(x) = [2/3-\delta, 2/3+\delta]$ and $I(y) = [3/2-\delta, 3/2+\delta]$. We are promised that $V(\Phi) \subset I(x_1) \times \cdots \times I(x_n)$. The goal is to decide whether $V(\Phi) \neq \emptyset$.*

Theorem 3. *RANGE-ETR-INV is ∃ℝ-complete, even when $\delta = O(n^{-c})$ for any constant $c > 0$.*

The promise that we only need to look for solutions to an ETR-INV formula in some tiny ranges of the variables will be crucial in our reduction. Like in the proof from [2] that deciding if an ETR-INV-formula is feasible, the proof of Theorem 3 is a reduction from ETR and relies of a careful substitution of some variables by other variables combined with tools from real algebraic geometry. To obtain the new proof, we developed a more general way to analyze the ranges in which these derived variables belong.

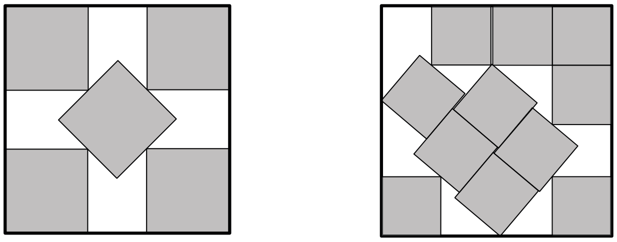

Figure 3. Left: The optimal packing of five unit squares already requires rotations. Right: The currently best known packing of eleven unit squares into a larger square [29].

Related work.: Alt [5] provides a survey of the literature on packing problems from a theoretical point of view. Most papers on geometric packing problems in theoretical computer science allow translations only despite the high relevance of problems with rotations allowed. One line of research deals with packing axis-parallel rectangles into an axis-parallel rectangle with translations only [4], [6], [27], [33], [36], [37]. Most often an optimization version is studied where the rectangles also get weights and the aim is to maximize the total weight of the packed rectangles, in which case the problem is called 2-dimensional knapsack. The current best approximation factor is 1.89 using involved geometric arguments based on the idea to use an L-shaped layout [27]. There exists also a QPTAS [4].

Very recently, Merino and Wiese studied a version of 2-dimensional knapsack where the pieces are convex polygons and arbitrary rotations are allowed [46].

Another line of research deals with strip-packing, where we are given a set of pieces that have to be placed inside a strip of a certain width in an axis-parallel fashion [3], [25], [31], [32], [35], [48]. Rotations by 90 degrees are sometimes allowed. The aim is to minimize the total height. The best polynomial time approximation algorithm has approximation ratio $5/3 + \varepsilon$ [31], whereas the best pseudopolynomial time algorithm achieves an approximation ratio of $5/4 + \varepsilon$ [32] and this is best possible [25].

Several other packing variants are known to be NP-hard. Here we mention the problem of packing squares into a square by translation [42], packing segments into a simple polygon by translation [39], packing circles into a square [18], packing identical simple polygons into a simple polygon by translation, and packing unit squares into a polygon with holes by translation [9], [26]. Alt [5] proves by a simple reduction from the partition problem that packing rectangles into a rectangle is NP-hard, and this reduction works with and without rotations allowed (note that *a priori*, it is not clear that rotations make problems more difficult, and it is straightforward to define (artificial) problems that even get easer with rotations). It is easy to modify the reduction to the problem of packing rectangles into a square, so this implies NP-hardness of all problems in Table 2.

A fundamental problem related to packing is to find the smallest square containing a given number of *unit* squares, with rotations allowed. A long line of mathematical research has been devoted to this problem, initiated by Erdős and Graham [22] in 1975, and it is still an active research area [17]. Even for *eleven* unit squares, the exact answer is unknown [29]; see Figure 3. Other packing problems have much older roots, for instance Kepler's conjecture on the densest packings of spheres from 1611, famously proven by Hales in 2005 [30]. The 2D analog, i.e., finding the densest packings of unit disks, was solved already in 1773 by Joseph Louis Lagrange under the assumption that the disk configurations are lattices, and the general case was solved by László Fejes Tóth in 1940 [57] (Axel Thue already published a proof in 1890 which is considered incomplete).

There is a staggering amount of papers in operations research on packing problems. The research is mainly experimental and focuses on the development of heuristics to solve benchmark instances efficiently. We refer to some surveys for an overview [7], [8], [21], [34], [41], [56]. In contrary to theoretical work, there is a lot of experimental work on packing pieces with irregular shapes and with arbitrary rotations allowed.

II. A Glimpse of an Idea.

Here, we give only some of the high-level ideas of the techniques we use. The core idea is that pieces are used to encode variables. Where the exact position of a piece represents in a linear way the exact value of the corresponding variable. Pieces are arranged in a way that values of variables are represented by more than one piece at different parts of the whole construction. See Figure 7 for a complete example. Furthermore, we build specific gadgets that give constraints on the possible values of the pieces.

Finding those gadgets is one of the keys for the proof. Depending on the packing variant that we are considering, we had to find different gadgets, see Figure 4.

Another key idea is that we force pieces to be adjacent to one another. This is relatively easy to ensure using a jigsaw-style technique, if the pieces are allowed to be non-convex. We manage to do the same with convex pieces by using a so-called fingerprinting technique. See Figure 5 for an illustration. This part of the proof is technically most intricate.

At last, we have an idea to ensure that the container is a square. In order to do that, we take the polygon of the previous reduction add a square around it and fill in the exterior of the original polygon with well designed pieces that can only fit in a unique way. See Figure 6 for an illustration.

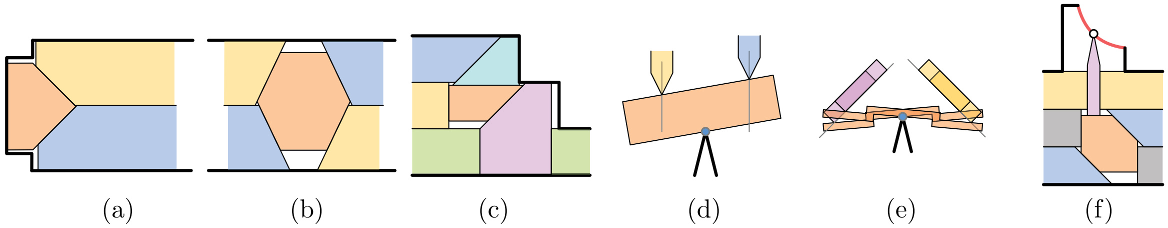

Figure 4. A conceptual illustration of the different gadgets that we used. The actual gadgets appear different, but follow the same principle.

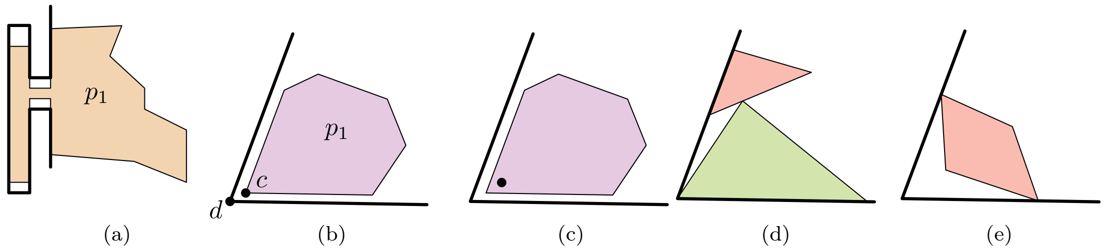

Figure 5. (a) The piece is forced to a specific position as the left part of the pieces fits only at the described position. (b) Only the piece p_1 fits at the corner and is thus forced to be placed there. Any other piece would create a too large empty space that cannot be filled.

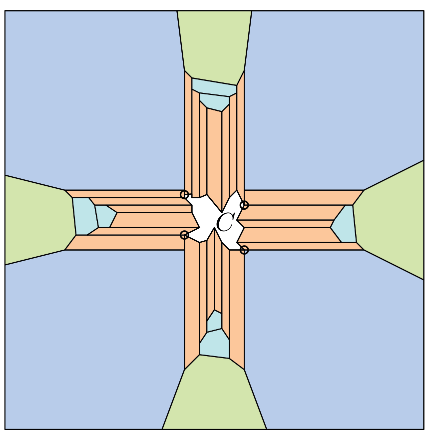

Figure 6. We create pieces that fill up the exterior part of the polygon. In this way, we can assume that the container is a square.

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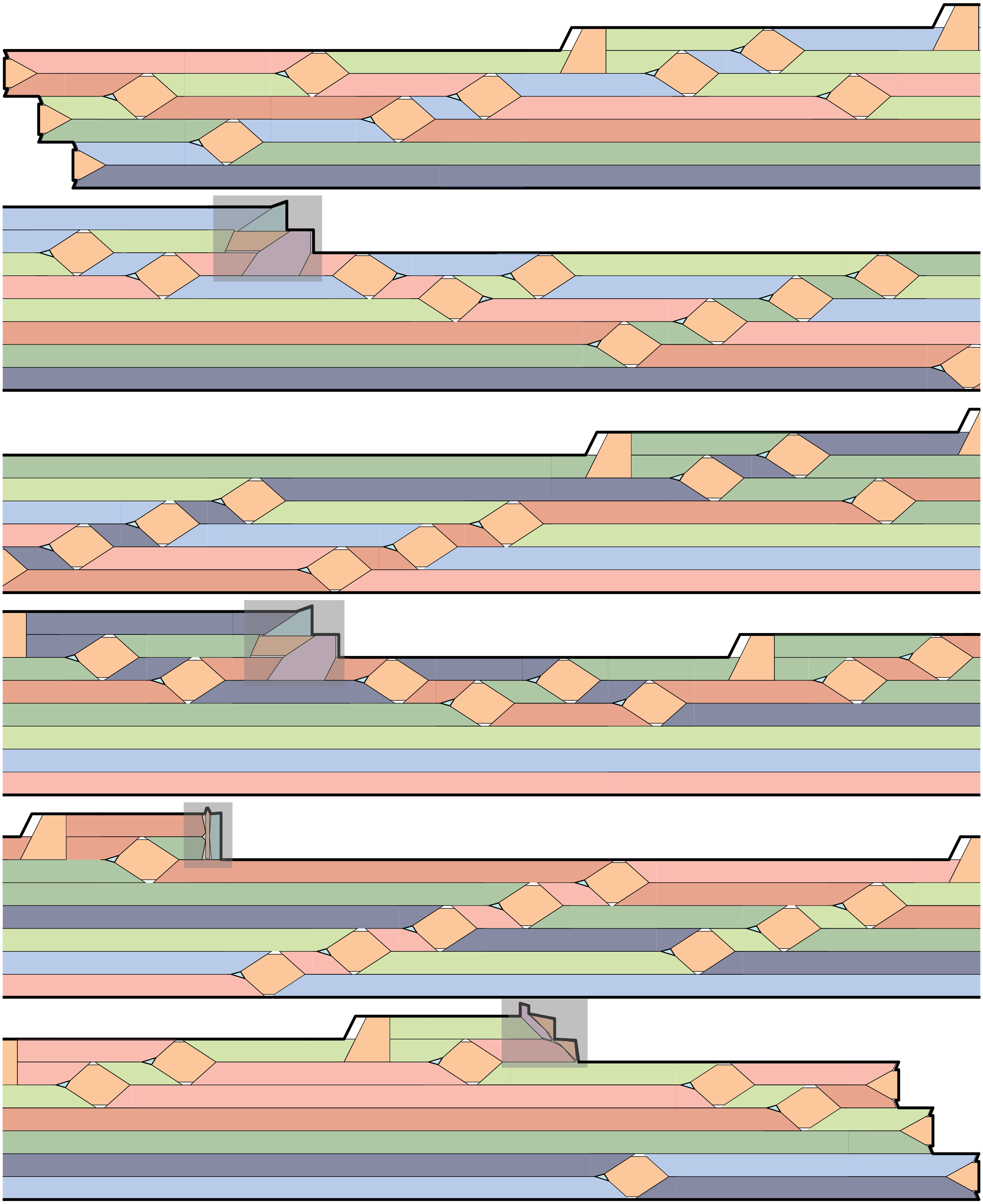

Figure 7. A sketch of the instance of **Pack**(▷ → ▷, ◌ +) when reduce from the ETR-INV formula $x + y = z \wedge x \cdot y = 1$. The adders and inverters are marked with gray boxes.

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