A Proof of CSP Dichotomy Conjecture

Dmitriy Zhuk
Department of Mechanics and Mathematics
Lomonosov Moscow State University
Moscow, Russia
Email: zhuk@intsys.msu.ru

***Abstract*—Many natural combinatorial problems can be expressed as constraint satisfaction problems. This class of problems is known to be NP-complete in general, but certain restrictions on the form of the constraints can ensure tractability. The standard way to parameterize interesting subclasses of the constraint satisfaction problem is via finite constraint languages. The main problem is to classify those subclasses that are solvable in polynomial time and those that are NP-complete. It was conjectured that if a core of a constraint language has a weak near unanimity polymorphism then the corresponding constraint satisfaction problem is tractable, otherwise it is NP-complete.**

In the paper we present an algorithm that solves Constraint Satisfaction Problem in polynomial time for constraint languages having a weak near unanimity polymorphism, which proves the remaining part of the conjecture.

***Keywords*-Constraint satisfaction problem; CSP dichotomy; computational complexity**

I. Introduction

Formally, the *Constraint Satisfaction Problem (CSP)* is defined as a triple $\langle \mathbf{X}, \mathbf{D}, \mathbf{C}\rangle$, where

- $\mathbf{X} = \{x_1, \dots, x_n\}$ is a set of variables,
- $\mathbf{D} = \{D_1, \dots, D_n\}$ is a set of the respective domains,
- $\mathbf{C} = \{C_1, \dots, C_q\}$ is a set of constraints,

where each variable x_i can take on values in the nonempty domain D_i, every *constraint* $C_j \in \mathbf{C}$ is a pair (t_j, ρ_j) where t_j is a tuple of variables of length m_j, called the *constraint scope*, and ρ_j is an m_j-ary relation on the corresponding domains, called the *constraint relation*.

The question is whether there exists *a solution* to $\langle \mathbf{X}, \mathbf{D}, \mathbf{C}\rangle$, that is a mapping that assigns a value from D_i to every variable x_i such that for each constraints C_j the image of the constraint scope is a member of the constraint relation.

In this paper we consider only CSP over finite domains. The general CSP is known to be NP-complete [1], [2]; however, certain restrictions on the allowed form of constraints involved may ensure tractability (solvability in polynomial time) [3], [4], [5], [6], [7], [8]. Below we provide a formalization to this idea.

To simplify the presentation we assume that all the domains $D_1, \dots, D_n$ are subsets of a finite set A. By R_A we denote the set of all finitary relations on A, that is, subsets of A^m for some m. Then all the constraint relations can be viewed as relations from R_A.

For a set of relations $\Gamma \subseteq R_A$ by $\mathrm{CSP}(\Gamma)$ we denote the Constraint Satisfaction Problem where all the constraint relations are from Γ. The set Γ is called *a constraint language*. Another way to formalize the Constraint Satisfaction Problem is via conjunctive formulas. Every h-ary relation on A can be viewed as a predicate, that is, a mapping $A^h \to \{0, 1\}$. Suppose $\Gamma \subseteq R_A$, then $\mathrm{CSP}(\Gamma)$ is the following decision problem: given a formula

$$\rho_1(x_{1,1}, \dots, x_{1,n_1}) \wedge \dots \wedge \rho_s(x_{s,1}, \dots, x_{1,n_s})$$

where $\rho_i \in \Gamma$ for every i; decide whether this formula is satisfiable.

It is well known that many combinatorial problems can be expressed as $\mathrm{CSP}(\Gamma)$ for some constraint language Γ. Moreover, for some sets Γ the corresponding decision problem can be solved in polynomial time; while for others it is NP-complete. It was conjectured that $\mathrm{CSP}(\Gamma)$ is either in P, or NP-complete [9].

Conjecture 1. *Suppose $\Gamma \subseteq R_A$ is a finite set of relations. Then $\mathrm{CSP}(\Gamma)$ is either solvable in polynomial time, or NP-complete.*

We say that an operation $f\colon A^n \to A$ *preserves* the relation $\rho \in R_A$ of arity m if for any tuples $(a_{1,1}, \dots, a_{1,m}), \dots, (a_{n,1}, \dots, a_{n,m}) \in \rho$ the tuple $(f(a_{1,1}, \dots, a_{n,1}), \dots, f(a_{1,m}, \dots, a_{n,m}))$ is in ρ. We say that an operation *preserves a set of relations* Γ if it preserves every relation in Γ. A mapping $f : A \to A$ is called *an endomorphism* of Γ if it preserves Γ.

Theorem 1. *[7] Suppose $\Gamma \subseteq R_A$. If f is an endomorphism of Γ, then $CSP(\Gamma)$ is polynomially reducible to $CSP(f(\Gamma))$ and vice versa, where $f(\Gamma)$ is a constraint language with domain $f(\Gamma)$ defined by $f(\Gamma) = \{f(\rho)\colon \rho \in \Gamma\}$.*

A constraint language is *a core* if every endomorphism of Γ is a bijection. It is not hard to show that if f is an endomorphism of Γ with minimal range, then $f(\Gamma)$ is a core. Another important fact is that we can add all singleton unary relations to a core constraint language without increasing the complexity of its CSP. By $\sigma_{=a}$ we denote the unary relation $\{a\}$.

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Theorem 2. *[7] Let $\Gamma \subseteq R_A$ be a core constraint language, and $\Gamma' = \Gamma \cup \{\sigma_{=a} \mid a \in A\}$, then $CSP(\Gamma')$ is polynomially reducible to $CSP(\Gamma)$.*

Therefore, to prove Conjecture 1 it is sufficient to consider only the case when Γ contains all unary singleton relations. In other words, all the predicates $x = a$, where $a \in A$, are in the constraint language Γ.

In [10] Schaefer classified all tractable constraint languages over two-element domain. In [11] Bulatov generalized the result for three-element domain. His dichotomy theorem was formulated in terms of a G-set. Later, the dichotomy conjecture was formulated in several different forms (see [7]).

The result of Mckenzie and Maróti [12] allows us to formulate the dichotomy conjecture in the following nice way. An operation f is called *a weak near-unanimity operation (WNU)* if $f(x, x, \dots, x) = x$ and $f(y, x, \dots, x) = f(x, y, x, \dots, x) = \dots = f(x, x, \dots, x, y)$.

Conjecture 2. *Suppose $\Gamma \subseteq R_A$ is a finite set of relations, $\{\sigma_{=a} \mid a \in A\} \subseteq \Gamma$. Then $\mathrm{CSP}(\Gamma)$ can be solved in polynomial time if there exists a WNU preserving Γ; $CSP(\Gamma)$ is NP-complete otherwise.*

One direction of this conjecture follows from [12].

Theorem 3. *[12] Suppose $\Gamma \subseteq R_A$, $\{\sigma_{=a} \mid a \in A\} \subseteq \Gamma$. If there exists no WNU preserving Γ, then $\mathrm{CSP}(\Gamma)$ is NP-complete.*

The dichotomy conjecture was proved for many special cases: for CSPs over undirected graphs [13], for CSPs over digraphs with no sources or sinks [14], for constraint languages containing all unary relations [15], and many other. Recently, a proof of the dichotomy conjecture was announced by Andrei Bulatov [16]. Note that Bulatov's algorithm also works for infinite constraint languages. More information about the algebraic approach to CSP can be found in [17].

In this paper we present an algorithm that solves $\mathrm{CSP}(\Gamma)$ in polynomial time if Γ is preserved by a WNU, and therefore prove the dichotomy conjecture. This is a short version of the paper published online [18] with some auxiliary statements and proofs omitted.

The paper is organized as follows. In Section II we give main definitions, in Section III we explain the algorithm. In Section IV we prove a theorem that explains the main idea of the algorithm and formulate theorems that prove correctness of the algorithm. In Section V we give an example that explains how the algorithm works for a system of linear equations in $\mathbb{Z}_4$.

In the next section we give the remaining definitions. In Section VII we formulate statements we will need in the proof of main theorems (see [18] for the proof).

In the last section we prove the main theorems of this paper formulated in Section IV. First, we explain how a linear variable can be added and prove the existence of a bridge. Finally, we use simultaneous induction to prove the main theorems.

II. Definitions

A set of operations is called *a clone* if it is closed under composition and contains all projections. For a set of operations M by $\mathrm{Clo}(M)$ we denote the clone generated by M.

A WNU w is called *special* if $x \circ (x \circ y) = x \circ y$, where $x \circ y = w(x, \dots, x, y)$. It is not hard to show that for any WNU w on a finite set there exists a special WNU $w' \in \mathrm{Clo}(w)$.

A relation $\rho \subseteq A_1 \times \dots \times A_n$ is called *subdirect* if for every i the projection of ρ onto the i-th coordinate is A_i. For a relation ρ by $\mathrm{pr}_{i_1,\dots,i_s}(\rho)$ we denote the projection of ρ onto the coordinates $i_1, \dots, i_s$.

A. Algebras

An algebra is a pair $\mathbf{A} := (A; F)$, where A is a finite set, called *universe*, and F is a family of operations on A, called *basic operations of* $\mathbf{A}$. In the paper we always assume that we have a special WNU preserving all constraint relations. Therefore, every domain D can be viewed as an algebra $(D; w)$. By $\mathrm{Clo}(\mathbf{A})$ we denote the clone generated by all basic operations of $\mathbf{A}$.

An equivalence relation σ on the universe of an algebra $\mathbf{A}$ is called *a congruence* if it is preserved by every operation of the algebra. A congruence (an equivalence relation) is called *proper*, if it is not equal to the full relation $A \times A$. We use standard universal algebraic notions of a term operation, a subalgebra, a factor algebra, a product of algebras, see [19]. We say that a subalgebra $\mathbf{R} = (R; F_R)$ is *a subdirect subalgebra* of $\mathbf{A} \times \mathbf{B}$ if R is a subdirect relation in $A \times B$.

B. Polynomially complete algebras

An algebra is called *polynomially complete (PC)* if the clone generated by F_A and all constants on A is the clone of all operations on A.

C. Linear algebra

A finite algebra $(A; w_A)$ is called *linear* if it is isomorphic to $(\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_s}; x_1 + \ldots + x_n)$ for prime numbers $p_1, \dots, p_s$. It is not hard to show that for every algebra $(B; w_B)$ there exists a minimal congruence σ, called *the minimal linear congruence*, such that $(B; w_B)/\sigma$ is linear.

D. Absorption

Let $\mathbf{B} = (B; F_B)$ be a subalgebra of $\mathbf{A} = (A; F_A)$. We say that B absorbs $\mathbf{A}$ if there exists $t \in \mathrm{Clo}(\mathbf{A})$ such that $t(B, B, \dots, B, A, B, \dots, B) \subseteq B$ for any position of A. In this case we also say that B is an absorbing subuniverse of $\mathbf{A}$. If the operation t can be chosen binary then we say that B is a binary absorbing subuniverse of $\mathbf{A}$.

E. Center

Suppose $\mathbf{A} = (A; w_A)$ is a finite algebra with a WNU operation. $C \subseteq A$ is called a *center* if there exists an algebra $\mathbf{B} = (B; w_B)$ with a WNU operation of the same arity and a subdirect subalgebra $(R; w_R)$ of $\mathbf{A} \times \mathbf{B}$ such that there is no binary absorbing subuniverse in $\mathbf{B}$ and

$$C = \{a \in A \mid \forall b \in B \colon (a, b) \in R\}.$$

F. CSP instance

An instance of the constraint satisfaction problem is called *a CSP instance*. Sometimes we use the same letter for a CSP instance and for the set of constraints of this instance. For a variable z by D_z we denote the domain of the variable z.

We say that $z_1 - C_1 - z_2 - \cdots - C_{l-1} - z_l$ is *a path* in Θ if z_i, z_{i+1} are in the scope of C_i for every i. We say that *a path* $z_1 - C_1 - z_2 - \ldots C_{l-1} - z_l$ *connects* b *and* c if there exists $a_i \in D_{z_i}$ for every i such that $a_1 = b$, $a_l = c$, and the projection of C_i onto z_i, z_{i+1} contains the tuple (a_i, a_{i+1}).

A CSP instance is called *cycle-consistent* if for every i and $a \in D_i$, any path starting and ending with x_i in Θ connects a and a.

A CSP instance Θ is called *linked* if for every variable x_i appearing in a constraint of Θ and every $a, b \in D_i$ there exists a path in Θ that connects a and b. Suppose $\mathbf{X}' \subseteq \mathbf{X}$. Then we can define a projection of Θ onto $\mathbf{X}'$, that is a CSP instance where variables are elements of $\mathbf{X}'$ and constraints are projections of constraints of Θ onto $\mathbf{X}'$. We say that an instance Θ is *fragmented* if the set of variables $\mathbf{X}$ can be divided into 2 nonempty disjoint sets $\mathbf{X_1}$ and $\mathbf{X_2}$ such that the constraint scope of any constraint of $C \in \Theta$ either has variables only from $\mathbf{X_1}$, or only from $\mathbf{X_2}$.

A CSP instance Θ is called *irreducible* if for any subset of constraints $\Theta' \subseteq \Theta$ and any subset of variables $\mathbf{X}' \subseteq \mathbf{X}$ the projection of Θ' onto $\mathbf{X}'$ is fragmented, linked, or its solution set is subdirect.

We say that a constraint $((y_1, \ldots, y_t); \rho_1)$ is *weaker than* a constraint $((z_1, \ldots, z_s); \rho_2)$ if $\{y_1, \ldots, y_t\} \subseteq \{z_1, \ldots, z_s\}$, $\rho_2(z_1, \ldots, z_s) \rightarrow \rho_1(y_1, \ldots, y_t)$, and $\rho_1(y_1, \ldots, y_t) \not\rightarrow \rho_2(z_1, \ldots, z_s)$.

Let $D_i' \subseteq D_i$ for every i. A constraint C of Θ is called *crucial in* $(D_1', \ldots, D_n')$ if Θ has no solutions in $(D_1', \ldots, D_n')$ but the replacement of $C \in \Theta$ by all weaker constraints gives an instance with a solution in $(D_1', \ldots, D_n')$. A CSP instance Θ is called *crucial in* $(D_1', \ldots, D_n')$ if every constraint of Θ is crucial in $(D_1', \ldots, D_n')$.

Remark 1. *Suppose Θ has no solutions in $(D_1', \ldots, D_n')$. Then we can replace constraints of Θ by all weaker constraints until we get a CSP instance that is crucial in $(D_1', \ldots, D_n')$.*

III. Algorithm

A. Main part

Suppose we have a constraint language Γ_0 that is preserved by a WNU operation. As it was mentioned before, Γ_0 is also preserved by a special WNU operation w. Let k_0 be the maximal arity of the relations in Γ_0. By Γ we denote the set of all relations of arity at most k_0 that are preserved by w. Obviously, $\Gamma_0 \subseteq \Gamma$, therefore CSP($\Gamma_0$) can be reduced to CSP($\Gamma$).

In this section we provide an algorithm that solves CSP(Γ) in polynomial time. Suppose we have a CSP instance $\Theta = \langle \mathbf{X}, \mathbf{D}, \mathbf{C} \rangle$, where $\mathbf{X} = \{x_1, \ldots, x_n\}$ is a set of variables, $\mathbf{D} = \{D_1, \ldots, D_n\}$ is a set of the respective domains, $\mathbf{C} = \{C_1, \ldots, C_q\}$ is a set of constraints. Let the arity of the WNU w be equal to m.

The algorithm is recursive, the list of all possible recursive calls is given in the end of this subsection. One of the main recursive calls is the reduction of a subuniverse D_i to D_i' such that either Θ has a solution with $x_i \in D_i'$, or it has no solutions at all.

Step 1. *Check whether Θ is cycle-consistent. If not then we reduce a domain D_i for some i or state that there are no solutions.*

Step 2. *Check whether Θ is irreducible. If not then we reduce a domain D_i for some i or state that there are no solutions.*

Step 3. *Replace every constraint of Θ by all weaker constraints. Recursively calling the algorithm, check that the obtained instance has a solution with $x_i = b$ for every $i \in \{1, 2, \ldots, n\}$ and $b \in D_i$. If not, reduce D_i to the projection onto x_i of the solution set of the obtained instance.*

By Theorem 6 we cannot loose the only solution while doing the following two steps.

Step 4. *If D_i has a binary absorbing subuniverse $B_i \subsetneq D_i$ for some i, then we reduce D_i to B_i.*

Step 5. *If D_i has a center $C_i \subsetneq D_i$ for some i, then we reduce D_i to C_i.*

By Theorem 7 we can do the following step.

Step 6. *If there exists a congruence σ on D_i such that the algebra $(D_i; w)/\sigma$ is polynomially complete, then we reduce D_i to any equivalence class of σ.*

By Theorem 4, it remains to consider the case when for every domain D_i there exists a congruence σ_i on D_i such that $(D_i; w)/\sigma_i$ is linear, i.e. it is isomorphic to $(\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_l}; x_1 + \cdots + x_m)$ for prime numbers $p_1, \ldots, p_l$. Moreover, σ_i is proper if $|D_i| > 1$.

We denote D_i/σ_i by L_i. We define a new CSP instance Θ_L with domains $L_1, \ldots, L_n$. To every constraint $((x_{i_1},\ldots,x_{i_s});\rho)\in\Theta$ we assign a constraint $((x'_{i_1},\ldots,x'_{i_s});\rho')$, where $\rho'\subseteq L_{i_1}\times\cdots\times L_{i_s}$ and $(E_1,\ldots,E_s)\in\rho'\Leftrightarrow(E_1\times\cdots\times E_s)\cap\rho\neq\varnothing$. The constraints of Θ_L are all constraints that are assigned to the constraints of Θ.

Since every relation on $\mathbb{Z}_{p_1}\times\cdots\times\mathbb{Z}_{p_l}$ preserved by $x_1+\ldots+x_m$ is known to be a conjunction of linear equations, the instance Θ_L can be viewed as a system of linear equations in $\mathbb{Z}_p$ for different p. To simplify the explanation we include variables with different domains in one equation. Note that all essential variables of every equation have the same domain.

Our general idea is to add some linear equations to Θ_L so that for any solution of Θ_L there exists the corresponding solution of Θ. We start with the empty set of equations Eq, which is a set of constraints on $L_1,\ldots,L_n$.

Step 7. *Put $Eq:=\varnothing$.*

Step 8. *Solve the system of linear equations $\Theta_L\cup Eq$ and choose independent variables $y_1,\ldots,y_k$. If it has no solutions then Θ has no solutions. If it has just one solution, then, recursively calling the algorithm, solve the reduction of Θ to this solution. Either we get a solution of Θ, or Θ has no solutions.*

Then there exist $Z=\mathbb{Z}_{q_1}\times\cdots\times\mathbb{Z}_{q_k}$ and a linear mapping $\phi\colon Z\to L_1\times\cdots\times L_n$ such that any solution of $\Theta_L\cup Eq$ can be obtained as $\phi(a_1,\ldots,a_k)$ for some $(a_1,\ldots,a_k)\in Z$.

Note that for any tuple $(a_1,\ldots,a_k)\in Z$ we can check recursively whether Θ has a solution in $\phi(a_1,\ldots,a_k)$. To do this, we just need to solve an easier CSP instance (on smaller domains). Similarly, we can check whether Θ has a solution in $\phi(a_1,\ldots,a_k)$ for every $(a_1,\ldots,a_k)\in\mathbb{Z}$. To do this, we just need to check the existence of a solution in $\phi(0,\ldots,0,1,0,\ldots,0)$ and $\phi(0,\ldots,0)$ for any position of 1.

Step 9. *If Θ has a solution in $\phi(0,\ldots,0)$, then Θ has a solution.*

Step 10. *Put $\Theta':=\Theta$. Iteratively remove from Θ' all constraints that are weaker than some other constraints of Θ'.*

Step 11. *For every constraint $C\in\Theta'$*

1) *Let Ω be obtained from Θ' by replacing a constraint $C\in\Theta'$ by all weaker constraints without dummy variables. Remove from Ω all constraints that are weaker than some other constraints of Ω.*
2) *If Ω has no solutions in $\phi(a_1,\ldots,a_k)$ for some $(a_1,\ldots,a_k)\in Z$, then put $\Theta':=\Omega$. Repeat Step 11.*

At this moment, the CSP instance Θ' has the following property. Θ' has no solutions in $\phi(b_1,\ldots,b_k)$ for some $(b_1,\ldots,b_k)\in Z$, but if we replace any constraint $C\in\Theta'$ by all weaker constraints, then we get an instance that has a solution in $\phi(a_1,\ldots,a_k)$ for every $(a_1,\ldots,a_k)\in Z$. Therefore, Θ' is crucial in $\phi(b_1,\ldots,b_k)$.

In the remaining steps we will find a new linear equation that can be added to Θ_L. Suppose V is an affine subspace of $\mathbb{Z}_p^h$ of dimension $h-1$, thus V is the solution set of a linear equation $c_1x_1+\cdots+c_hx_h=c_0$. Then the coefficients $c_0,c_1,\ldots,c_h$ can be learned (up to a multiplicative constant) by $(p\cdot h+1)$ queries of the form "$(a_1,\ldots,a_h)\in V$?" as follows. First, we need at most $(h+1)$ queries to find a tuple $(d_1,\ldots,d_h)\notin V$. Then, to find this equation it is sufficient to check for every a and every i whether the tuple $(d_1,\ldots,d_{i-1},a,d_{i+1},\ldots,d_h)$ satisfies this equation.

Step 12. *Suppose Θ' is not linked. For each i from 1 to k*

1) *Check that for every $(a_1,\ldots,a_i)\in\mathbb{Z}_{q_1}\times\cdots\times\mathbb{Z}_{q_i}$ there exist $(a_{i+1},\ldots,a_k)\in\mathbb{Z}_{q_{i+1}}\times\cdots\times\mathbb{Z}_{q_k}$ and a solution of Θ' in $\phi(a_1,\ldots,a_k)$.*
2) *If yes, go to the next i.*
3) *If no, then find an equation $c_1y_1+\cdots+c_iy_i=c_0$ such that for every $(a_1,\ldots,a_i)\in\mathbb{Z}_{q_1}\times\cdots\times\mathbb{Z}_{q_i}$ satisfying $c_1a_1+\cdots+c_ia_i=c_0$ there exist $(a_{i+1},\ldots,a_k)\in\mathbb{Z}_{q_{i+1}}\times\cdots\times\mathbb{Z}_{q_k}$ and a solution of Θ' in $\phi(a_1,\ldots,a_k)$.*
4) *Add the equation $c_1y_1+\cdots+c_iy_i=c_0$ to Eq.*
5) *Go to Step 8.*

If Θ' is linked, then by Theorem 8 there exists a constraint $((x_{i_1},\ldots,x_{i_s}),\rho)$ in Θ' and a subuniverse σ of $\mathbf{D_{i_1}}\times\cdots\times\mathbf{D_{i_s}}\times\mathbf{Z_p}$ such that the projection of σ onto the first s coordinates is bigger than ρ but the projection of $\sigma\cap(D_{i_1}\times\cdots\times D_{i_s}\times\{0\})$ onto the first s coordinates is equal to ρ. Then we add a new variable z with domain $\mathbb{Z}_p$ and replace $((x_{i_1},\ldots,x_{i_s}),\rho)$ by $((x_{i_1},\ldots,x_{i_s},z),\sigma)$. We denote the obtained instance by Υ. Let L be the set of all tuples $(a_1,\ldots,a_k,b)\in\mathbb{Z}_{q_1}\times\cdots\times\mathbb{Z}_{q_k}\times\mathbb{Z}_p$ such that Υ has a solution with $z=b$ in $\phi(a_1,\ldots,a_k)$. We know that the projection of L onto the first n coordinates is a full relation. Therefore L is defined by one linear equation. If this equation is $z=b$ for some $b\neq 0$, then both Θ' and Θ have no solutions. Otherwise, we put $z=0$ in this equation and get an equation that describes all $(a_1,\ldots,a_k)$ such that Θ' has a solution in $\phi(a_1,\ldots,a_k)$. It remains to find this equation.

Step 13. *Suppose Θ' is linked.*

1) *Find an equation $c_1y_1+\cdots+c_ky_k=c_0$ such that for every $(a_1,\ldots,a_k)\in(\mathbb{Z}_{q_1}\times\cdots\times\mathbb{Z}_{q_k})$ satisfying $c_1a_1+\cdots+c_ka_k=c_0$ there exists a solution of Θ' in $\phi(a_1,\ldots,a_k)$.*
2) *If the equation was not found then Θ has no solutions.*
3) *Add the equation $c_1a_1+\cdots+c_ka_k=c_0$ to Eq.*
4) *Go to Step 8.*

Note that every time we reduce our domains, we get constraint relations that are still from Γ.

We have four types of recursive calls of the algorithm:

1) we reduce one domain D_i, for example to a binary absorbing subuniverse or to a center (Steps 1, 4, 5, 6).
2) we solve an instance that is not linked. In this case we divide the instance into the linked parts and solve each of them independently (Steps 2, 12).
3) we replace every constraint by all weaker constraints and solve an easier CSP instance (Step 3).
4) we reduce every domain D_i such that $|D_i| > 1$ (Steps 8, 9, 11, 13).

Lemma 5 states the depth of the recursive calls of type 3 is at most $|\Gamma|$. It is easy to see that the depth of the recursive calls of type 2 and 4 is at most $|A|$.

B. Remaining parts

In this section we explain Steps 1, 2, and 12 of the algorithm, which were not clarified in the previous section.

Provide cycle-consistency. To provide cycle-consistency it is sufficient to use constraint propagation providing (2,3)-consistency. Formally, it can be done in the following way. First, for every pair of variables (x_i, x_j) we consider the intersections of projections of all constraints onto these variables. The corresponding relation we denote by $\rho_{i,j}$. For every $i, j, k \in \{1, 2, \ldots, n\}$ we replace $\rho_{i,j}$ by $\rho'_{i,j}$ where $\rho'_{i,j}(x, y) = \exists z\ \rho_{i,j}(x, y) \wedge \rho_{i,k}(x, z) \wedge \rho_{k,j}(z, y)$. It is not hard to see that this replacement does not change the solution set.

We repeat this procedure while we can change some $\rho_{i,j}$. If at some moment we get a relation $\rho_{i,j}$ that is not subdirect in $D_i \times D_j$, then we can either reduce D_i or D_j, or, if $\rho_{i,j}$ is empty, state that there are no solutions. If we cannot change any relation $\rho_{i,j}$ and every $\rho_{i,j}$ is subdirect in $D_i \times D_j$, then the original CSP instance is cycle-consistent.

Solve the instance that is not linked. Suppose the instance Θ is not linked and not fragmented, then it can be solved in the following way. We say that an element $d_i \in D_i$ and an element $d_j \in D_j$ are *linked* if there exists a path that connects d_i and d_j. Let P be the set of pairs $(i; a)$ such that $i \in \{1, 2, \ldots, n\}$, $a \in D_i$. Then P can be divided into the linked components.

It is easy to see that it is sufficient to solve the problem for every linked component and join the results. Precisely, for a linked component by D'_i we denote the set of all elements d such that (i, d) is in the component. It is easy to see that $\varnothing \subsetneq D'_i \subsetneq D_i$ for every i. Therefore, the reduction to $(D'_1, \ldots, D'_n)$ is a CSP instance on smaller domains.

Check irreducibility. For every $k \in \{1, 2, \ldots, n\}$ and every maximal congruence σ_k on D_k we do the following.

1) Put $I = \{k\}$.
2) Choose a constraint C having the variable x_i in the scope for some $i \in I$, choose another variable x_j from the scope such that $j \notin I$.
3) Denote the projection of C onto (x_i, x_j) by δ.
4) Put $\sigma_j(x, y) = \exists x' \exists y' \delta(x, x') \wedge \delta(y, y') \wedge \sigma_i(x', y')$. If σ_j is a proper equivalence relation, then add j to I.
5) go to the next C, x_i, and x_j in 2).

As a result we get a set I and a congruence σ_i on D_i for every $i \in I$. Put $\mathbf{X}' = \{x_i \mid i \in I\}$. It follows from the construction that for every equivalence class E_k of σ_k and every $i \in I$ there exists a unique equivalence class E_i of σ_i such that there can be a solution with $x_k \in E_k$ and $x_i \in E_i$. Thus, for every equivalence class of σ_k we have a reduction to the instance on smaller domains. Then for every i and $a \in E_i$ we consider the corresponding reduction and check whether there exists a solution with $x_i = a$.

Thus, we can check whether the solution set of the projection of the instance onto $\mathbf{X}'$ is subdirect or empty. If it is empty then we state that there are no solutions. If it is not subdirect, then we can reduce the corresponding domain. If it is subdirect, then we go to the next $k \in \{1, 2, \ldots, n\}$ and next maximal congruence σ_k on D_k, and repeat the procedure.

IV. Correctness of the Algorithm

A. Rosenberg completeness theorem

The main idea of the algorithm is based on a beautiful result obtained by Ivo Rosenberg in 1970, who found all maximal clones on a finite set. Applying this result to the clone generated by a WNU together with all constant operations, we can show that every algebra with a WNU operation has a binary absorption, a center, or it is polynomially complete or linear modular some congruence.

Theorem 4. *Suppose $\mathbf{A} = (A; w)$ is an algebra, w is a special WNU of arity m. Then one of the following conditions hold:*

1) *there exists a binary absorbing set $B \subsetneq A$,*
2) *there exists a center $C \subsetneq A$,*
3) *there exists a proper congruence σ on A such that $(A; w)/\sigma$ is polynomially complete,*
4) *there exists a proper congruence σ on A such that $(A; w)/\sigma$ is isomorphic to $(\mathbb{Z}_p; x_1 + \cdots + x_m)$.*

Proof: Let us prove this statement by induction on the size of A. If we have a binary absorbing subuniverse in A then there is nothing to prove. Let M be the clone generated by w and all constant operations on A. If M is the clone of all operations, then $(A; w)$ is polynomially complete.

Otherwise, by Rosenberg Theorem [20], M belongs to one of the following maximal clones.

1) Maximal clone of monotone operations;
2) Maximal clone of autodual operations;
3) Maximal clone defined by an equivalence relation;
4) Maximal clone of quasi-linear operations;
5) Maximal clone defined by a central relation;
6) Maximal clone defined by an h-universal relation.

Let us consider all the cases.

1) The minimal element of the partial order can be viewed as a center. Since there is no binary absorbing subuniverse, we have a center in A.

2) Constants are not autodual operations. This case cannot happen.
3) Let δ be a maximal congruence on $\mathbf{A}$. We consider a factor algebra $(A; w)/\delta$ and apply the inductive assumption.
   a) If $\mathbf{A}/\delta$ has a binary absorbing subuniverse $B' \subseteq A/\delta$, then we can check that $\bigcup_{E \in B'} E$ is a binary absorbing subuniverse of A.
   b) If $\mathbf{A}/\delta$ has a center $C' \subseteq A/\sigma$, then we can check that $\bigcup_{E \in C'} E$ is a center of A.
   c) Suppose $(\mathbf{A}/\delta)/\sigma$ is polynomially complete. Since δ is a maximal congruence, σ is an equality relation and $\mathbf{A}/\delta$ is polynomially complete.
   d) Suppose $(\mathbf{A}/\delta)/\sigma$ is isomorphic to $(\mathbb{Z}_p; x_1 + \ldots + x_m)$. Since δ is a maximal congruence, σ is an equality relation and $\mathbf{A}/\delta$ is isomorphic to $(\mathbb{Z}_p; x_1 + \ldots + x_m)$.
4) By Lemma 6.4 from [21], we know that $w(x_1, \ldots, x_m) = x_1 + \ldots + x_m$, where $+$ is the operation in an abelian group. We assume that $\mathbf{A}$ has no nontrivial congruences, otherwise we refer to case 3). Then the algebra $\mathbf{A}$ is simple and isomorphic to $(\mathbb{Z}_p; x_1 + \cdots + x_m)$ for a prime number p.
5) We consider the central relation ρ. Let k be the arity of ρ. It is not hard to see that the existence of a binary absorbing subuniverse on $\underbrace{\mathbf{A} \times \cdots \times \mathbf{A}}_{k-1}$ implies the existence of a binary absorbing subuniverse on $\mathbf{A}$. Therefore, the center of ρ can be viewed as a center.
6) By Corollary 5.10 from [21] this case cannot happen. ∎

B. Correctness of the algorithm

Lemma 5. *The depth of the recursive calls of type 3 in the algorithm is less than $|\Gamma|$.*

Proof: First, we introduce a partial order on the set of relations in Γ in the following way. We say that $\rho_1 \leq \rho_2$ if one of the following conditions hold

1) the arity of ρ_1 is less than the arity of ρ_2.
2) the arity of ρ_1 equals the arity of ρ_2, $\mathrm{pr}_i(\rho_1) \subseteq \mathrm{pr}_i(\rho_2)$ for every i, $\mathrm{pr}_j(\rho_1) \neq \mathrm{pr}_j(\rho_2)$ for some j.
3) the arity of ρ_1 equals the arity of ρ_2, $\mathrm{pr}_i(\rho_1) = \mathrm{pr}_i(\rho_2)$ for every i, and $\rho_1 \supseteq \rho_2$.

It is easy to see that any reduction makes every relation smaller or does not change it. Since our constraint language Γ is finite, there can be at most $|\Gamma|$ recursive calls of type 3. ∎

The following three theorems will be proved in Section VIII.

Theorem 6. *Suppose Θ is a cycle-consistent irreducible CSP instance, B is a binary absorbing set or a center of D_i. Then Θ has a solution if and only if Θ has a solution with $x_i \in B$.*

Theorem 7. *Suppose Θ is a cycle-consistent irreducible CSP instance, there does not exist a binary absorbing subuniverse or a center on D_j for every j, $(D_i; w)/\sigma$ is a polynomially complete algebra, E is an equivalence class of σ. Then Θ has a solution if and only if Θ has a solution with $x_i \in E$.*

Theorem 8. *Suppose the following conditions hold:*

1) *Θ is a cycle-consistent irreducible CSP instance with domain set $(D_1, \ldots, D_n)$;*
2) *there does not exist a binary absorbing subuniverse or a center on D_j for every j;*
3) *if we replace every constraint of Θ by all weaker constraints then the obtained instance has a solution with $x_i = b$ for every i and $b \in D_i$;*
4) *Θ_L is Θ factorized by minimal linear congruences;*
5) *$(D'_1, \ldots, D'_n)$ is a solution of Θ_L, and Θ is crucial in $(D'_1, \ldots, D'_n)$.*

Then there exists a constraint $((x_{i_1}, \ldots, x_{i_s}), \rho)$ in Θ and a subuniverse ζ of $\mathbf{D_{i_1}} \times \cdots \times \mathbf{D_{i_s}} \times \mathbb{Z}_\mathbf{p}$ such that the projection of ζ onto the first s coordinates is bigger than ρ but the projection of $\zeta \cap (D_{i_1} \times \cdots \times D_{i_s} \times \{0\})$ onto the first s coordinates is equal to ρ.

V. An example in $\mathbb{Z}_4$

In this section we demonstrate the main part of the algorithm for a system of linear equations in $\mathbb{Z}_4$. Suppose we have a system

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_3 + 2x_4 = 0 \end{cases} \tag{1}$$

The minimal congruence σ such that $(\mathbb{Z}_4; x_1 + \ldots + x_5)/\sigma$ is linear is an equivalence relation modulo 2.

We write the corresponding system of linear equations in $\mathbb{Z}_2$, where $x'_i = x_i \mod 2$.

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ x'_1 + x'_2 = 0 \end{cases} \tag{2}$$

We choose independent variables x'_1 and x'_3, and write the general solution: $x'_1 = x'_1, x'_2 = x'_1, x'_3 = x'_3, x'_4 = x'_1 + x'_3$. We check that (1) doesn't have a solution, corresponding to $x'_1 = x'_3 = 0$. Let us remove the last equation from (1).

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \end{cases} \tag{3}$$

We check that (3) still has no solutions corresponding to $x'_1 = x'_3 = 0$.

We check that if we remove any equation from (3), then for any $a_1, a_3 \in \mathbb{Z}_2$ there will be a solution corresponding to $x'_1 = a_1$ and $x'_3 = a_3$. Hence we need to add exactly one equation to describe all pairs (a_1, a_3) such that (3) has a solution corresponding to $x'_1 = a_1$ and $x'_3 = a_3$. Let the equation be $c_1 x'_1 + c_3 x'_3 = c_0$. We need to find c_1, c_3, and c_0.

Since (3) has a solution corresponding to $x'_1 = 1, x'_3 = 0$, but no solutions for $x'_1 = 0, x'_3 = 1$, the equation is $x'_1 = 1$.

We add this equation to (2) and solve the new system of linear equations in $\mathbb{Z}_2$.

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ \qquad x'_1 + x'_2 = 0 \\ \qquad\qquad x'_1 = 1 \end{cases} \tag{4}$$

The general solution of this system is $x'_1 = 1$, $x'_2 = 1$, $x'_3 = x'_3$, $x'_4 = x'_3 + 1$, where x'_3 is an independent variable. We go back to (1), and check whether it has a solution corresponding to $x'_3 = 0$. Thus, we find a solution $(1, 1, 0, 1)$.

While solving the system of equations, we just solved systems of linear equations in the field $\mathbb{Z}_2$ and constraint satisfaction problems on 2 element set (which are also equivalent to system of equations in $\mathbb{Z}_2$).

VI. The Remaining Definitions

A. Additional notations

We say that the i-th variable of a relation ρ is *compatible with the congruence* σ if $(a_1, \dots, a_n) \in \rho$ and $(a_i, b_i) \in \sigma$ implies $(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) \in \rho$. We say that a relation is *compatible* with σ if every variable of this relation is compatible with σ.

We say that a congruence σ is *irreducible* if it cannot be represented as an intersection of other binary relations $\delta_1, \dots, \delta_s$ compatible with σ. For an irreducible congruence σ on a set A by σ^* we denote the minimal binary relation $\delta \supsetneq \sigma$ compatible with σ.

For a relation ρ by $\mathrm{Con}(\rho, i)$ we denote the binary relation $\sigma(y, y')$ defined by

$$\exists x_1 \dots \exists x_{i-1} \exists x_{i+1} \dots \exists x_n \rho(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \\ \wedge \rho(x_1, \dots, x_{i-1}, y', x_{i+1}, \dots, x_n).$$

For a constraint $C = \rho(x_1, \dots, x_n)$, by $\mathrm{Con}(C, x_i)$ we denote $\mathrm{Con}(\rho, i)$.

A subuniverse A' of $\mathbf{A}$ is called a *PC subuniverse* if $A' = E_1 \cap \dots \cap E_s$, where E_i is an equivalence class of a congruence σ_i such that $\mathbf{A}/\sigma_i$ is a PC algebra.

For an algebra $\mathbf{A}$ by $\mathrm{ConLin}(\mathbf{A})$ we denote the minimal linear congruence. A subuniverse of $\mathbf{A}$ is called a *linear subuniverse* if it is compatible with $\mathrm{ConLin}(\mathbf{A})$.

B. Variety of algebras

We consider the variety of all algebras $\mathbf{A} = (A; w)$ such that w is a special WNU operation of arity m. In the paper every algebra and every domain is considered as an algebra in this variety. Every relation $\rho \subseteq A_1 \times \dots \times A_n$ appearing in the paper is a subalgebra of $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$ for some algebras $\mathbf{A}_1, \dots, \mathbf{A}_n$ of this variety.

C. Formulas

Every variable x appearing in the paper has its domain, which we denote by D_x. A set of constraints is called a *formula*. Sometimes we write a formula as $C_1 \wedge \dots \wedge C_n$. For example, a CSP instance can be viewed as a formula.

For a formula Ω by $\mathrm{Var}(\Omega)$ we denote the set of all variables of Ω. For a formula Ω by $\mathrm{Expanded}(\Omega)$ we denote the set of all formulas Ω' such that there exists a mapping $S : \mathrm{Var}(\Omega') \to \mathrm{Var}(\Omega)$ satisfying the following conditions:

1) for every constraint $(\rho; (x_1, \dots, x_n))$ of Ω' either variables $S(x_1), \dots, S(x_n)$ are different and the constraint $(\rho; (S(x_1), \dots, S(x_n)))$ is weaker than or equal to some constraint of Ω, or ρ is a binary reflexive relation and $S(x_1) = S(x_2)$;
2) if a variable x appears in Ω and Ω' then $S(x) = x$.

Remark 2. *It is easy to check for every cycle-consistent irreducible CSP instance Θ that any instance $\Theta' \in$* $\mathrm{Expanded}(\Theta)$ *is also cycle-consistent and irreducible.*

For a formula Θ and a variable x of this formula by $\mathrm{LinkedCon}(\Theta, x)$ we denote the congruence on the set D_x defined as follows: $(a, b) \in \mathrm{LinkedCon}(\Theta, x)$ if there exists a path in Θ that connects a and b.

D. Critical relations and parallelogram property

We say that a relation *has parallelogram property* if any permutation of variables in ρ satisfies the following implication

$$\forall \alpha_1, \beta_1, \alpha_2, \beta_2 \colon (\alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2 \in \rho \Rightarrow \alpha_1\alpha_2 \in \rho).$$

We say that *the i-th variable of a relation ρ is rectangular*, if for every $(a_i, b_i) \in \mathrm{Con}(\rho, i)$ and $(a_1, \dots, a_n) \in \rho$ we have $(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) \in \rho$. We say that a relation is *rectangular* if all of its variables are rectangular. The following facts can be easily seen: if the i-th variable of ρ is rectangular then $\mathrm{Con}(\rho, i)$ is a congruence; if a relation has parallelogram property then it is rectangular.

A relation $\rho \subseteq A_1 \times \dots \times A_n$ is called *critical* if it cannot be represented as an intersection of other subalgebras of $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$ and it has no dummy variables.

A constraint is called *critical* if the constraint relation is critical.

E. Reductions

A CSP instance is called *1-consistent* if every constraint of the instance is subdirect.

Suppose the domain set of the instance Θ is $D = (D_1, \dots, D_n)$. The domain set $D' = (D'_1, \dots, D'_n)$ is called *a reduction* if D'_i is a subuniverse of D_i for every i.

The reduction $D' = (D'_1, \dots, D'_n)$ is called *1-consistent* if the instance obtained after reduction of every domain is 1-consistent.

We say that D' is *an absorbing reduction*, if D'_i is a binary absorbing subuniverse of D_i with a term operation t for every i. We say that D' is *a central reduction*, if D'_i is a center of D_i for every i. We say that D' is *a PC/linear reduction*, if D'_i is a PC/linear subuniverse of D_i and D_i does not have a center or binary absorbing subuniverse for every i. Additionally, we say that D' is *a minimal central/PC/linear reduction* if D' is a minimal center/PC/linear subuniverse of D_i for every i. We say that D' is *a minimal absorbing reduction* for a term operation t if D' is a minimal absorbing subuniverse of D_i with t for every i.

A reduction is called *nonlinear* if it is an absorbing, central, or PC reduction. A reduction D' is called *proper* if it is an absorbing, central, PC, or linear reduction such that $D' \neq D$.

We usually denote reductions by $D^{(j)}$ for some j (or by $D^{(\top)}$). In this case by $C^{(j)}$ we denote the constraint obtained after reduction of the constraint C. Similarly, by $\Theta^{(j)}$ we denote the instance obtained after reduction of Θ. For a relation ρ by $\rho^{(j)}$ we denote the relation ρ restricted to the corresponding domains of $D^{(j)}$. Sometimes we write $(a_1, \dots, a_n) \in D^{(j)}$ to say that every a_i belongs to the corresponding $D_x^{(j)}$.

A *strategy* for a CSP instance Θ with a domain set D is a sequence of reductions $D^{(0)}, \dots, D^{(s)}$, where $D^{(i)} = (D_1^{(i)}, \dots, D_n^{(i)})$, such that $D^{(0)} = D$ and $D^{(i)}$ is a proper 1-consistent reduction of $\Theta^{(i-1)}$ for every $i \geq 1$. A strategy is called *minimal* if every reduction in the sequence is minimal.

F. Bridges

Suppose σ_1 and σ_2 are congruences on D_1 and D_2, correspondingly. A relation $\rho \subseteq D_1^2 \times D_2^2$ is called *a bridge* from σ_1 to σ_2 if the first two variables of ρ are compatible with σ_1, the last two variables of ρ are compatible with σ_2, $\mathrm{pr}_{1,2}(\rho) \supsetneq \sigma_1$, $\mathrm{pr}_{3,4}(\rho) \supsetneq \sigma_2$, and $(a_1, a_2, a_3, a_4) \in \rho$ implies

$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$

Suppose σ_1, σ_2, σ_3 are irreducible congruences, we have a bridge ρ_1 from σ_1 to σ_2 and a bridge ρ_2 from σ_2 to σ_3. Then we can define a bridge from σ_1 to σ_3 by $\exists y_1 \exists y_2 \rho_1(x_1, x_2, y_1, y_2) \wedge \rho_2(y_1, y_2, z_1, z_2)$.

A bridge $\rho \subseteq D^4$ is called *reflexive* if $(a, a, a, a) \in \rho$ for every $a \in D$.

We say that two congruences σ_1 and σ_2 on a set D are *adjacent* if there exists a reflexive bridge from σ_1 to σ_2.

Remark 3. *Since we can always put $\rho(x_1, x_2, x_3, x_4) = \sigma(x_1, x_3) \wedge \sigma(x_2, x_4)$, any congruence σ is adjacent with itself.*

We say that two constraints C_1 and C_2 are *adjacent* in a common variable x if $\mathrm{Con}(C_1, x)$ and $\mathrm{Con}(C_2, x)$ are adjacent. A formula is called *connected* if every constraint in the formula is rectangular and for every two constraints there exists a path that connects them. It can be shown (see Theorem 22) that every two constraints with a common variable in a connected instance are adjacent.

Then a CSP instance, whose constraints are rectangular, can be divided into *the connected components*.

VII. Auxiliary Statements without Proof

A. Absorption, Center, PC Subuniverse, and Linear Subuniverse

In this subsection we formulate the common property of a binary absorption, a center, a PC subuniverse, and a linear subuniverse, that is, if we restrict all but one variables of a subdirect relation to binary absorbing subuniverses, centers, PC subuniverses, or linear subuniverses, then we restrict the remaining variable correspondingly. The proof of Lemma 9 can be found in [22], the proof of remaining lemmas are in the full proof [18].

Lemma 9. *Suppose $\rho \subseteq A_1 \times \dots \times A_n$ is a relation such that $\mathrm{pr}_1(\rho) = A_1$, $C = \mathrm{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$, where C_i is a binary absorbing subuniverse in A_i with a term operation t for every i. Then C is a binary absorbing subuniverse in A_1 with the term operation t.*

Lemma 10. *Suppose $\rho \subseteq A_1 \times \dots \times A_n$ is a relation such that $\mathrm{pr}_1(\rho) = A_1$, $C = \mathrm{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$, where C_i is a center in A_i for every i. Then C is a center in A_1.*

Lemma 11. *Suppose $\rho \subseteq A_1 \times \dots \times A_n$ is a subdirect relation, there is no binary absorption and center on A_i for every i, $C = \mathrm{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$, where C_i is a PC subuniverse in A_i for every i. Then C is a PC subuniverse in A_1.*

Lemma 12. *Suppose $\rho \subseteq A_1 \times \dots \times A_n$ is a relation such that $\mathrm{pr}_1(\rho) = A_1$, there is no binary absorption on A_1, $C = \mathrm{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$, where C_i is a linear subuniverse in A_i for every i. Then C is a linear subuniverse in A_1.*

B. Properties of reductions

The next two lemmas summarize some properties of minimal reductions (see the proof in [18]).

Lemma 13. *Suppose $D^{(1)}$ is a proper minimal reduction, the constraint $\rho(x_1, \dots, x_n)$ is subdirect, $\rho^{(1)}(x_1, \dots, x_n)$ is not empty. Then $\rho^{(1)}(x_1, \dots, x_n)$ is subdirect.*

Lemma 14. *Suppose $D^{(1)}$ is a proper minimal reduction for a cycle-consistent irreducible CSP instance Θ, $\Theta^{(1)}$ has a solution. Then $\Theta^{(1)}$ is cycle-consistent and irreducible.*

The next theorem allows us to find the next minimal reduction whenever there exists a binary absorption, a center, or a PC subuniverse. Combining this with Theorem 4, we obtain that the difficulties with finding the next reduction can be only if $\mathrm{ConLin}(D_i)$ is proper for any domain D_i such that $|D_i| > 1$ (see the proof in [18]).

Theorem 15. *Suppose $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ is a strategy for a cycle-consistent CSP instance Θ.*

- *If $D_x^{(s)}$ has a binary absorbing set B then there exists a 1-consistent minimal absorbing reduction $D^{(s+1)}$ of $\Theta^{(s)}$ with $D_x^{(s+1)} \subseteq B$.*
- *If $D_x^{(s)}$ has a center B then there exists a 1-consistent minimal central reduction $D^{(s+1)}$ of $\Theta^{(s)}$ with $D_x^{(s+1)} \subseteq B$.*
- *If $D_y^{(s)}$ has no binary absorption and center for every y but there exists a proper PC subuniverse B in $D_x^{(s)}$ for some x, then there exists a 1-consistent minimal PC reduction of $\Theta^{(s)}$ with $D_x^{(s+1)} \subseteq B$.*

The next lemma shows an important property of a relation without parallelogram property.

Lemma 16. *Suppose $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ is a strategy for the constraint $\rho(x_1, \dots, x_n)$, $D^{(s+1)}$ is a linear reduction,*

$$(b_1, \dots, b_t, a_{t+1}, \dots, a_n) \in \rho,$$
$$(a_1, \dots, a_t, b_{t+1}, \dots, b_n) \in \rho,$$
$$(b_1, \dots, b_t, b_{t+1}, \dots, b_n) \in \rho,$$
$$(a_1, \dots, a_t, a_{t+1}, \dots, a_n) \in D^{(s+1)}.$$

Then there exists $(d_1, d_2, \dots, d_n) \in \rho^{(s+1)}$.

VIII. Proof of the Main Theorems

A. Adding linear variable

First, we prove a property of critical relations with a rectangular variable. Then, we prove the main property of a bridge, that is, we explain how a bridge can be used to add a new linear variable to a CSP instance.

Lemma 17. *Suppose ρ is a critical subdirect relation, the i-th variable of ρ is rectangular. Then $\mathrm{Con}(\rho, i)$ is an irreducible congruence.*

Proof: Assume the converse. To simplify notations assume that $i = 1$. Put $\sigma = \mathrm{Con}(\rho, i)$. Consider binary relations $\delta_1, \dots, \delta_s$ compatible with σ such that $\delta_1 \cap \dots \cap \delta_s = \sigma$. Put

$$\rho_i(x_1, \dots, x_n) = \exists x_1' \ \rho(x_1', x_2, \dots, x_n) \wedge \delta_i(x_1, x_1').$$

It is easy to see that the intersection of $\rho_1, \dots, \rho_s$ gives ρ, which contradicts the fact that ρ is critical. ■

Below we formulate few statements from [21] that will help us to prove the main property of a bridge. A relation $\rho \subseteq A^n$ is called *strongly rich* if for every tuple $(a_1, \dots, a_n)$ and every $j \in \{1, \dots, n\}$ there exists a unique $b \in A$ such that $(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n) \in \rho$. We will need two statements from [21].

Theorem 18. *[21] Suppose $\rho \subseteq A^n$ is a strongly rich relation preserved by a WNU. Then there exists an abelian group $(A; +)$ and bijective mappings $\phi_1, \phi_2, \dots, \phi_n : A \to A$ such that*

$$\rho = \{(x_1, \dots, x_n) \mid \phi_1(x_1) + \phi_2(x_2) + \dots + \phi_n(x_n) = 0\}.$$

Lemma 19. *[21] Suppose $(G; +)$ is a finite abelian group, the relation $\sigma \subseteq G^4$ is defined by $\sigma = \{(a_1, a_2, a_3, a_4) \mid a_1 + a_2 = a_3 + a_4\}$, σ is preserved by a WNU f. Then $f(x_1, \dots, x_n) = t \cdot x_1 + t \cdot x_2 + \dots + t \cdot x_n$ for some $t \in \{1, 2, 3, \dots\}$.*

Theorem 20. *Suppose $\sigma \subseteq A^2$ is a congruence, $\rho(x_1, x_2, y_1, y_2)$ is a bridge from σ to σ such that $\rho(x, x, y, y)$ defines a full relation, $\mathrm{pr}_{1,2}(\rho) = \omega$, ω is a minimal relation compatible with σ such that $\omega \supsetneq \sigma$. Then there exists a prime number p and a relation $\zeta \subseteq A \times A \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \zeta \Leftrightarrow (x_1, x_2) \in \sigma$ and $\mathrm{pr}_{1,2} \zeta = \omega$.*

Proof: Since the relations ρ and ω are compatible with σ, we consider A/σ instead of A and assume that σ is the equality relation, ρ and ω are relations on A/σ.

Without loss of generality we assume that $\rho(x_1, x_2, y_1, y_2) = \rho(y_1, y_2, x_1, x_2)$ and $(a, b, a, b) \in \rho$ for any $(a, b) \in \omega$. Otherwise, we consider the relation ρ' instead of ρ, where

$$\rho'(x_1, x_2, y_1, y_2) = \exists z_1 \exists z_2 \rho(x_1, x_2, z_1, z_2) \wedge \rho(y_1, y_2, z_1, z_2).$$

We prove by induction on the size of A. Assume that for some subuniverse $A' \subsetneq A$ we have $(A' \times A') \cap (\omega \setminus \sigma) \neq \varnothing$. By ρ', σ' we denote the restriction of ρ, σ to A' correspondingly. By ω' we denote a minimal relation compatible with σ' such that $\sigma' \subsetneq \omega' \subseteq (A' \times A') \cap \omega$. By the inductive assumption for $\rho \cap (\omega' \times \omega')$ there exists a relation $\zeta' \subseteq A' \times A' \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \zeta' \Leftrightarrow (x_1, x_2) \in \sigma'$ and $\mathrm{pr}_{1,2}(\zeta) = \omega'$. Put

$$\zeta(x_1, x_2, z) = \exists y_1 \exists y_2 \ \rho(x_1, x_2, y_1, y_2) \wedge \zeta'(y_1, y_2, z).$$

It is easy to see that ζ satisfies the necessary conditions.

Thus, we assume that for any subuniverse $A' \subsetneq A$ we have $(A' \times A') \cap (\omega \setminus \sigma) = \varnothing$.

Consider a pair $(a_1, a_2) \in \omega \setminus \sigma$. Then $\{a \mid (a_1, a) \in \omega\} = \{a \mid (a, a_2) \in \omega\} = A$. Hence, any element connected in ω to some other element is connected to all elements. Since $(a_1, a), (a, a_2) \in \omega$ for every $a \in A \setminus \{a_1, a_2\}$, if $|A| > 2$ then $\omega = A \times A$.

If $|A| = 2$ and $\omega \neq A \times A$ then $\omega = \{(a, a), (a, b), (b, b)\}$. This case cannot happen because the corresponding relation ρ is not preserved by any WNU.

Thus, we assume that $\omega = A \times A$.

Let us show that for any $a_1, a_2, a_3 \in A$ there exists a unique a_4 such that $(a_1, a_2, a_3, a_4) \in \rho$. For every $a \in A$ put $\lambda_a(x_1, x_2) = \exists y_2 \rho(x_1, x_2, a, y_2)$. It is easy to see that $\sigma \subsetneq \lambda_a \subseteq \omega$. Therefore $\lambda_a = \omega = A \times A$ for every a. We consider the unary relation defined by $\delta(x) = \rho(a_1, a_2, a_3, x)$. By the above fact δ is not empty. If δ contains more than one element, then we get a contradiction with the fact that there are no proper subuniverses.

Then ρ is a strongly rich relation. By Theorem 18, there exist an Abelian group $(A; +)$ and bijective mappings $\phi_1, \phi_2, \phi_3, \phi_4 \colon A \to A$ such that

$$\rho = \{(a_1, a_2, b_1, b_2) \mid \phi_1(a_1) + \phi_2(a_2) + \phi_3(b_1) + \phi_4(b_2) = 0\}.$$

We know that $(a, a, b, b) \in \rho$ for any $a, b \in A$, $\rho(x_1, x_2, y_1, y_2) = \rho(y_1, y_2, x_1, x_2)$. Then without loss of generality we can assume that $\phi_1(x) = \phi_3(x) = x$, $\phi_2(x) = \phi_4(x) = -x$.

Since w is a special WNU, it follows from Lemma 19 that w on A is defined by $x_1 + \ldots + x_m$. Therefore, the relation $\zeta \subseteq A \times A \times A$ defined by $\zeta = \{(b_1, b_2, b_3) \mid b_1 - b_2 + b_3 = 0\}$ is preserved by w. If $(A; +)$ is not simple, then there exists a subuniverse $A' \subsetneq A$ contradicting our assumption. Therefore, $(A; +)$ is a simple Abelian group. ∎

Corollary 20.1. *Suppose $\sigma \subseteq A^2$ is an irreducible congruence, $\rho(x_1, x_2, y_1, y_2)$ is a bridge from σ to σ such that $\rho(x, x, y, y)$ defines a full relation. Then there exists a prime number p and a relation $\zeta \subseteq A \times A \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \zeta \Leftrightarrow (x_1, x_2) \in \sigma$ and $\mathrm{pr}_{1,2}\, \zeta = \sigma^*$.*

B. Existence of a bridge

In this subsection we explain how we to get a bridge from a rectangular relation and join bridges appeared in the instance together.

Lemma 21. *Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a subdirect relation, the first and the last variables of ρ are rectangular, there exist $(b_1, a_2, \ldots, a_n), (a_1, \ldots, a_{n-1}, b_n) \in \rho$ such that $(a_1, a_2, \ldots, a_n) \notin \rho$. Then there exists a bridge δ from $\mathrm{Con}(\rho, 1)$ to $\mathrm{Con}(\rho, n)$ such that $\delta(x, x, y, y)$ is equal to the projection of ρ onto the first and the last variables.*

Proof: The required bridge can be defined by

$$\delta(x_1, x_2, y_1, y_2) = \exists z_2 \ldots \exists z_{n-1}\ \rho(x_1, z_2, \ldots, z_{n-1}, y_1) \wedge \rho(x_2, z_2, \ldots, z_{n-1}, y_2).$$

∎

Theorem 22. *Suppose Θ is a cycle-consistent connected formula such that every constraint relation is a critical rectangular relation. Then for every constraints C, C' with a common variable x there exists a bridge δ from $\mathrm{Con}(C, x)$ to $\mathrm{Con}(C', x)$ such that $\delta(x, x, y, y)$ contains the relation $\mathrm{LinkedCon}(\Theta, x)$.*

Proof: Since C and C' are connected, there exists a path $z_0 C_1 z_1 C_2 z_2 \ldots C_{t-1} z_{t-1} C_t z_t$, where $z_0 = z_t = x$, $C_1 = C$, $C_t = C'$, and C_i and C_{i+1} are adjacent in z_i for every i.

By Lemma 17, every relation defined by $\mathrm{Con}(C_0, x_0)$ for some C_0 and x_0 is an irreducible congruence. Suppose σ_i is a reflexive bridge from $\mathrm{Con}(C_i, z_i)$ to $\mathrm{Con}(C_{i+1}, z_i)$, δ_i is a bridge from $\mathrm{Con}(C_i, z_{i-1})$ to $\mathrm{Con}(C_i, z_i)$ from Lemma 21 for every i. Then we join all bridges together and define a new bridge $\delta(u_0, u_0', v_t, v_t')$ by

$$\exists u_1 \exists u_1' \exists v_1 \exists v_1' \ldots \exists u_{t-1} \exists u_{t-1}' \exists v_{t-1} \exists v_{t-1}' \delta_1(u_0, u_0', v_1, v_1') \wedge \bigwedge_{i=1}^{t-1} (\sigma_i(v_i, v_i', u_i, u_i') \wedge \delta_{i+1}(u_i, u_i', v_{i+1}, v_{i+1}')).$$

Since Θ is cycle-consistent, δ is a reflexive bridge from $\mathrm{Con}(C, x)$ to $\mathrm{Con}(C', x)$. Thus we proved that any two constraints with a common variable are adjacent.

It is not hard to show that there exists a path in Θ starting and ending at x that connects any pair of elements $(a, b) \in \mathrm{LinkedCon}(\Theta, x)$. Since every pair of constraints with common variable are adjacent, we can assume that the above path $z_0 C_1 z_1 C_2 z_2 \ldots C_{t-1} z_{t-1} C_t z_t$ satisfies this property. Then it is easy to check that $\delta(x, x, y, y)$ contains $\mathrm{LinkedCon}(\Theta, x)$. ∎

C. Three main statements

In this subsection we prove that all constraints in a crucial instance have the parallelogram property, show that we can always find a linked connected component with required properties, prove that we cannot loose the only solution while applying a minimal nonlinear reduction.

We prove theorems of this subsection simultaneously by the induction on the size of the reductions (domain sets). First, we need to introduce an order on the reductions. Suppose we have two domain sets $D^{(\top)}$ and $D^{(\perp)}$. We say that $D^{(\perp)} \leq D^{(\top)}$ if for every $D_y^{(\perp)}$ one of the following conditions hold

1) there exists a variable x such that $D_y^{(\perp)} = D_x^{(\top)}$.
2) there exists a variable x such that $D_y^{(\perp)} \subsetneq D_x^{(\top)}$; there does not exist a variable z such that $D_z^{(\perp)} = D_x^{(\top)}$.

We say that $D^{(\perp)} < D^{(\top)}$ if $D^{(\perp)} \leq D^{(\top)}$ and $D^{(\top)} \not\leq D^{(\perp)}$. It is not hard to see that the relation $\leq$ is transitive and there does not exist an infinite descending chain of reductions.

Let $D^{(\perp)}$ be a domain set. Assume that Theorems 24 and 25 hold if $D^{(1)} < D^{(\perp)}$, and Theorem 23 holds if $D^{(s)} < D^{(\perp)}$. We omit the proof of Theorem 24 (see [18]) and prove Theorem 25 for $D^{(1)} = D^{(\perp)}$, and Theorem 23 for $D^{(s)} = D^{(\perp)}$.

Theorem 23. *Suppose $D^{(0)}, \ldots, D^{(s)}$ is a minimal strategy for a cycle-consistent irreducible CSP instance Θ, the constraint $\rho(x_1, \ldots, x_n)$ is crucial in $D^{(s)}$. Then ρ is a critical relation with the parallelogram property.*

Proof: Since $\rho(x_1, \dots, x_n)$ is crucial, ρ is a critical relation. Let Θ' be obtained from Θ by replacement of $\rho(x_1, \dots, x_n)$ by all weaker constraints.

Assume that $|D_x^{(s)}| = 1$ for every variable x. Since the reduction $D^{(s)}$ is 1-consistent, we get a solution, which contradicts the fact that Θ has no solutions in $D^{(s)}$.

If we have a binary absorption, or a center, or a proper PC subuniverse on some domain $D_x^{(s)}$, then by Theorem 15 there exists a minimal nonlinear reduction $D^{(s+1)}$ for Θ. By Lemma 14, $\Theta'^{(s)}$ is cycle-consistent and irreducible. Hence, by Theorem 25 Θ' has a solution in $D^{(s+1)}$. Hence, $\rho(x_1, \dots, x_n)$ is crucial in $D^{(s+1)}$. By the inductive assumption ρ has parallelogram property.

It remains to consider the case when $\mathrm{ConLin}(D_x^{(s)})$ is proper for every x such that $|D_x^{(s)}| > 1$. Let α be a solution of Θ' in $D^{(s)}$. Let the projection of α onto the variables $x_1, \dots, x_n$ be $(a_1, \dots, a_n)$.

Assume that ρ does not have the parallelogram property. Without loss of generality we can assume that there exist $c_1, \dots, c_n$ and $d_1, \dots, d_n$ such that

$$\begin{aligned}(c_1, \dots, c_k, c_{k+1}, \dots, c_n) &\notin \rho,\\ (c_1, \dots, c_k, d_{k+1}, \dots, d_n) &\in \rho,\\ (d_1, \dots, d_k, c_{k+1}, \dots, c_n) &\in \rho,\\ (d_1, \dots, d_k, d_{k+1}, \dots, d_n) &\in \rho.\end{aligned}$$

Put

$$\rho'(x_1, \dots, x_n) = \exists y_1 \dots \exists y_n \; \rho(x_1, \dots, x_k, y_{k+1}, \dots, y_n) \wedge \rho(y_1, \dots, y_k, x_{k+1}, \dots, x_n) \wedge \rho(y_1, \dots, y_k, y_{k+1}, \dots, y_n).$$

Obviously, $\rho \subsetneq \rho'$ and $\rho' \in \Gamma$, therefore $(a_1, \dots, a_n) \in \rho'$. Hence, there exist $b_1, \dots, b_n$ such that

$$\begin{aligned}(a_1, \dots, a_k, b_{k+1}, \dots, b_n) &\in \rho,\\ (b_1, \dots, b_k, a_{k+1}, \dots, a_n) &\in \rho,\\ (b_1, \dots, b_k, b_{k+1}, \dots, b_n) &\in \rho.\end{aligned}$$

By Lemma 16, there exists a tuple $(e_1, \dots, e_n) \in \rho$ such that $(a_i, e_i) \in \mathrm{ConLin}(D_{x_i}^{(s)})$ for every i. It is easy to see that $\Theta^{(s)}$ factorized by $\mathrm{ConLin}(D_x^{(s)})$ for every x has a solution corresponding to α. By Lemma 12, the minimal linear reduction containing this solution is 1-consistent. We denote this reduction by $D^{(s+1)}$. Since Θ' has a solution in $D^{(s+1)}$, $\rho(x_1, \dots, x_n)$ is crucial in $D^{(s+1)}$. We get a longer minimal strategy with smaller $D^{(s+1)}$, hence by the inductive assumption the relation ρ is a critical relation with the parallelogram property. ■

Theorem 24. *Suppose $D^{(1)}$ is a proper minimal 1-consistent reduction of a cycle-consistent irreducible CSP instance Θ, Θ is linked and crucial in $D^{(1)}$. Then there exists an instance $\Theta' \in \mathrm{Expanded}(\Theta)$ that is crucial in $D^{(1)}$ and contains a linked connected component such that it has no solutions in $D^{(1)}$ or its solution set is not subdirect.*

Theorem 25. *Suppose $D^{(1)}$ is a minimal 1-consistent nonlinear reduction of a cycle-consistent irreducible CSP instance Θ. If Θ has a solution then it has a solution in $D^{(1)}$.*

Proof: Assume the converse. Suppose $D^{(1)}$ is a PC reduction. Then we replace constraints of Θ by all weaker constraints while there exists a 1-consistent minimal PC reduction such that the instance has no solutions in it. Thus, we can assume that if we replace any constraint of Θ by all weaker constraints then we get an instance with a solution in every 1-consistent minimal PC reduction.

By Remark 1, we weaken the instance to get an instance that is crucial in $D^{(1)}$. If the obtained instance is not linked, then we consider a linked component Υ having a nonempty intersection with $D^{(1)}$ and apply the inductive assumption (see details in [18]). Therefore, by Theorem 23, every constraint in the obtained instance has the parallelogram property. By Theorem 24, there exists an instance $\Theta' \in \mathrm{Expanded}(\Theta)$ that is crucial in $D^{(1)}$ and contains a linked connected component Ω.

Choose a variable x appearing in a constraint $C \in \Omega$. By Lemma 17, $\mathrm{Con}(C, x)$ is irreducible. By Theorem 22, there exists a bridge δ from $\mathrm{Con}(C, x)$ to $\mathrm{Con}(C, x)$ such that $\delta(x, x, y, y)$ is a full relation. By Corollary 20.1, there exists a relation $\zeta \subseteq D_x \times D_x \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \zeta \Leftrightarrow (x_1, x_2) \in \mathrm{Con}(C, x)$ and $\mathrm{pr}_{1,2}(\zeta) = \mathrm{Con}(C, x)^*$. Let us replace the variable x of C in Θ' by x' and add the constraint $\zeta(x, x', z)$. The obtained instance we denote by Θ''. By the assumption, Θ'' has a solution with $z = 0$, and a solution in $D^{(1)}$ with $z \neq 0$.

If $D^{(1)}$ is an absorbing or central reduction, then by Corollaries 9, 10 the restriction of all variable of Θ'' but z to $D^{(1)}$ implies the corresponding restriction of the variable z. This contradicts the fact that the domain of z is $\mathbb{Z}_p$.

It remains to consider the case when $D^{(1)}$ is a PC reduction. Combining our assumption for the PC case and Theorem 15, we can show that for every variable y and a PC subuniverse U of D_y the instance Θ'' has a solution with $y \in U$. Hence, by Corollary 11, the restriction of Θ'' to $D^{(1)}$ implies the corresponding restriction of z, which contradicts the fact that the domain of z is $\mathbb{Z}_p$. ■

D. Proof of Theorems from Section IV

Proof of Theorem 6 and Theorem 7. By Theorem 15, there exists a smaller minimal reduction. By Theorem 25, there exists a solution in this reduction.

Proof of Theorem 8. Assume the converse. We denote the reduction $(D_1', \dots, D_n')$ by $D^{(1)}$. By Theorem 23, every constraint in Θ has the parallelogram property. By Theorem 24, there exists an instance $\Theta' \in \mathrm{Expanded}(\Theta)$ that is crucial in $D^{(1)}$ and contains a linked connected component Ω such that the solution set of Ω is not subdirect or $\Omega^{(1)}$ has no solutions. By condition 3), if the solution set of Ω is not subdirect then Ω contains a constraint relation from Θ. Since Θ is crucial in $D^{(1)}$, if $\Omega^{(1)}$ has no solutions then Ω contains a constraint relation from Θ. Let $((x_{i_1}, \ldots, x_{i_s}), \rho) \in \Omega$ be a constraint such that ρ is a constraint relation from Θ.

By Lemma 17, $\mathrm{Con}(\rho, 1)$ is an irreducible congruence. By Theorem 22, there exists a bridge δ from $\mathrm{Con}(\rho, 1)$ to $\mathrm{Con}(\rho, 1)$ such that $\delta(x, x, y, y)$ is a full relation. By Corollary 20.1, there exists a relation $\xi \subseteq D_{i_1} \times D_{i_1} \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \xi \Leftrightarrow (x_1, x_2) \in \mathrm{Con}(\rho, 1)$ and $\mathrm{pr}_{1,2}(\xi) = \mathrm{Con}(\rho, 1)^*$.

Put $\zeta(x_{i_1}, \ldots, x_{i_s}, z) = \exists x'_{i_1} \quad \rho(x'_{i_1}, x_{i_2}, \ldots, x_{i_s}) \wedge \xi(x_{i_1}, x'_{i_1}, z)$.

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