2018 IEEE 59th Annual Symposium on Foundations of Computer Science

Efficient Polynomial-Time Approximation Scheme for the Genus of Dense Graphs

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***Abstract*—The results of this paper provide an Efficient Polynomial-Time Approximation Scheme (EPTAS) for approximating the genus (and non-orientable genus) of dense graphs. The running time of the algorithm is quadratic. Moreover, we extend the algorithm to output an embedding (rotation system), whose genus is arbitrarily close to the minimum genus. This second algorithm is an Efficient Polynomial-time Randomized Approximation Scheme (EPRAS) and the expected running time is also quadratic.**

***Keywords*-genus of a graph; graph embedding; regularity lemma; approximation algorithm; PTAS**

I. Introduction

A. The graph genus problem

For a simple graph G, let $\mathrm{g}(G)$ be the *genus* of G, that is, the minimum h such that G embeds into the orientable surface $\mathbb{S}_h$ of genus h, and let $\widetilde{\mathrm{g}}(G)$ be the *non-orientable genus* of G which is the minimum c such that G embeds into the non-orientable surface $\mathbb{N}_c$ with crosscap number c. If orientability of a surface is not a concern, the notion of the *Euler genus*, which is defined as $\widehat{\mathrm{g}}(G) = \min\{2\mathrm{g}(G), \tilde{\mathrm{g}}(G)\}$, can be used. The genus is a natural measure how far is G from being planar. Determining the genus of a graph is one of fundamental problems in graph theory with wide range of applications in computing and design of algorithms. Algorithmic interest comes from the fact that graphs of bounded genus share many properties with planar graphs and thus admit efficient algorithms for many problems that are difficult for general graphs [1]–[3].

The genus of graphs played an important role in the developments of graph theory through its relationship to the *Heawood problem* asking what is the largest chromatic number of graphs embedded in a surface of genus g. This problem was eventually reduced to the genus computation for complete graphs and resolved by Ringel and Youngs in 1968, see [4]. Further importance of the genus became evident in the graph minors theory of Robertson and Seymour with developments of structural graph theory that plays an important role in stratification of complexity classes [3].

The *genus problem* is the computational task of deciding whether the genus of a given graph G is smaller than a given integer k. The question about its computational complexity was listed among the 12 open problems in the monograph by Garey and Johnson [5] in 1979. Half of these problems were resolved by 1981, three of them (including graph isomorphism) are still unresolved, while three of them have been answered with considerable delay. The genus problem was among the latter three problems. It was resolved in 1989 when Thomassen [6] proved that it is NP-complete. Later, Thomassen simplified his proof in [7] by showing that the question whether G triangulates an (orientable) surface is NP-complete. In 1997 he also proved that the genus problem for cubic graphs is NP-complete [8]. Mohar [9] proved that the genus is NP-complete even if we restrict our attention to *apex graphs*, i.e. graphs which become planar by removing a single vertex.

Measuring graphs by their genus is *fixed parameter tractable*. It follows from the Robertson and Seymour theory of graph minors (and their $O(n^3)$ algorithm for testing H-minor inclusion for any fixed graph H [10], [11]) that for every fixed k, there is an $O(n^3)$ algorithm for testing whether a given graph G has genus at most k. The time complexity in their cubic-time algorithm involves a huge constant depending on H, and the algorithm needs the list of the forbidden minors for genus k. Notably, this is "an impossible task" since the number of surface obstructions is huge (see, e.g., [12] for the up to date results about the surface of genus 1). Moreover, the Robertson–Seymour theory has a non-constructive element. The constants involved in their estimates about forbidden minors are not computable through their results. This deficiency was repaired with the results of Mohar [13], [14], who found a linear-time algorithm for embedding graphs in any fixed surface. His result generalizes the seminal linear-time algorithms for planarity testing by Hopcroft and Tarjan [15] and by Booth and Lueker [16]. It also generalizes, to any surface, the linear-time algorithms that actually construct an embedding in the plane [17] or find a Kuratowski subgraph when an embedding does not exist [18]. Mohar's algorithm gives a constructive proof for the finite number of forbidden minors for surface embeddability. The price paid for this is that

2575-8454/18/$31.00 ©2018 IEEE
DOI 10.1109/FOCS.2018.00074

the algorithms are complicated and hard to implement. A different linear-time FPT algorithm based on structural graph theory (reducing a graph to have bounded tree-width) has been found by Kawarabayashi, Mohar, and Reed [19]. This algorithm includes as a subroutine a linear-time algorithm for computing the genus of graph of bounded tree-width, which turned out to be a difficult task by itself.

A large body of research has been done on approximating the genus by means of polynomial-time algorithms. Graphs whose genus is $\Theta(n)$ (n being the number of vertices) admit a constant factor approximation algorithm. This is an easy consequence of Euler's formula, see [20]. This case includes graphs whose (average) degree is at least d for some $d > 6$.

For graphs of bounded degree, $\Delta = \Delta(G) \leq \Delta_0$, other approaches have been found. Chen et al. [20] describe a factor $O(\sqrt{n})$ algorithm. Chekuri and Sidiropoulos [21] found a polynomial-time algorithm that returns an embedding into a surface whose Euler genus is at most $(\Delta\, \widehat{\mathsf{g}}(G) \log n)^{O(1)}$. Here the approximation factor depends on Δ, polylog factor in n and polynomial factor of the Euler genus itself.

Some other results give additional insight into approximating the genus when the average degree is bounded. For example, the aforementioned paper of Mohar [9] yields a polynomial-time constant factor approximation for the genus of apex graphs (whose maximum degree can be arbitrarily large, but their average degree is less than 8). This result was extended to k-apex graphs in [22].

Kawarabayashi and Sidiropoulos [22] removed the dependence on the maximum degree needed in Chekuri and Sidiropoulos approximation. With a very clever approach they were able to design a polynomial-time algorithm that approximates the Euler genus of any graph within a factor of $O(\widehat{\mathsf{g}}^{255} \log^{189} n)$. A corollary of their result is that the genus can be approximated within factor $O(n^{1-\alpha})$ for some constant $\alpha > 0$, see [22]. A predecessor to this result was published by Makarychev, Nayyeri, and Sidiropoulos [23], who proved that for a graph G possessing a Hamiltonian path (which, unfortunately, needs to be given as part of the input), one can efficiently approximate the Euler genus within factor $(\mathsf{g}(G) \log n)^{O(1)}$. Here the quality of approximation depends on the orientable genus together with a $polylog(n)$ factor.

B. Our results

The main results of this paper provide an Efficient Polynomial-Time Approximation Scheme (EPTAS) for approximating the genus of dense graphs. A graph is α-*dense* if $\mathsf{e}(G) := |E(G)| \geq \alpha n^2$. By saying a graph G is *dense* we mean it to be α-dense for some fixed $\alpha > 0$. While a constant factor approximation is trivial for this class of graphs, approximations with factor arbitrarily close to 1 provided in this paper need a sophisticated algorithm and complicated mathematical justification.

Given a (dense) graph G of order n and the allowed approximation error $\varepsilon > 0$, we want to find an integer g and an embedding of G into a surface of genus g which is close to a minimum genus embedding, i.e. $\mathsf{g}(G) \leq g \leq (1+\varepsilon)\mathsf{g}(G)$. It is easy to see that (after appropriate rescaling of ε) this problem is equivalent to the following one, where the assumption on density is left out.

> APPROXIMATING GENUS DENSE.
> *Input:* A graph G of order n and a real number $\varepsilon > 0$.
> *Output:* An integer g and either a conclusion that $\mathsf{g}(G) \leq g < \varepsilon n^2$, or that $\mathsf{g}(G) \leq g \leq (1+\varepsilon)\mathsf{g}(G)$.

In order to obtain an EPTAS for the genus of dense graphs, we outline an algorithm whose time complexity is $O(f(\varepsilon)n^{O(1)})$, where $f(\cdot)$ is an arbitrary positive function. In fact, the polynomial dependence on n in our algorithm is quadratic.

Theorem I.1. *The problem* APPROXIMATING GENUS DENSE *can be solved in time $O(f(\varepsilon)\, n^2)$, where $f(\cdot)$ is an explicit positive function.*

The dependence on ε^{-1} in the time complexity of our algorithms is super-exponential. In fact, the value $f(\varepsilon)$ is a tower of exponents of height $O(\varepsilon^{-1})$. There are two steps where non-polynomial dependence on ε^{-1} occurs. The main one is with "small" graphs, where n is considered small in terms of a huge function of ε^{-1}. This case has linear complexity in terms of n [13], [14], but it involves a large constant factor which is increasing super-exponentially fast with ε^{-1}. This is where we are prevented of designing an FPTAS.

Our second computational result extends the previous one by constructing an embedding whose genus is close to the minimum genus. Formally, we consider the problem:

> APPROXIMATE GENUS EMBEDDING DENSE.
> *Input:* A graph G of order n and a real number $\varepsilon > 0$.
> *Output:* Rotation system of a 2-cell embedding of G, whose genus g is close to $\mathsf{g}(G)$: either $\mathsf{g}(G) \leq g \leq (1+\varepsilon)\mathsf{g}(G)$, or $\mathsf{g}(G) \leq g < \varepsilon n^2$.

As a solution we provide an Efficient Polynimial-time Randomized Approximation Scheme (EPRAS) [24].

Theorem I.2. *There is an explicit randomized algorithm for* APPROXIMATE GENUS EMBEDDING DENSE *which returns an embedding of the input graph G of genus g, such that either $\mathsf{g}(G) \leq g \leq (1+\varepsilon)\mathsf{g}(G)$, or $\mathsf{g}(G) \leq g < \varepsilon n^2$. The time spent by the algorithm is $O(f_1(\varepsilon)\, n^2)$, where $f_1(\cdot)$ is an explicit positive function.*

There are two parts in the embedding algorithm that are nondeterministic. One of them uses random partition of the edges of G. This part can be derandomized (yielding a cubic polynomial dependence on n), but for the other one we do not see how to derandomize it. This part finds a large matching in a 3-uniform (or 4-uniform) hypergraph, and

existence of a large matching and its construction relies on the Lovász Local Lemma (for which one could use a randomized algorithm by Moser and Tardos [25]). We make use of another randomized solution involving Rödl nibble [26] which yields quadratic dependence on n.

Algorithms in Theorems I.1 and I.2 are based on analysis of minimum genus embeddings of quasirandom graphs. We partition the input graph into a bounded number of quasirandom subgraphs, which are preselected in such a way that they admit embeddings using as many triangles and quadrangles as faces as possible. The starting partition is obtained through an algorithmic version of the Szemerédi Regularity Lemma (due to Frieze and Kannan [27] and to Fox, Lovász, and Zhao [28], [29]).

We use the notion of quasirandomness inspired by the seminal paper of Chung, Graham, and Wilson [30] (see also [31]), and we need it in two special cases of bipartite and tripartite graphs, respectively. In order to define it, we need some notation.

Let G be a graph, and $X, Y \subseteq V(G)$. We define the *edge density* between X and Y as the number

$$d(X,Y) = \frac{\mathsf{e}(X,Y)}{|X||Y|},$$

where $\mathsf{e}(X,Y)$ is the number of edges with one end in X and another end in Y. If G is a (large) bipartite graph with balanced bipartition $V(G) = V_1 \cup V_2$ ($||V_1| - |V_2|| \leq 1$ and $n = |V_1| + |V_2|$), we say that G is *ε-quasirandom* if for every $X \subseteq V_1$ and every $Y \subseteq V_2$, we have

$$|\mathsf{e}(X,Y) - |X||Y|d(V_1,V_2)| \leq \varepsilon |V_1||V_2|.$$

This is equivalent (see [30]) to saying that the number of 4-cycles with vertices in $X \cup Y$ is close to what one would expect in a random bipartite graph with the same edge density, i.e. $d^4(V_1,V_2)|V_1|^2|V_2|^2$, with an error of at most $4\varepsilon|V_1|^2|V_2|^2$.

Theorem I.3. *Suppose that G is a bipartite ε-quasirandom graph with edge density $d = d(V_1,V_2)$. If $\varepsilon < d^8/10$ and $n = |V(G)| \geq \Theta(\varepsilon^{-3/2})$, then*

$$\tfrac{1}{4}(1-10\varepsilon)\mathsf{e}(G) \leq \mathsf{g}(G) \leq \tfrac{1}{4}(1+10\varepsilon)\mathsf{e}(G).$$

The above theorem says, roughly speaking, that G admits an embedding in some orientable surface in which almost all faces are quadrilaterals. More precisely, almost all edges are contained in two quadrangular faces.

A similar result holds for *tripartite ε-quasirandom graphs*. Here we have three almost equal parts V_1, V_2, V_3, and the graph between any two of them is bipartite ε-quasirandom. Here we only need the corresponding embedding result when the densities between the three parts are the same.

Theorem I.4. *Suppose that G is a tripartite ε-quasirandom graph with edge densities $d = d(V_1,V_2) = d(V_1,V_3) = d(V_2,V_3)$. If $\varepsilon < d^8/10$ and $n = |V(G)| \geq \Theta(\varepsilon^{-3/2})$, then*

$$\tfrac{1}{6}(1-10\varepsilon)\mathsf{e}(G) \leq \mathsf{g}(G) \leq \tfrac{1}{6}(1+10\varepsilon)\mathsf{e}(G).$$

Similarly as for Theorem I.3, the outcome of the above theorem is that G admits an embedding in which almost all edges are contained in two triangular faces.

Theorems I.3 and I.4 have extensions to multipartite case with possibly non-equal edge densities and non-empty quasirandom graphs in the parts of the vertex partition. For this extension see the main part of the paper.

Proofs of Theorems I.3 and I.4 build on the approach introduced by Archdeacon and Grable [32] and Rödl and Thomas [33]. The main ingredient is to find two disjoint almost perfect matchings in a 3-uniform (or in a 4-uniform) hypergraph associated with short cycles in G. One difference is that there may be too many short cycles, in which case the matchings obtained from these hypergraphs may not form a set which could be realized as facial cycles of an embedding of the graph. This has to be dealt with accordingly. The proof uses an old result of Frankl and Rödl [34].

From the above expressions about the genus of quasirandom graphs, there is just one major step left. We show that G can be partitioned into a constant number of bipartite and tripartite ε-quasirandom subgraphs such that almost all triangles of G belong to the tripartite ε-quasirandom subgraphs in the partition. For each of these ε-quasirandom subgraphs, Theorem I.4 or Theorem I.3 can be applied. The main result is the following version of Szemerédi Regularity Lemma.

Theorem I.5. *There exists a computable function $s : \mathbb{N} \times [0,1] \to \mathbb{N}$ such that the following holds. For every $\varepsilon > 0$ and every positive integer m there is an integer K, where $m \leq K \leq s(m,\varepsilon)$ such that every graph of order $n \geq m$ has an equitable partition of its vertices into K parts, $V(G) = V_1 \cup \cdots \cup V_K$, and G admits a partition into $O(K^2)$ bipartite ε-quasirandom subgraphs G_{ij} $(1 \leq i < j \leq K)$ with $V(G_{ij}) = V_i \cup V_j$, and into $O(K^3)$ tripartite ε-quasirandom subgraphs G_{ijk} $(1 \leq i < j < k \leq K)$ with $V(G_{ijk}) = V_i \cup V_j \cup V_k$ with equal densities between their parts, and one additional subgraph G_0 with at most εn^2 edges. Moreover, the union of all bipartite constituents G_{ij} is triangle-free.*

To obtain such a partition $V(G) = V_1 \cup \cdots \cup V_K$ we start with an ε-regular partition obtained from the Szemerédi Regularity Lemma. Such a partition can be constructed in quadratic time by using an algorithm of Fox, Lovász, and Zhao [29]. The edges in irregular pairs and all edges in subgraphs $G[V_i]$ $(1 \leq i \leq K)$ are put into the subgraph G_0. All the remaining edges belong to bipartite subgraphs joining pairs V_i and V_j $(1 \leq i < j \leq K)$. Let d_{ij} be the edge density for each such bipartite subgraph. We represent the partition by a weighted graph H on vertices $\{1,\ldots,K\}$, where each edge ij has weight d_{ij}.

Let $\mathcal{T}$ be the set of all triangles in the quotient graph H (of positive edge weight). For every triangle $T = abc \in \mathcal{T}$, let $d(T) = \min\{d_{ab}, d_{bc}, d_{ac}\}$. Now we consider the following

linear program with indeterminates $\{t(T) \mid T \in \mathcal{T}\}$:

$$\begin{aligned} \nu(H) &= \max \sum_{T \in \mathcal{T}} t(T), \\ &\sum_{T \ni e, T \in \mathcal{T}} t(T) \le d_e, \quad \text{for every edge } e \text{ of } H, \qquad (1) \\ t(T) &\ge 0, \quad \text{for every } T \in \mathcal{T}. \end{aligned}$$

We consider an optimum solution $(t(T) \mid T \in \mathcal{T})$ of this linear program. For each $T = abc \in \mathcal{T}$ we now define G_T as a subgraph of $G[V_a \cup V_b \cup V_c]$ by taking a random set of edges with density $t(T)$ from each of the three bipartite graphs between V_a, V_b, V_c. The edges between the sets V_i and V_j $(1 \le i < j \le K)$ that remain after removing all tripartite subgraphs G_T form bipartite quasirandom subgraphs.

From the Partition Theorem I.5 it is not hard to see that

$$\mathsf{g}(G) \le \sum_{i,j} \mathsf{g}(G_{ij}) + \sum_{i,j,k} \mathsf{g}(G_{ijk}) + nK^2 + \varepsilon n^2.$$

By using Theorems I.3 and I.4 we derive the main result:

Corollary I.6. *Let G be a graph that is partitioned as stated in Theorem I.1, let $\nu = \nu(H)$ be the optimum value of the linear program (1), and let $s : \mathbb{N} \times [0,1] \to \mathbb{N}$ be the function from Theorem I.5. If $n = |V(G)| \ge \Theta(s(4\varepsilon^{-1}, \varepsilon) \cdot \varepsilon^{-3/2})$, then the genus of G satisfies:*

$$\begin{aligned} \frac{1}{4}(1-\varepsilon)\left(\mathsf{e}(G) - \frac{\nu n^2}{K^2}\right) &\le \mathsf{g}(G) \\ &\le \frac{1}{4}(1+\varepsilon)\left(\mathsf{e}(G) - \frac{\nu n^2}{K^2}\right) + nK^2 + \varepsilon n^2. \end{aligned}$$

Application of Theorems I.3 and I.4 requires that the densities are not too small. If this is not the case, we just add the edges between pairs (V_i, V_j), whose density is too small, to G_0.

In order to apply the corollary to obtain an ε-approximation to the genus, we use the corollary with the value $\frac{1}{2}\varepsilon$ playing the role of ε. If $n \ge \Theta(s^2(4\varepsilon^{-1}, \varepsilon) \cdot \varepsilon^{-1})$, then the last two terms in the corollary are bounded by $\frac{3}{4}\varepsilon n^2$. Now, if ε is much smaller than the lower bound α on the density of G, we get an ε-approximation of the genus of G.

Although we have not mentioned anything about the nonorientable genus before, the same results hold for the nonorientable genus, where all formulas about the genus need to be multiplied by 2.

C. Overview of the algorithms

On a high level, our EPTAS for APPROXIMATING GENUS DENSE works as follows.

PHASE 0. Check whether the graph is dense enough: If $\mathsf{e}(G) \le \varepsilon n^2$, we return the information that $\mathsf{g}(G) < \varepsilon n^2$ and stop.

PHASE 1. Let $m = 2\varepsilon^{-1}$ and let $M = s(m, \varepsilon)$ where the function s is from Theorem I.5. If $|G| = O(M^2\varepsilon^{-1})$, we compute the genus of G exactly and return the result. Otherwise we proceed with the next step.

PHASE 2. We find a Szemerédi partition of G into K parts, where $m \le K \le M$, and according to Theorem I.5 partition G into edge-disjoint subgraphs $G_0, G_1, \ldots, G_N$, where $N = O(K^3)$, each of them except G_0 of order $\Theta(n/K)$ such that the following holds:

(i) G_0 has at most $\frac{1}{2}\varepsilon n^2$ edges.
(ii) All other subgraphs are either bipartite or tripartite ε-quasirandom.
(iii) The union of bipartite subgraphs contains no triangles.

PHASE 3. Determine the densities d_{ij}, $1 \le i < j \le K$, and solve the linear program (1). Let $\nu = \nu(H)$ be the optimal value computed. Return the value $g = \frac{1}{4}\mathsf{e}(G) - \frac{\nu n^2}{4K^2}$.

The heart of the algorithm lies in PHASE 2. However, PHASE 3 is the most challenging mathematical part and has complicated justification. For the partition of G into $G_0, G_1, \ldots, G_N$ we could use the algorithmic version of the Szemerédi Regularity Lemma due to Frieze and Kannan [27]. But it is more convenient to use a recent strengthening of Frieze-Kannan partitions due to Fox, Lovász, and Zhao [28]. Their result provides an *ε-regular partition* (in the sense of Szemerédi) of $V(G)$ into sets $V_1, \ldots, V_K$ of size n/K (we neglect rounding of non-integral values as they are not important for the exposition) such that the majority of pairs (V_i, V_j) $(1 \le i < j \le K)$ are *ε-regular*. Each such pair induces a bipartite ε-quasirandom graph. The edges in pairs that are not ε-regular can be added to G_0 together with all edges in $\bigcup_{i=1}^{K} G[V_i]$. So from now on, we assume that all edges of the graph are in ε-regular pairs (V_i, V_j) $(1 \le i < j \le K)$.

The second, most difficult step, is to analyse the quotient graph determined by the partition. In this step we use a linear programming approach to find for each triple $T = (i, j, k)$, $1 \le i < j < k \le K$, the number $t(T) \ge 0$ and an ε-quasirandom subgraph $G_T \subseteq G[V_i \cup V_j \cup V_k]$ with $t(T)(n/K)^2$ edges between each pair $(V_i, V_j), (V_i, V_k)$, and (V_j, V_k). The graphs G_T are then used to obtain as many triangular faces as possible (up to the allowed error) for the embedding of G. The edges that remain form quasi-random bipartite parts G_{ij} between pairs (V_i, V_j) $(1 \le i < j \le K)$. We use those to obtain as many quadrangular faces as possible (up to the allowed error) for the embedding of G.

Finally, the near-triangular embeddings of all G_T and near-quadrangular embeddings of all G_{ij} are used to produce a near-optimal embedding of G. The description of this part is in the main part of the paper.

Let us now comment on the main issues in the algorithmic part and in the theoretical justification. First, we use known regularity partition results to find a partition $V_1, \ldots, V_K$. The algorithm runs in quadratic time with a decent (but superexponential) dependence on $1/\varepsilon$. The linear programming part to determine triangle densities $t(T)$ is done on a constant

size linear program and a rounding error of magnitude $O(\varepsilon)$ is allowed. Having gathered all the information about the required edge densities in the partition, the partition of G into subgraphs G_T and G_{ij} uses a randomized scheme, although derandomization is possible. For the computation of the approximate value for the genus (Corollary I.6), the partition is not needed. We just need to know that it exists, and we need to know the edge densities between the regular pairs of the partition. Thus this part is deterministic.

For the justification that the graphs G_T and G_{ij} admit almost triangular and almost quadrangular embeddings, respectively, we use the quasirandomness condition. The proof is based on a theorem by Frankl and Rödl [34] giving a large matching in a dense 3- or 4-uniform hypergraph (respectively). The hyperedges in the hypergraph correspond to cycles of length 3 and 4 (respectively) in the considered subgraph G_T or G_{ij}. Two such matchings are needed in order to combine them into an embedding of the graph, most of whose faces will be the triangles or quadrangles of the two hypergraph matchings. Quasirandomness is used to show that such matchings exists and that they have additional properties needed for them to give rise to an embedding. To obtain such a matching, we can follow the proof of Frankl and Rödl, but the proof uses the Lovasz Local Lemma. In order to make a construction, we may apply the algorithmic version of the Lovasz Local Lemma that was obtained by Moser and Tardos [25]. Alternatively, we may apply the randomized algorithm of Rödl and Thoma [26] which uses the Rödl nibble method. A similar algorithm was obtained by Spencer [35]. Both of these latter algorithms use greedy selection and run in quadratic time, but they are both randomized. This is the essential part where we are not able to provide corresponding derandomized version.

II. Preliminaries

This section contains basic definitions and theoretical background needed for the rest of the paper.

A. Minimum genus embeddings

We say that a graph G embedded in a surface is *2-cell embedded* if each face of G is homeomorphic to an open disk. We say an embedding of G is *triangular* if every face is bounded by a triangle, and an embedding is *quadrangular* if every face is bounded by a cycle of length 4. We say $\Pi = \{\pi_v \,|\, v \in V(G)\}$ is a *rotation system* if for each vertex v, π_v is a cyclic permutation of the edges incident with v. The *Heffter-Edmonds-Ringel rotation principle* [36, Theorem 3.2.4] shows that every 2-cell embedding of a graph G in an orientable surface is determined (up to homeomorphisms of the surface) by its rotation system. Let $\mathrm{g}(G)$ be the orientable genus of G and let $\widetilde{\mathrm{g}}(G)$ be the non-orientable genus of G. For 2-cell embeddings we have the famous *Euler's Formula*: if G is 2-cell embedded in a surface S and has n vertices, e edges and f faces in S, then $n-e+f = \chi(S)$, where $\chi(S)$ is the *Euler characteristic* of S (i.e. $\chi(S) = 2-2h$ when $S = \mathbb{S}_h$ and $\chi(S) = 2-c$ when $S = \mathbb{N}_c$).

Let G be a simple graph. The *corresponding digraph* of G is a random simple digraph $\mathcal{D}$ obtained from G by randomly orienting each edge. Specifically, each digraph $D \in \mathcal{D}$ has $V(D) = V(G)$ and if $uv \in E(G)$ then either $\overrightarrow{uv}$ or $\overrightarrow{vu}$ is an edge of D, each with probability $1/2$, and the two events are exclusive. For a digraph D, we let D^{-1} be the digraph obtained from D by replacing each arc $\overrightarrow{xy}$ with the reverse arc $\overrightarrow{yx}$.

Given a digraph D, a *blossom* of length l with *center* v and *tips* $\{v_1, v_2, \ldots, v_l\}$ is a set $\mathcal{C}$ of l directed cycles $\{C_1, C_2, \ldots, C_l\}$, where $\overrightarrow{v_i v}, \overrightarrow{v v_{i+1}} \in C_i$, for $i = 1, 2, \ldots, l$, with $v_{l+1} = v_1$. A *k-blossom* is a blossom, all of whose elements are directed k-cycles. A blossom of length l is *simple* if either $l \geq 3$ or $l = 2$ and $C_1 \neq C_2^{-1}$. In this paper, we mainly consider 3-blossoms and 4-blossoms.

Let $\mathcal{C}$ be a family of arc-disjoint closed trails in $D \cup D^{-1}$. We say that $\mathcal{C}$ is *blossom-free* if no subset of $\mathcal{C}$ forms a blossom centered at some vertex. The following lemma [37, Lemma 2.2] (see also [33] where its original version appears) is the main tool used to construct embeddings of a graph from blossom-free families of cycles.

Lemma II.1. *Let G be a graph and let D be the corresponding digraph. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are sets of arc-disjoint cycles in D and D^{-1} (respectively) such that their union $\mathcal{C}_1 \cup \mathcal{C}_2$ is blossom-free in $D \cup D^{-1}$. Then there exists a rotation system Π of G such that every cycle in $\mathcal{C}_1 \cup \mathcal{C}_2$ is a face of Π.*

Given a 2-cell embedding with rotation system Π, let $f_k(\Pi)$ be the number of faces of length k in Π. For every $\varepsilon > 0$, an *ε-near triangular embedding* and an *ε-near quadrangular embedding* is a rotation system of G such that $3f_3(\Pi) \geq 2(1-\varepsilon)|E(G)|$ and $4f_4(\Pi) \geq 2(1-\varepsilon)|E(G)|$, respectively. Lemma II.1 shows that in order to find an ε-near triangular embedding (or an ε-near quadrangular embedding), it suffices to find a large set of directed triangles (or directed 4-cycles) in $D \cup D^{-1}$ which is also blossom-free.

B. Matchings in hypergraphs

In order to construct an ε-near triangular (or quadrangular) embedding, one way is that we view the edge-set of G as a vertex-set of a hypergraph $\mathcal{H}$. We randomly orient the edges of G and then consider all directed triangles (or directed 4-cycles) in G as hyperedges of $\mathcal{H}$. Note that $\mathcal{H}$ is a 3-uniform (or 4-uniform) hypergraph. The following result from [34] (see also [33], [38] where its current formulation appears) on hypergraph matching theory will play an important role for constructing near-optimal embeddings of graphs. Recall that a *matching* in a hypergraph $\mathcal{H}$ is a set of pairwise disjoint

edges of $\mathcal{H}$. A matching $M \subseteq E(\mathcal{H})$ is *ε-near perfect* if all but $\varepsilon|V(\mathcal{H})|$ vertices of $\mathcal{H}$ are contained in the edges in M.

Theorem II.2. *Let $\varepsilon > 0$ be a real number and $d \geq 2$ be an integer. Then there exist a positive real number δ and an integer N_0 such that for every $N \geq N_0$ the following holds. If Δ is a real number and if $\mathcal{H}$ is a d-uniform hypergraph with $|V(\mathcal{H})| = N$ such that*

1) *$|\{x \in V(\mathcal{H}) \mid (1-\delta)\Delta \leq \deg(x) \leq (1+\delta)\Delta\}| \geq (1-\delta)N$,*
2) *for every $x, y \in V(\mathcal{H})$, $|\{e \in E(H) \mid x, y \in e\}| < \delta\Delta$,*
3) *at most $\delta N\Delta$ hyperedges of $\mathcal{H}$ contain a vertex $v \in V(\mathcal{H})$ with $\deg(v) > (1+\delta)\Delta$,*

then $\mathcal{H}$ has a matching of size at least $(1-\varepsilon)N/d$. Moreover, for every matching M in $\mathcal{H}$, there exists a matching M' in $\mathcal{H}$ with $M \cap M' = \varnothing$, and with $|M'| \geq (1-\varepsilon)N/d$.

For nearly regular uniform hypergraphs, Rödl and Thoma provide a random greedy algorithm [26] (see also [35] for a different proof) which can output an ε-near perfect matching in $\mathcal{H}$.

C. Cut metric and Szemerédi partitions

Szemerédi Regularity Lemma [39] is one of the most powerful tools in understanding large dense graphs. Szemerédi first used the lemma in his celebrated theorem on long arithmetic progressions in dense subset of integers [40]. Nowadays, the regularity lemma has many connections to other areas of mathematics, for example, analysis, number theory and theoretical computer science. For an overview of applications, we refer to [41], [42]. Regularity Lemma gives us a structural characterization of graphs. Roughly speaking, it says that every large graph can be partitioned into a bounded number of parts such that the graphs between almost every pair of parts is random-like. To make this precise we need some definitions.

Let G be a graph and $X, Y \subseteq V(G)$. We define the *edge density* $d(X,Y) = \mathsf{e}(X,Y)/(|X||Y|)$. We say the pair (X,Y) is *ε-regular* if for all $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$, we have $|d(X',Y') - d(X,Y)| < \varepsilon$. A vertex partition $\mathcal{P} = \{V_i\}_{i=1}^K$ is *equitable* if $\big||V_i| - |V_j|\big| \leq 1$ for every $1 \leq i < j \leq K$. An equitable vertex partition $V_1, \ldots, V_K$ with K parts is *ε-regular* if all but εK^2 pairs of parts (V_i, V_j) $(1 \leq i < j \leq K)$ are ε-regular.

Theorem II.3 (Szemerédi Regularity Lemma)**.** *There exists a computable function $s : \mathbb{N} \times [0,1] \to \mathbb{N}$ such that the following holds. For every $\varepsilon > 0$ and every positive integer m, every graph G of order $n \geq m$ has an ε-regular partition into K parts, where $m \leq K \leq s(m,\varepsilon)$.*

For $\varepsilon > 0$, a partition obtained from the regularity lemma is also called an *ε-Szemerédi partition*. In the original proof of the regularity lemma, the bound $s(m,\varepsilon)$ on the number of parts is a tower of twos. Unfortunately, this is not far away from the truth. Gowers [43] showed that such an enormous bound is indeed necessary. In this paper, we will always assume that $m \gg 1/\varepsilon$, so $K \gg 1/\varepsilon$ as well.

Recall that the cut metric $\mathsf{d}_\square$ between two (edge-weighted) graphs G and H on the same vertex-set V is defined by

$$\mathsf{d}_\square(G,H) = \max_{U,W \subseteq V} \frac{|\mathsf{e}_G(U,W) - \mathsf{e}_H(U,W)|}{|V|^2}. \quad (2)$$

Here $\mathsf{e}_G(U,W)$ denotes the total weight of the edges with one end vertex in U and the other end vertex in W. When G and H are bipartite graphs with bipartition $X \sqcup Y$, we can define the cut metric as

$$\mathsf{d}_\square(G,H) = \max_{U \subseteq X, W \subseteq Y} \frac{|\mathsf{e}_G(U,W) - \mathsf{e}_H(U,W)|}{|X||Y|}. \quad (3)$$

If $|X| \sim |Y|$ is large, definitions (2) and (3) differ only by a small constant factor.

If G and H are graphs of the same order and $\sigma : V(H) \to V(H)$ is a bijection, we consider the graph H^σ isomorphic to H whose vertex-set is $V(G)$ and edges are $\{\sigma(e) \mid e \in E(H)\}$. Then we define the cut distance $\mathsf{d}_\square(G,H)$ as the minimum of $\mathsf{d}_\square(G,H^\sigma)$ taken over all bijections $\sigma : V(H) \to V(H)$.

By using the language of cut metrics, a weaker condition than that of the Szemerédi partition $\mathcal{P} = \{V_1, \ldots, V_K\}$ is that, for all but at most εK^2 pairs (V_i, V_j), we have $\mathsf{d}_\square(G[V_i \sqcup V_j], K(V_i, V_j; p_{ij})) < \varepsilon$, where $K(V_i, V_j; p_{ij})$ is the complete bipartite graph defined on the vertex-set $V_i \sqcup V_j$ with all edges weighted $p_{ij} = d_G(V_i, V_j)$.

The cut distance gives us a way to describe the similarity between two graphs, and it is widely used in graph limit theory [44]. Another widely used way to describe the similarity between two large graphs is comparing the homomorphism densities of small graphs. Let $\mathsf{hom}(F,G)$ denote the number of homomorphisms of F into G. Then we define the *homomorphism density*:

$$t(F,G) = \frac{\mathsf{hom}(F,G)}{|V(G)|^{|V(F)|}}.$$

To compare the cut distance and homomorphism densities, we have the following fundamental relation. For the more general version on graphons, see [44], [45].

Lemma II.4 (Counting Lemma)**.** *Let G and G' be two graphs defined on the same vertex-set. Then for any graph F,*

$$|t(F,G) - t(F,G')| \leq |E(F)|\mathsf{d}_\square(G,G').$$

D. Quasirandomness

Quasirandom graphs are graphs which share many properties with random graphs. The definition of quasirandomness was first introduced in a seminal paper by Chung, Graham and Wilson [30]. In that paper, they listed several equivalent

definitions of quasirandom graphs, but essentially, quasirandom graphs are graphs close to random graphs in the sense of cut distance.

We will introduce a general form of quasirandomness. In order to avoid to use probability, we define the corresponding edge-weighted (complete) graphs.

We will focus on a more general setting of quasirandom graphs. Let P be an $m \times m$ symmetric matrix with non-negative entries. Let $K(n^{(m)}, P)$ be the complete edge weighted graph, which is defined on the vertex-set $V_1 \sqcup \cdots \sqcup V_m$, and for every $i \in [m]$ we have $|V_i| = n$ and the weight of edges between V_i and V_j is given by the (i, j)-entry of P. Although by using the same method, we can deal with more general graphs, for the convenience we only consider the case that each part of the graph has the same size. We will always let the diagonal of P be 0 (then $K(n^{(m)}, P)$ is actually a m-partite graph), and all the entries in P are between 0 and 1. We will use p_{ij} to denote the (i, j)-entry of P.

Recall [44] that a *graphon* is a symmetric measurable function $W : [0, 1]^2 \to [0, 1]$. Let $\widetilde{K}_m = K_m(P)$ be the *quotient graph* of $K(n^{(m)}, P)$, that is, $\widetilde{K}_m$ has m vertices, and the weight of the edge ij is p_{ij}. Let $W_m = W_m(P)$ be a *step function* of $\widetilde{K}_m$, that means $\langle W_m, (\frac{i-1}{m}, \frac{i}{m}] \times (\frac{j-1}{m}, \frac{j}{m}]\rangle = p_{ij}$. We say a sequence of graphs $\{G_n\}$ is W_m*-quasirandom* if $G_n \to W_m$ as $n \to \infty$ where the convergence is in the cut-distance metric. Given a graph G of order mn, we say G is ε-W_m*-quasirandom* if $\mathsf{d}_\square(G, K(n^{(m)}, P)) < \varepsilon$, and we write $G \in \mathcal{Q}(n^{(m)}, P, \varepsilon)$ in such a case. If $m = 1$, we write just $\mathcal{Q}(n, p, \varepsilon)$.

Given a graph G of order n and a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$, we say $\mathcal{P}$ is an ε*-quasirandom partition* if for every $i \in [k]$ we have $|V_i| = n/k$, and for every $1 \le i < j \le k$, $\mathsf{d}_\square\big(G[V_i \cup V_j], K((n/k)^{(2)}, d(V_i, V_j))\big) < \varepsilon$. Clearly, one can obtain an ε-quasirandom partition from an ε-Szemerédi partition $\mathcal{P}' = \{V_1, \ldots, V_k\}$. By removing all the edges between irregular pairs, we obtain an ε-quasirandom partition of the resulting graph.

A Szemerédi partition of a large graph can be obtained by means of an efficient algorithm. See, for example, [46]. Recently, Tao [47] provided a probabilistic algorithm which produces an ε-Szemerédi partition with high probability in constant time (depending on ε). Frieze and Kannan [27] provided a quadratic-time algorithm for constructing an ε-quasirandom partition. In this paper, we will use a more recent deterministic PTAS due to Fox et al. [28].

Theorem II.5. *There exists an algorithm, which, given $\varepsilon > 0$, and $0 < \alpha < 1$, an integer k, and a graph G on n vertices that admits an ε-Szemerédi partition with k parts, outputs a $(1 + \alpha)\varepsilon$-Szemerédi partition of G into k parts in time $O_{\varepsilon,\alpha,k}(n^2)$.*

E. Notation

We will use standard graph theory terminology and notation [48], as well as that of topological graph theory and graph limit theory as given in [36] and [44], respectively. We will also use the following notation: $A(n) \sim B(n)$ means $\lim_{n\to\infty} A(n)/B(n) = 1$, and $A(n) \ll B(n)$ means $\lim_{n\to\infty} A(n)/B(n) = 0$. By $X \sqcup Y$ we denote the disjoint union of X and Y, and we set $X \oplus Y = (X \times Y) \cup (Y \times X)$. We say an event $A(n)$ happens asymptotically almost surely (abbreviated a.a.s.) if $\mathbb{P}(A(n)) \to 1$ as $n \to \infty$.

Given a graph G, suppose $X, Y \subseteq V(G)$. We use $E(X, Y)$ denote the set of edges between X and Y, and $\mathsf{e}(X, Y) = |E(X, Y)|$. We also use $\mathsf{e}(G)$ to denote $|E(G)|$. We use $[n]$ to denote the set of integers $\{1, 2, \ldots, n\}$.

III. Algorithms and Analysis

In this section, we will analyse properties on a large dense graph G. Section 4.1 provides a deterministic EPTAS for the problem Approximating Genus Dense. In the rest of the paper, we will discuss how to construct a near minimum genus embedding for the problem Approximate Genus Embedding Dense.

A. Genus of dense graphs

Suppose G has n vertices and suppose we have an equitable partition $\mathcal{P}$ of $V(G)$ into K parts, $V(G) = V_1 \sqcup \cdots \sqcup V_K$. We use $H = G/\mathcal{P}$ to denote the edge-weighted complete graph with K vertices, $V(H) = \{v_1, \ldots, v_K\}$, in which the edge $v_i v_j$ has weight equal to the edge density $d_{ij} = d(V_i, V_j)$ between V_i and V_j in G.

The optimum value $\nu(H)$ of the linear program (1) gives an upper bound on the packing of triangles in G.

Lemma III.1. *Suppose that $\mathcal{F}$ is a set of triangles in G such that each edge lies in at most t of them. Then*

$$|\mathcal{F}| \le t\,\nu(H)(\tfrac{n}{K})^2.$$

Lemma III.1 is used in the proof of the following theorem with $\mathcal{F}$ being the set of facial triangles of an embedding of the graph G (with $t = 2$) in order to obtain a lower bound on the genus of G.

Theorem III.2. *Let $\varepsilon > 0$ and let G be a graph of order $n = |G| \gg \varepsilon^{-1}$. Then there exist a positive number $\varepsilon' > 0$ and an ε'-Szemerédi partition $\mathcal{P}$ with $K \gg \varepsilon^{-1}$ parts such that the following holds. Let $n_0 = n/K$, let $H = G/\mathcal{P}$ be the quotient graph, and let $\nu(H)$ be an optimal solution of the linear program (1). Then*

$$(1-\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n_0^2}{4} \le \mathsf{g}(G) \le (1+\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n_0^2}{4}.$$

With all tools in hand, we obtain an algorithm for Approximating Genus Dense, whose running time is quadratic. Given $\varepsilon > 0$ and a graph G of order n, we do the following:

STEP 1. Let $\tau = 3\varepsilon'/2$, where ε' is defined in Theorem III.2. Pick $\alpha = 1/2$, $m = 4/\varepsilon$ and apply the algorithm in Theorem II.5 with integer k taking values from m to $s(\tau, m)$. Then the algorithm will output an ε'-Szemerédi partition into K parts, where $m \leq K \leq s(\tau, m)$.

STEP 2. Consider the quotient graph $H = G/\mathcal{P}$. Solve the linear program (1) on H to obtain $\nu(H)$.

STEP 3. Output $g = \frac{1}{4}(1+\varepsilon)(\mathsf{e}(G) - \nu(H)n_0^2)$, where $n_0 = n/K$.

Corollary III.3. *The problem* APPROXIMATING GENUS DENSE *can be solved in $O(n^2)$ time, where n is the order of the input graph.*

B. Embeddings of dense graphs

Now, we turn to our algorithm for APPROXIMATE GENUS EMBEDDING DENSE where the added feature is to construct an embedding. Given $\varepsilon > 0$ and a graph G of order n, we can apply APPROXIMATING GENUS DENSE to get g such that $\mathsf{g}(G) \leq g \leq (1+\varepsilon)\mathsf{g}(G)$. We are going to construct a rotation system Π of G, whose genus satisfies the same bound. Our algorithm proceeds as follows:

STEP 1. Apply APPROXIMATING GENUS DENSE, we obtain an $r(\varepsilon)$-Szemerédi partition $\mathcal{P}$ into K parts, where $r(\varepsilon)$ is the value of ε' in Theorem III.2. We will partition the graph into $\mathsf{b} = O(K^2)$ bipartite graphs $\mathcal{B}_{ij}$ and $\mathsf{t} = O(K^3)$ tripartite graphs $\mathcal{T}_{ijk}$ and a small leftover graph G_0 with less than $\frac{\varepsilon}{10}(n/K)^2$ edges. In G_0 we put all edges in irregular parts of the Szemerédi partition and all edges between parts whose density is too small. For the remaining edges we determine the quotient graph H and compute the value $\nu(H)$ as well as the family of triangles $\mathfrak{T}$ in H and their balanced edge densities $t(T)$, $T \in \mathfrak{T}$. Then we partition the edges randomly in order to obtain, for each $T_{ijk} \in \mathfrak{T}$, a quasirandom tripartite graph $\mathcal{T}_{ijk}$ between parts V_i, V_j, V_k with edge-density $t(T)$. For each edge $ij \in E(H)$, if $d'(ij) = d_{ij} - \sum_{T \ni ij, T \in \mathcal{T}} t(T) > 0$, we put the remaining edges to form a bipartite quasirandom subgraph $\mathcal{B}_{ij}$ between parts V_i, V_j with edge-density $d'(ij)$. With high probability, for any $\tau > 0$, at least $(1 - r(\varepsilon) - \tau)\mathsf{b}$ of the resulting bipartite graphs are quasirandom and at least $(1 - r(\varepsilon) - \tau)\mathsf{t}$ tripartite graphs are quasirandom. Note that $r(\varepsilon)$ appears here because of the irregular pairs.

STEP 2. Let $t_1 = n^{\frac{2-\varepsilon}{4-\varepsilon}}$. For every $1 \leq i < j < k \leq K$, we partition the edge-set of the graph $\mathcal{T}_{ijk}$ into t_1 parts uniformly at random. These t_1 sets of edges give us t_1 graphs ${}^1\mathcal{T}_{ijk}, \ldots, {}^{t_1}\mathcal{T}_{ijk}$. For any $\tau > 0$, with probability at least $(1-\tau)t_1$, the graphs are still quasirandom. For every $x \in [t_1]$, let ${}^xD_{ijk}$ be the corresponding digraph of ${}^x\mathcal{T}_{ijk}$, and let ${}^x\mathcal{H}_{ijk}$ be the hypergraph whose vertices are the arcs of ${}^x\mathcal{T}_{ijk}$ and whose edges are the directed 3-cycles in ${}^x\mathcal{T}_{ijk}$.

STEP 3. Let $t_2 = n^{\frac{4-\varepsilon}{6-\varepsilon}}$. For every $1 \leq i < j \leq K$, we partition the edge-set of the graph $\mathcal{B}_{ij}$ into t_2 parts uniformly at random. These t_2 sets of edges give us t_2 graphs ${}^1\mathcal{B}_{ij}, \ldots, {}^{t_2}\mathcal{B}_{ij}$. For any $\tau > 0$, with probability at least $(1-\tau)t_2$, the graphs are still quasirandom. For every $y \in [t_2]$, let ${}^yD_{ij}$ be the corresponding digraph of ${}^y\mathcal{B}_{ij}$, and let ${}^y\mathcal{H}_{ij}$ be the 4-uniform hypergraph, whose vertices are the arcs of ${}^x\mathcal{B}_{ij}$ and whose edges are the directed 4-cycles in ${}^x\mathcal{B}_{ij}$. Let h be the total number of hypergraphs.

STEP 4. Apply random greedy algorithm [26] on 3-uniform hypergraphs ${}^x\mathcal{H}_{ijk}$ for every $x \in [t_1]$ and every $1 \leq i < j < k \leq K$, and on 4-uniform hypergraphs ${}^y\mathcal{H}_{ij}$ for every $y \in [t_2]$ and every $1 \leq i < j \leq K$. Note that the expected running time here is still quadratic even though we need to run the algorithm $\Theta(t_1 + t_2)$ times. This is because the running time is linear in the number of vertices of the hypergraph, which means it is linear in the number of edges in the corresponding digraphs ${}^xD_{ijk}$ and ${}^yD_{ij}$. With high probability, we have $|V({}^x\mathcal{H}_{ijk})| = O(\frac{n^2}{t_1})$ and $|V({}^y\mathcal{H}_{ij})| = O(\frac{n^2}{t_2})$. Then for every $\tau > 0$, the algorithm will output a τ-near perfect matching in at least $(1 - r(\varepsilon) - \tau)\mathsf{h}$ hypergraphs with high probability. Let $\mathfrak{M}$ be the set of hyperedges (triangles and 4-cycles) such that for every $e \in \mathfrak{M}$, e is output by the algorithm as an element of a τ-near perfect matching in a hypergraph $\mathcal{H}$.

STEP 5. For each hypergraph $\mathcal{H}$ we defined in STEP 4, consider $\mathcal{H}^{-1}$. Delete all the edges contained in $\mathfrak{M}$ (with inverse direction) from $\mathcal{H}^{-1}$. By [33, Theorem 3.3], the resulting hypergraph still satisfies Conditions (1)–(3) in Theorem II.2. Apply random greedy algorithm again. For every $\tau > 0$, the algorithm will output a τ-near perfect matching in at least $(1 - r(\varepsilon) - \tau)\mathsf{h}$ hypergraphs with high probability. We also put edges in these near perfect matchings into $\mathfrak{M}$.

STEP 6. Output the rotation $\Pi = \{\pi_v \mid v \in V(G)\}$ which is constructed as follows. Every hyperedge $C \in \mathfrak{M}$ corresponds to a 3-cycle or a 4-cycle in G. Let V_C and E_C be the vertex and the edge-set of that cycle. Now define

$$F(v) = \{C \in \mathfrak{M} \mid v \in V_C\}.$$

The elements in $\mathfrak{M}$ determine which pairs of edges should be consecutive in the local rotation around v. Each arc of the corresponding digraph D of G is in at most one of the cycles $C \in \mathfrak{M}$ since the digraphs ${}^xD_{ijk}$ and ${}^yD_{ij}$ are edge-disjoint. Thus the cycles $C \in \mathfrak{M}$ are arc disjoint cycles in $D \cup D^{-1}$. For each blossom created by $\mathfrak{M}$, we remove one of the cycles from $\mathfrak{M}$. As proved in the paper, there is only a small number of blossoms all together with high probability. Obtaining a blossom-free subset, we apply Lemma II.1 (following its proof). By Theorem III.2, we have $(1+\varepsilon)\mathsf{g}(G) \geq \mathsf{g}(G, \Pi)$.

IV. OUTLINES OF THE PROOFS

In this section we sketch the proof for the genus of quasirandom multipartite graphs, which is the key part in the proof of Theorem III.2, and provides the theoretical basis for our algorithm.

A. Quasirandom bipartite graphs

Given $\varepsilon > 0$, we consider a graph $G \in \mathcal{Q}(n^{(2)}, p, \varepsilon)$. This means G is defined on the vertex-set $V_1 \sqcup V_2$, each V_i of size n, and $\mathsf{d}_\square(G, K(n^{(2)}, p)) < \varepsilon$. Let $\mathsf{p}_3(u, v)$ be the number of walks of length 3 between u and v. We have the following lemma.

Lemma IV.1. *Let $\varepsilon > 0$ and let $G_2 \in \mathcal{Q}(n^{(2)}, p, \varepsilon)$. For every $u \in V_1$ and $v \in V_2$, we have*

$$2 \sum_{uv \in E(G_2)} |\mathsf{p}_3(u, v) - n^2 p^3| \le \sqrt{17\varepsilon}\, n^4.$$

Proof: By Counting Lemma, we have

$$2 \sum_{uv \in E(G_2)} \mathsf{p}_3(u, v) = \mathsf{hom}(C_4, G_2) \ge (p^4 - 4\varepsilon) n^4.$$

Therefore, by the Cauchy-Schwartz inequality,

$$\begin{aligned}
&\left(2 \sum_{uv \in E(G_2)} |\mathsf{p}_3(u, v) - n^2 p^3|\right)^2 \\
\le\, & 2n^2 \sum_{uv \in E(G_2)} \left|\mathsf{p}_3(u, v) - n^2 p^3\right|^2 \\
\le\, & n^2 (n^6 p^7 + 8\varepsilon n^6 + n^6 p^7 + \varepsilon n^6 p^6 - 2n^6 p^7 + 8\varepsilon n^6 p^3) \\
<\, & 17\varepsilon n^8,
\end{aligned}$$

and this proves the lemma. ∎

In order to bound the number of short blossoms, we choose an integer $t = t(n)$ and let $p_1 = p/t$, such that $n^{-\frac{2}{3}} \ll p_1 \ll n^{-\frac{2-\varepsilon}{3-\varepsilon}}$. Let $D \in \mathcal{D}(G_2)$ be the corresponding digraph of G_2. We partition its edges into t parts uniformly at random, the resulting edge disjoint digraphs are $D_1, \ldots, D_t$. For each D_i, let $\mathcal{H}_i$ be the 4-uniform hypergraph, whose vertex-set is $V(\mathcal{H}_i) = E(D_i)$ and whose edges are all sets of edges of directed 4-cycles in D_i. The following lemma shows that condition (1) of Theorem II.2 is true.

Lemma IV.2. *Let $\Delta_2 = n^2 p_1^3/8$. There exists a real number $\delta > 0$ such that a.a.s.*

$$\begin{aligned}
&|\{x \in V(\mathcal{H}_i) \mid (1-\delta)\Delta_2 \le \deg(x) \le (1+\delta)\Delta_2\}| \\
&\qquad \ge (1-\delta)|V(\mathcal{H}_i)|.
\end{aligned}$$

Proof: Recall that $p = tp_1$. Suppose $\lambda^2 = \frac{\sqrt{18\varepsilon}}{p^3(p-\varepsilon)}$ and uv is an edge in G_2. We say uv is *balanced* if $\mathsf{p}_3(u, v)$ is at least $(1-\lambda)n^2p^3$ and at most $(1+\lambda)n^2p^3$, otherwise it is *unbalanced*. Assume that at least $\lambda|E(G_2)|$ edges are unbalanced. Then we have

$$\begin{aligned}
2 \sum_{uv \in E(G_2)} |\mathsf{p}_3(u, v) - n^2 p^3| &\ge \lambda^2 n^2 p^3 (n^2 p - \varepsilon n^2) \\
&\ge \sqrt{18\varepsilon}\, n^4,
\end{aligned}$$

which contradicts Lemma IV.1. Then in the graph G_2, at least $(1-\lambda)|E(G_2)|$ edges are balanced.

For the graph D_i, by Chebyshev's inequality, we have

$$(1-\varepsilon)\frac{|E(G_2)|}{t} \le |E(D_i)| \le (1+\varepsilon)\frac{|E(G_2)|}{t}, \quad \text{a.a.s.}$$

Let U be the number of balanced edges in G_2. Then a.a.s. D_i contains at least $(1-\varepsilon)U/t$ edges which are balanced in G_2, and most of them are still in D_i. To be more precise, at least $(1-\varepsilon)^2 U/t$ edges are contained in at least $(1-\varepsilon)(1-\lambda)n^2p_1^3/8$ directed cycles of length 4, and are contained in at most $(1+\varepsilon)(1+\lambda)n^2p_1^3/8$ directed cycles of length 4. Now we let $\delta \ge \max\{\lambda+\varepsilon+\varepsilon\lambda, \psi(\varepsilon, \lambda)\}$, where $\psi(\varepsilon, \lambda) = \frac{\lambda(1-\varepsilon)^2+\varepsilon(3-\varepsilon)}{1+\varepsilon}$. We have a.a.s.

$$\begin{aligned}
(1-\varepsilon)^2 \frac{U}{t} &\ge (1-\varepsilon)^2(1-\lambda)\frac{|E(G_2)|}{t} \\
&\ge \frac{(1-\varepsilon)^2(1-\lambda)}{1+\varepsilon}|V(\mathcal{H}_i)| \ge (1-\delta)|V(\mathcal{H}_i)|.
\end{aligned}$$

Also for the number of 4-cycles, we have $1+\delta > (1+\varepsilon)(1+\lambda)$ and $1-\delta < (1-\varepsilon)(1-\lambda)$. This completes the proof. ∎

For the Condition (2) in Theorem II.2, a.a.s. D_i contains at most $t^2 n$ cycles of length 4 that contain two edges. Since $t \ll n^{1/3}$, this is definitely less than $\delta\Delta_2 = \Theta(n^2)$. In order to verify Condition (3), we have the following lemma.

Lemma IV.3. *Let F_i be the number of directed cycles of length 4 in D_i which contain at least one directed edge $\overrightarrow{uv} \in P^\delta$, where P^δ is the set of pairs of vertices $(u, v) \in V_1 \oplus V_2$ such that the number of directed paths from v to u of length 3 is at least $(1+\delta)\Delta_2$. Then with high probability, $F_i < \delta\Delta_2|V(\mathcal{H}_i)|$.*

By the above-stated facts, each $\mathcal{H}_i$ satisfies conditions (1)–(3) a.a.s. Then we can apply Theorem II.2, once for each D_i and once for D_i^{-1}, and combine it with Lemma II.1, to obtain a rotation system.

Theorem IV.4. *Let $\varepsilon > 0$ and $G_2 \in \mathcal{Q}(n^{(2)}, P, \varepsilon)$, then G_2 has a 10ε-near quadrangular embedding.*

Proof: For every $i \in [t]$, $\mathcal{H}_i$ satisfies conditions (1)–(3) in Theorem II.2 a.a.s. That means, if q is the probability that $\mathcal{H}_i$ does not satisfy all of the conditions, then $q \to 0$ as $n \to \infty$. Let $I \subseteq [t]$ be the set of those values $j \in [t]$, for which $\mathcal{H}_j$ fails to satisfy any of the three conditions listed in Theorem II.2. By Markov's inequality we obtain $|I| \le \varepsilon t$ a.a.s. Note that "a.a.s." here comes from the way we construct D_i, that means there exists a construction of D_i for every $i \in [t]$ such that $|I| \le \varepsilon t$. For the convenience, we denote $J = [t] \setminus I$.

Then for every $i \in J$, there exists a matching M_i of $\mathcal{H}_i$ of size at least $(1-\varepsilon)\frac{|E(D_i)|}{4}$, and another matching M_i' in $\mathcal{H}_i^{-1}$ disjoint with M_i^{-1}, and of size at least $(1-\varepsilon)\frac{|E(D_i)|}{4}$. In $D_i \cup D_i^{-1}$, $M_i \cup M_i'$ has size at least $(1-\varepsilon)\frac{|E(D_i)|}{2}$, and has no non-simple blossoms of length 2.

Now we proceed with removing all the blossoms in $M_i \cup M_i'$. For all of the blossoms of length at most $1/\varepsilon$, let $\overrightarrow{\mathcal{B}_j}$ be the 4-blossom-graphs of length j in $D \cup D^{-1}$, and $\mathcal{B}_j$ is the underlying simple graph in G_2, where $2 \le j \le 1/\varepsilon$. Therefore,

$$\mathsf{hom}(\mathcal{B}_j, G) \le n^{2j+1}p^{3j} + 3j\varepsilon n^{2j+1}.$$

For every $i \in [t]$, giving a 4-blossom simple graph $\mathcal{B}_j$, $\mathbb{P}(\overrightarrow{\mathcal{B}_j} \in D_i \cup D_i^{-1}) = \frac{1}{2^{j-1}t^{3j}}$. Let $N_i(j)$ be the number of $\overrightarrow{\mathcal{B}_j}$ in $D_i \cup D_i^{-1}$. Then we have a.a.s.

$$N_i(j) < (1+\varepsilon)\Big(\frac{n^{2j+1}p_1^{3j}}{2^{2j-1}} + \frac{3j\varepsilon n^{2j+1}p_1^{3j}}{p^{3j}2^{2j-1}}\Big) \ll n^2 p_1.$$

This means $\sum_{j=1}^{1/\varepsilon} N_i(j) < \varepsilon(1-\varepsilon)n^2p_1 < \varepsilon|E(D_i)|$, a.a.s.

Now we are going to remove one cycle from each of the blossoms. Since the number of 4-blossoms of length at least $1/\varepsilon$ is bounded by $2\varepsilon|E(D_i)|$, we obtain a blossom-free subset of $M_i \cup M_i'$ of size at least $(1-7\varepsilon)\frac{|E(D_i)|}{2}$, a.a.s. By Lemma II.1 there exists a rotation system of D_i for every $i \in [t]$, and $(1-\varepsilon)^2 t$ of them have almost the correct number of edges, and they give rise to a 7ε-near quadrangular embedding. Then G_2 has an embedding Π such that

$$4f_4(\Pi) \ge 2(1-7\varepsilon)(1-\varepsilon)^3|E(G_2)| \ge 2(1-10\varepsilon)|E(G_2)|,$$

which completes the proof. ∎

Theorem IV.4 and its analogue for the tripartite case are used to determine the genus of quasirandom bipartite and tripartite graphs, see Theorems I.3 and I.4.

B. Multipartite quasirandom graphs

Now, we consider partitioning of general quasirandom graphs into several bipartite and tripartite quasirandom subgraphs. We need this process throughout our approximation algorithm when we try to construct an embedding. Given a quasirandom graph G, the following lemma shows that we can partition the graph into a number of edge disjoint graphs with prescribed edge densities, each of which is also quasirandom.

Lemma IV.5. *Let $\varepsilon > 0$ and $G \in \mathcal{Q}(n^{(2)}, p, \varepsilon)$. Suppose k is a constant and $c_1, c_2, \ldots, c_k$ are positive real numbers such that $\sum_{i=1}^k c_i = 1$. Then there exists an edge partition of G into k edge-disjoint graphs $G_1, G_2, \ldots, G_k$, such that for every $i \in [k]$, $G_i \in \mathcal{Q}(n^{(2)}, c_i p, 3c_i\varepsilon)$.*

Proof: We first consider a random partition, that is, for every edge $e \in E(G)$, we have $\mathbb{P}(e \in E(G_i)) = c_i$. Suppose G is defined on the vertex-set $V_1 \sqcup V_2$. Then for every $X \subseteq V_1$ and $Y \subseteq V_2$, by Chebyshev's inequality, we have a.a.s.

$$(1-\varepsilon)c_i\mathsf{e}_G(X,Y) \le \mathsf{e}_{G_i}(X,Y) \le (1+\varepsilon)c_i\mathsf{e}_G(X,Y).$$

That is

$$|\mathsf{e}_{G_i}(X,Y) - \mathsf{e}_G(X,Y)c_i| \le c_i\varepsilon n^2 + c_i\varepsilon^2 n^2 + c_i\varepsilon\mathsf{e}_G(X,Y) < 3c_i\varepsilon n^2.$$

This implies that $\mathsf{d}_\square(G_i, K(n^{(2)}, c_i p)) < 3c_i\varepsilon$. Thus, if n is large enough, there exists a random partition such that for every $i \in [k]$, $G_i \in \mathcal{Q}(n^{(2)}, c_i p, 3c_i\varepsilon)$. ∎

For an edge-weighted simple graph H of order n, let $d_e \ge 0$ denote the weight of the edge e. Let $\mathcal{T}$ be the set of all triangles in H (of positive edge-weight). Now we consider the linear program (1) with optimum solution $\{t(T) \mid T \in \mathcal{T}\}$ and its maximum $\nu(H)$. We extend the solution to all triangles by setting $t(T) = 0$ when $T \notin \mathcal{T}$.

Given $G \in \mathcal{Q}(n^{(m)}, P, \varepsilon)$, let p_{ij} be the (i,j)-entry of P. Suppose G is defined on the vertex-set $V_1 \sqcup \cdots \sqcup V_m$ and let H be the quotient graph of G. That is, $|V(H)| = m$, and for every $i, j \in V(H)$, the edge-weight $w(ij) = p_{ij}$. With all tools in hand, we have the following theorem on the genus of ε-W_m-quasirandom graphs.

Theorem IV.6. *Let $\varepsilon > 0$ and $G \in \mathcal{Q}(n^{(m)}, P, \varepsilon/60)$. Suppose H is the quotient graph of G. Then we have*

$$(1-\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n^2}{4} \le \mathsf{g}(G) < (1+\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n^2}{4} + nm^2$$

and

$$(1-\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n^2}{2} \le \widetilde{\mathsf{g}}(G) < (1+\varepsilon)\frac{\mathsf{e}(G) - \nu(H)n^2}{2} + 2nm^2.$$

Proof: We first consider the upper bound. Suppose G is defined on the vertex-set $V_1 \sqcup V_2 \sqcup \cdots \sqcup V_m$. Suppose $\nu(H)$ is defined as above together with a set of triangles $\{T_{ijk}\}$ for every $1 \le i < j < k \le m$. Let $d_{ijk} = t(T_{ijk})$. For every $i, j \in [m]$, let $b_{ij} = p_{ij} - \sum_{k \ne i,j} d_{ijk}$. For every $1 \le i < j \le m$ with $p_{ij} > 0$, we randomly partition the edges between V_i and V_j into at most $m-1$ parts $E_1^{ij}, \ldots, E_m^{ij}, E_0^{ij}$ (excluding E_i^{ij} and E_j^{ij}), using probabilities $d_{ij1}/p_{ij}, \ldots, d_{ijm}/p_{ij}$, and b_{ij}/p_{ij}, respectively.

By Lemma IV.5 we partition the graph G into $O(m^2)$ $\varepsilon/20$-quasirandom bipartite graphs and $O(m^3)$ $\varepsilon/20$-quasirandom tripartite graphs. That is, pick V_i, V_j and V_k, there exist a subgraph G_k^{ij} defined on $V_i \cup V_j$, a subgraph G_i^{jk} defined on $V_j \cup V_k$ and a subgraph G_j^{ik} defined on $V_i \cup V_k$ such that $G_k^{ij}, G_i^{jk}, G_j^{ik} \in \mathcal{Q}(n^{(2)}, d_{ijk}, \varepsilon/20)$. Combining them together, we obtain a tripartite graph in $\mathcal{Q}(n^{(3)}, d_{ijk}, \varepsilon/20)$.

Now we embed the graph G by the partition we constructed. For the tripartite parts, we embed them as quasirandom tripartite graphs, and for the bipartite parts, we embed them as quasirandom bipartite graphs. By Theorems I.3 and

I.4 we obtain a rotation system Π for the disjoint union of these graphs whose genus is

$$\begin{aligned}\sum_i \mathsf{g}(G_i) &\le (1+\varepsilon)\Big(\sum_{i<j}\frac{n^2 b_{ij}}{4} + \sum_{i<j<k}\frac{3n^2 d_{ijk}}{6}\Big)\\ &= (1+\varepsilon)\Big(\sum_{i<j}\frac{n^2 p_{ij}}{4} - \sum_{i<j<k}\frac{n^2 d_{ijk}}{2}\Big)\\ &= (1+\varepsilon)\frac{\mathsf{e}(G)-\nu(H)n^2}{4}.\end{aligned}$$

To obtain an embedding of G we identify copies of the same vertex in different copies of the quasirandom subgraphs. Each identification can increase the genus by at most 1, and the number of identification is at most m for each of the vertices. This is then accounted for in the added term nm^2 in the upper bound.

Finally, let us justify the lower bound for $\mathsf{g}(G)$. By Lemma III.1, the number of faces of length 3 is at most $2\,\nu(H)n^2$, and then a similar calculation as above, using Euler's formula, we obtain the lower bound. ∎

ACKNOWLEDGEMENT

B.M. was supported in part by the NSERC Discovery Grant R611450 (Canada), by the Canada Research Chairs program, and by the Research Project J1-8130 of ARRS (Slovenia).

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