

Popular Conjectures as a Barrier for Dynamic Planar Graph Algorithms

Amir Abboud
 Department of Computer Science
 Stanford University
 abboud@cs.stanford.edu

Søren Dahlgaard
 Department of Computer Science
 University of Copenhagen
 soerend@di.ku.dk

Abstract—The *dynamic shortest paths* problem on planar graphs asks us to preprocess a planar graph G such that we may support insertions and deletions of edges in G as well as distance queries between any two nodes u, v subject to the constraint that the graph remains planar at all times. This problem has been extensively studied in both the theory and experimental communities over the past decades. The best known algorithm performs queries and updates in $\tilde{O}(n^{2/3})$ time, based on ideas of a seminal paper by Fakcharoenphol and Rao [FOCS’01]. A $(1 + \varepsilon)$ -approximation algorithm of Abraham *et al.* [STOC’12] performs updates and queries in $\tilde{O}(\sqrt{n})$ time. An algorithm with a more practical $O(\text{poly log } n)$ runtime would be a major breakthrough. However, such runtimes are only known for a $(1 + \varepsilon)$ -approximation in a model where only restricted weight updates are allowed due to Abraham *et al.* [SODA’16], or for easier problems like connectivity.

In this paper, we follow a recent and very active line of work on showing lower bounds for polynomial time problems based on popular conjectures, obtaining the first such results for natural problems in *planar graphs*. Such results were previously out of reach due to the highly non-planar nature of known reductions and the impossibility of “planarizing gadgets”. We introduce a new framework which is inspired by techniques from the literatures on distance labelling schemes and on parameterized complexity.

Using our framework, we show that no algorithm for dynamic shortest paths or maximum weight bipartite matching in planar graphs can support both updates and queries in amortized $O(n^{\frac{1}{2}-\varepsilon})$ time, for any $\varepsilon > 0$, unless the classical all-pairs-shortest-paths problem can be solved in truly subcubic time, which is widely believed to be impossible. We extend these results to obtain strong lower bounds for other related problems as well as for possible trade-offs between query and update time. Interestingly, our lower bounds hold even in very restrictive models where only weight updates are allowed.

Keywords—planar graphs; hardness in \mathbb{P} ; conditional lower bounds; dynamic distance oracles; all pairs shortest paths;

I. INTRODUCTION

The *dynamic shortest paths* problem on *planar graphs* is to preprocess a planar graph G , e.g. parts of the national road network, so that we are able to efficiently support the following two operations:

- At any point, we might *insert* or *remove* an edge (u, v) in G , e.g. in case a road gets congested due to an

accident. Such updates are subjected to the constraint that the planarity of the graph is not violated. We may also consider another natural variant in which we are only allowed to update the weights of existing edges.

- We want to be able to quickly answer *queries* that ask for the length of the shortest path between two given nodes u and v , in the most current graph G .

This is a problem that is very relevant to GPS navigation and gets solved many times every day on very large graphs. It is thus a very important question in both theory and practice whether there exists data structures that can perform updates and (especially) queries on graphs with n nodes in polylogarithmic or even $n^{o(1)}$ time.

Shortest paths problems on planar graphs provide an ideal combination of mathematical simplicity and elegance with faithful modeling of realistic applications of major industrial interest. The literature on the topic is too massive for us to survey in this paper: the current draft of the book “Optimization Problems in Planar Graphs” by Klein and Mozes [1] dedicates *four chapters* to the algorithmic techniques for shortest paths by the theory community. While near-optimal algorithms are known for most variants of shortest paths on static planar graphs, the *dynamic* setting has proven much more challenging.

Since an s, t -shortest path in a planar graph can be found in near-linear time (linear time for non-negative weights) [2], [3], there is a naïve algorithm for the dynamic problem that spends $\tilde{O}(n)$ time on queries. After progress on other related problems on dynamic planar graphs [2], [4]–[9], the first sublinear bound was obtained in the seminal paper of Fakcharoenphol and Rao [3], which introduced new techniques that led to major results for other problems like Max Flow (even on static graphs) [10], [11]. The amortized time per operation was $O(n^{2/3} \log^{7/3} n)$ and $O(n^{4/5} \log^{13/5} n)$ if negative edges are allowed, and follow up works of Klein [12], Italiano *et al.* [13], and Kaplan *et al.* [14] reduced the runtime to $O(n^{2/3} \log^{5/3} n)$ for both queries and updates (even allowing negative weights), and most recently, Gawrychowski and Karczmarz [15] reduced it further to $O(n^{2/3} \frac{\log^{5/3} n}{\log^{4/3} \log n})$. In fact these algorithms give a trade-off

on the update and query time of $\tilde{O}(n/\sqrt{r})$ and $\tilde{O}(r)$, for all r . The problem has also been extensively studied from an engineering viewpoint on real-world transportation networks (see [16] for a survey). State of the art algorithms [17]–[21] are able to exploit structure of road networks and process updates to networks with tens of millions of nodes in milliseconds.

In a recent SODA’16 paper, Abraham *et al.* [22] study worst case bounds under a restricted but realistic model of dynamic updates in which a base graph G is given and one is allowed to perform only weight updates subject to the following constraint: For any updated graph G' it must hold that $d_G(u, v) \leq d_{G'}(u, v) \leq M \cdot d_G(u, v)$ for all u, v and some parameter M . (Note that this will hold if, for example, the weight of each edge only changes to within a factor of M .) In this model, the authors obtain a $(1 + \varepsilon)$ -approximation algorithm that maintains updates in $O(\text{polylog } n \cdot M^4/\varepsilon^3)$ time. Without this restriction, the best known $(1 + \varepsilon)$ -approximation algorithms use $\tilde{O}(\sqrt{n})$ updates [9], [23]. Thus, when M is small, this model allows for a major improvement over the above results which require *polynomial time* updates. But is it enough to allow for *exact* algorithms with *subpolynomial* updates? Such a result would explain the impressive experimental performance of state of the art algorithms.

On the negative side, Eppstein showed that $\Omega(\log n / \log \log n)$ time is required in the cell probe model [24] for planar connectivity (and therefore also shortest path). However, an unconditional $\log^{\omega(1)} n$ lower bound is far beyond the scope of current techniques (see [25]). In recent years, much stronger lower bounds were obtained for dynamic problems under certain popular conjectures [26]–[32]. For example, Roditty and Zwick [26] proved an $n^{2-o(1)}$ lower bound for dynamic single source shortest paths in general graphs under the following conjecture.

Conjecture 1 (APSP Conjecture). *There exists no algorithm for solving the all pairs shortest paths (APSP) problem in general weighted (static) graphs in time $O(n^{3-\varepsilon})$ for any $\varepsilon > 0$.*

However, the reductions used in these results produce graphs that are fundamentally non-planar, such as dense graphs on three layers, and popular approaches for making them planar, e.g. by replacing each edge crossing with a small “planarizing gadget”, are provably impossible (this was recently shown for matching [33] and is easier to show for problems like reachability and shortest paths). Due to this and other challenges no (conditional) polynomial lower bounds were known for any natural problem on (static or dynamic) planar graphs.

On a more general note, an important direction for future research on the fine-grained complexity of polynomial time problems (a.k.a. Hardness in P) is to understand the com-

plexity of fundamental problems on restricted but realistic classes of inputs. A recent result along these lines is the observation that the $n^{2-o(1)}$ lower bound for computing the diameter of a sparse graph [34] holds even when the treewidth of the graph is $O(\log n)$ [35]. In this paper, we take a substantial step in this direction, proving the first strong (conditional) lower bounds for natural problems on planar graphs.

A. Our Results

We present the first conditional lower bounds for natural problems on planar graphs using a new framework based on several ideas for conditional lower bounds on dynamic graphs combined with ideas from parameterized complexity [36], [37] and labeling schemes [38]. We believe that this framework is of general interest and might lead to more interesting results for planar graphs. Our framework shows an interesting connection between dynamic problems and distance labeling and also slightly improves the result of [38] providing a tight lower bound for distance labeling in weighted planar graphs (this is discussed in Section I-B).

Our first result is a conditional *polynomial* lower bound for dynamic shortest paths on planar graphs. Like several recent results [31], [32], [39]–[42], our lower bound is based on the APSP conjecture. Perhaps the best argument for this conjecture is the fact that it has endured decades of extensive algorithmic attacks. Moreover, due to the known *subcubic equivalences* [39], [41], [42], the conjecture is false *if and only if* several other fundamental graph and matrix problems can be solved substantially faster.

Theorem 1. *No algorithm can solve the dynamic APSP problem in planar graphs on N nodes with amortized query time $q(N)$ and update time $u(N)$ such that $q(N) \cdot u(N) = O(N^{1-\varepsilon})$ for any $\varepsilon > 0$ unless Conjecture 1 is false. This holds even if we only allow weight updates to G .*

Thus, under the APSP conjecture, there is no hope for a very efficient dynamic shortest paths algorithm on planar graphs with provable guarantees. We show that an algorithm achieving $O(n^{1/2-\varepsilon})$ time for both updates and queries is unlikely, implying that the current upper bounds achieving $\tilde{O}(n^{2/3})$ time are not too far from being conditionally optimal. Furthermore, our result implies that any algorithm with subpolynomial query time must have *linear* update time (and the other way around). Thus, the naïve algorithm of simply computing the entire shortest path every time a query is made is (conditionally) optimal if we want $n^{o(1)}$ update time.

An important property of Theorem 1 is that our reduction does not even violate planarity with respect to a *fixed embedding*. Thus, we give lower bounds even for *plane graph* problems, which in many cases allow for improved upper bounds over flexible planar graphs (e.g. for reachability [43], [44]). Moreover, our graphs are *grid graphs* which are

subgraphs of the infinite grid, a special and highly structured subclass of planar graphs. Finally, as stated in Theorem 1 our lower bound holds even for the edge weight update model of Abraham *et al.* [22], where each edge only ever changes its weight to within a factor of $M > 1$. While they obtain fast $\text{polylog}(n)$ time $(1+\varepsilon)$ -approximation algorithm in this model, we show that an exact answer with the same query time likely requires *linear* update time and that an algorithm with $O(n^{1/2-\varepsilon})$ runtime for both is highly unlikely. Thus, further theoretical restrictions need to be added in order to explain the impressive performance on real road networks.

We also extend Theorem 1 to the case in which we only need to maintain one s, t distance (the s, t -shortest path problem) showing that the product of $q(N)$ and $u(N)$ has to be $N^{1-o(1)}$ if $q(N) \geq u(N)$ unless Conjecture 1 is false identically to the bounds for maximum weight matching discussed below in Theorem 3. While this problem is equivalent to the APSP version in general (as we may connect s and t to any two nodes u, v we wish to know the distance between) this may violate planarity and especially a fixed embedding. We show that this problem exhibits similar trade-offs under Conjecture 1 even if we are only allowed to update weights. Finally, we note that in the case of directed planar graphs allowing negative edge weights our techniques can be extended to show the same hardness result for any approximation under Conjecture 1.

Next, we seek a lower bound for the unweighted version of the problem, which arguably, is of more fundamental interest. Typically, a conditional lower bound under the APSP conjecture for a weighted problem can be modified into a lower bound for its unweighted version under the Boolean Matrix Multiplication (BMM) Conjecture [26], [28], [39], [41], [45]. While for combinatorial algorithms the complexity of BMM is conjectured to be cubic, it is known that using algebraic techniques there is an $O(n^\omega)$ algorithm, where $\omega < 2.373$ [46], [47]. When reducing to dynamic problems, however, lower bounds under BMM are often under a certain online version of BMM for which, Henzinger *et al.* [30] conjecture that there is no truly subcubic algorithms, even using algebraic techniques. This *Online Matrix Vector Multiplication* (OMv) Conjecture is stated formally in Section V.

The OMv conjecture implies strong lower bounds for many dynamic problems on general graphs [30], via extremely simple reductions [28], [30]. Our next result is a significantly more involved reduction from OMv to dynamic shortest paths on planar graphs, giving unweighted versions of the theorems above. The lower bounds are slightly weaker but they still rule out algorithms with subpolynomial update and query times, even in grid graphs. We remark that all lower bounds under the APSP conjecture in this paper, such as Theorem 1, also hold under OMv.

Theorem 2. *No algorithm can solve the dynamic APSP*

problem in unit weight planar graphs on N nodes with amortized query time $q(N)$ and update time $u(N)$ such that $\max(q(N)^2 \cdot u(N), q(N) \cdot u(N)^2) = O(N^{1-\varepsilon})$ for any $\varepsilon > 0$ unless the OMv conjecture of [30] is false. This holds even if we only allow weight updates.

For instance, Theorem 2 shows that no algorithm is likely to have $O(n^{\frac{1}{3}-\varepsilon})$ amortized time for both queries and updates. It also shows that if we want to have $n^{o(1)}$ for one we likely need $n^{\frac{1}{2}-o(1)}$ time for the other.

Combined with previous results, our theorems reveal a mysterious phenomenon: there are two contradicting separations between planar graphs and small treewidth graphs, in terms of the time complexity of dynamic problems related to shortest paths (under popular conjectures). To illustrate these separations, consider the dynamic s, t -shortest path problem and the dynamic approximate diameter problem. For s, t -shortest path, planar graphs are much harder, they require $n^{1/3-o(1)}$ update or query time by Theorems 5 and 2 (under OMv), while on small (polylog) treewidth graphs there is an algorithm achieving polylog updates and queries [22]. On the other hand, for approximate diameter, planar graphs are provably *easier* unless the Strong Exponential Time Hypothesis (SETH) is false. A naive algorithm, that runs the known $\tilde{O}(n)$ time static algorithm for $(1+\varepsilon)$ approximate diameter on planar graphs after each update [48], shatters an $n^{2-o(1)}$ SETH-based lower bound for a $(4/3 - \delta)$ approximation for diameter on graphs with treewidth $O(\log n)$ [28]¹.

We demonstrate the potential of our framework to yield further strong lower bounds for important problems in planar graphs by proving such a result for another well-studied problem in the graph theory literature, namely Maximum Weight Matching.

Maintaining a maximum matching in general dynamic graphs is a difficult task: the best known algorithm by Sankowski [49] has an $O(n^{1.495})$ amortized update time, and it is better than the simple $O(m)$ algorithm (that looks for an augmenting path after every update) only in dense graphs. Recent results show barriers for much faster algorithms via conjectures like OMv and 3-SUM [28]–[30], [32]. To our knowledge, this $O(m)$ update time is the best known for planar graphs and no lower bound is known. Meanwhile, there has been tremendous progress on approximation algorithms [50]–[59], both on general and planar graphs, as well as for the natural *Maximum Weight Matching* (see the references in [60] for the history of this variant). Planar graphs have proven easier to work with in this context: the state of the art deterministic algorithm for maintaining a $(1+\varepsilon)$ -maximum matching in general graphs has $O(\sqrt{m})$ update time [61],

¹This lower bound follows from observing that the reduction from CNF-SAT to dynamic diameter [28] produces graphs with logarithmic treewidth. For more details on an analogous observation w.r.t. the lower bound for diameter in static graphs, see [35].

while in planar graphs the bound is $O(1)$ [58].

We show a strong *polynomial* lower bound for Max Weight Matching on planar graphs, that holds even for bipartite graphs with a fixed embedding into the plane and even in grid graphs. The lower bound is similar to Theorem 1 and shows a trade-off between query and update time.

Theorem 3. *No algorithm can solve the dynamic maximum weight matching problem in bipartite planar graphs on N nodes with amortized update time $u(N)$ and query time $q(N)$ such that $\max(q(N), u(N)) = O(N^{\frac{1}{2}-\varepsilon})$ for any $\varepsilon > 0$ unless Conjecture 1 is false. Furthermore, if $q(N) \geq u(N)$ the algorithm cannot have $q(N) \cdot u(N) = O(N^{1-\varepsilon})$. This holds even if the planar embedding of G never changes.*

Finally, we use our framework to show lower bounds for various other problems, like dynamic girth and diameter. We also argue that our bounds can be turned into worst-case bounds for incremental and decremental versions of the same problems.

B. Techniques and relations to distance labeling

To prove the results mentioned above we introduce a new framework for reductions to optimization problems on planar graphs. As mentioned, we combine ideas from previous lower bound proofs for dynamic graph problems with an approach inspired by the framework of Marx for hardness of parameterized geometric problem (via the Grid Tiling problem) [36], [37] and a graph construction from the research on labelling schemes by Gavaille *et al.* [38].

Gavaille *et al.* [38] used a family of grid-like graphs to prove an (unconditional) lower bound of $\Omega(\sqrt{n})$ on the label size of distance labeling in weighted planar graphs along with a $O(\sqrt{n} \log n)$ upper bound. (A full discussion of distance labeling schemes is outside the scope of this paper. For details on this we refer to [38], [62], [63]). In this paper we generalize their family of graphs to a family of grid graphs capable of representing general matrices with weights in $[\text{poly}(n)]$ via shortest paths distances. Using our construction with the framework of [38], we obtain a *tight* $\Omega(\sqrt{n} \log n)$ lower bound on the size of distance labeling in weighted planar graphs (and even grid graphs).

Our main approach works by reducing from the $(\min, +)$ -Matrix-Multiplication problem which is known to be equivalent to APSP (see [39]): Given two $n \times n$ matrices A, B with entries in $[\text{poly}(n)]$, compute a matrix $C = A \oplus B$ such that $C[i, j] = \min_{k \in [n]} A[i, k] + B[k, j]$. By concatenating grid graphs from the family described above we are able to represent one of the matrices in the product and we can then simulate the multiplication process via updates and shortest paths queries.

In a certain intuitive sense, our connection between dynamic algorithms and labeling schemes is the reverse direction of the one shown by Abraham *et al.* [23] to obtain their $\tilde{O}(\sqrt{N})$ update time $(1 + \varepsilon)$ -approximation algorithm for

dynamic APSP. Their algorithm utilizes a clever upper bound for the so-called *forbidden set distance labeling* problem, while our lower bound constructions have a clever lower bound for labeling schemes embedded in them.

II. A GRID CONSTRUCTION

In order to reduce to problems on planar graphs we will need a planar construction, which is able to capture the complications of problems like OMv and APSP. To do this we will employ a grid construction based on the one used in [38] to prove lower bounds on distance labeling for planar graphs. Our construction takes a matrix as input and produces a grid graph representing that matrix. We first present a boolean version similar to the one from [38] and then modify it to obtain a version taking matrices with integer entries as input. This modified matrix also immediately leads to a tight $\Omega(\sqrt{n} \log n)$ lower bound for distance labeling in planar graphs with weights in $[\text{poly}(n)]$ when combined with the framework of [38].

Definition 1. Let M be a boolean $R \times C$ matrix. We will call the following construction the *grid embedding* of M :

Let G_M be a rectangular grid graph with R rows and C columns. Denote the node at intersection (i, j) by $u_{i,j}$ ($u_{1,1}$ is top-left and $u_{R,C}$ is bottom-right). Add C nodes a_1, \dots, a_C and edges $(u_{1,j}, a_j)$ above G_M . Similarly add the nodes b_1, \dots, b_R and edges $(u_{i,C}, b_i)$ to the right of G_M . Now subdivide each vertical edge adding the node $v_{i,j}$ above $u_{i,j}$, and subdivide each horizontal edge adding the node $w_{i,j}$ to the right of $u_{i,j}$. Finally, for each entry of M such that $M_{i,j} = 1$ add the node $x_{i,j}$ and edges $(v_{i,j}, x_{i,j})$ and $(w_{i,j}, x_{i,j})$ to the graph.

The weights of G_M are as follows: Each edge $(u_{i,j}, v_{i+1,j})$ and $(a_j, v_{1,j})$ has weight $2j - 1$. Each edge $(w_{i,j}, u_{i+1,j})$ and $(w_{i,C}, b_i)$ has weight $2R - 2$. The edge $(u_{i,j}, w_{i,j})$ has weight 2. All remaining edges have weight 1.

We will call the two-edge path $v_{i,j} \rightarrow x_{i,j} \rightarrow w_{i,j}$ a *shortcut* from $v_{i,j}$ to $w_{i,j}$ as it has length 1 less than the path $v_{i,j} \rightarrow u_{i,j} \rightarrow w_{i,j}$. Clearly, the grid embedding of a $R \times C$ matrix has $O(RC)$ nodes. It is also easy to see that such a grid embedding is a subgraph of a $2R + 1 \times 2C + 1$ rectangular grid. The construction of Definition 1 for a 3×3 matrix can be seen in Figure 1.

Proposition 1. *Let M be a boolean $R \times C$ matrix and let G_M be its grid embedding as defined in Definition 1. Then for any $1 \leq i \leq R, 1 \leq j \leq C$ and $i < k \leq R$ the shortest path distance from $u_{i,j}$ to b_k is exactly*

$$(k - i) \cdot 2j + 2R \cdot (C - j + 1)$$

if $M_{k,j} = 0$ and

$$(k - i) \cdot 2j + 2R \cdot (C - j + 1) - 1$$

required for updates. We may try to assign edges and weights differently, but such approaches run into similar issues.

Observe that the graph created has $N = O(n^2)$ nodes. Thus, if we were able to perform only $O(n^2)$ total queries and updates the result of Theorem 4 would follow. \diamond

In order to circumvent this dependence on j when assigning weights to the edges (b_k, t) we instead replace t by another grid whose purpose is to “normalize” the distance for each a_j . By doing this we can connect the grids with edges whose weight is independent of j . This step deviates significantly from the construction of [38] and is inspired by the grid tiling framework of Marx [36], [37].

Proof of Theorem 4: We follow the same approach as in Attempt 1, but with a few changes. Define the initial graph G as follows: Let G_B be as before and let G'_B be the grid embedding of B mirrored along the vertical axis with all shortcuts removed. Now for each $1 \leq k \leq n$ add the edge (b_k, b'_k) and define G to be this graph.

Now we perform a phase for each row i of A as follows:

- 1) For each $1 \leq k \leq n$ set the weight of the edge (b_k, b'_k) to be $X^2 \cdot (2(n+1)(n-k)) + A_{i,k}$.
- 2) For each j query the distance between a_j and a'_{n-j+1} .

An example of this construction for $n = 3$ can be seen in Figure 2.

From the query between nodes a_j and a'_{n-j+1} above during phase i we can determine the entry $C_{i,j}$ of the output matrix. To see this, consider the distance from a_j to a'_{n-j+1} at the time of query. This path has to go via some edge (b_k, b'_k) . From Corollary 2 we know that this distance is exactly

$$\begin{aligned} d_G(a_j, a'_{n-j+1}) &= d(a_j, b_k) + w(b_k, b'_k) + d(b'_k, a'_{n-j+1}) \\ &= X^2 \cdot 2n(n+1) + X^2 \cdot 2k(n+1) \\ &\quad + X^2 \cdot 2(n+1)(n-k) + B_{k,j} + A_{i,k} - X^2 \\ &= X^2 \cdot 4n(n+1) - X^2 + B_{k,j} + A_{i,k}. \end{aligned}$$

The crucial property that our construction achieves is that the dominant term of this expression is independent of k . Thus, the shortest path will choose to go through the edge (b_k, b'_k) that minimizes $B_{k,j} + A_{i,k}$, implicitly giving us $C_{i,j}$. Subtracting $X^2 \cdot (4n(n+1) - 1)$ from the queried distance gives exactly the value of $C_{i,j}$ and the algorithm therefore correctly computes C .

Following the analysis from Attempt 1 we have that any algorithm with an amortized running time of $O(N^{\frac{1}{2}-\varepsilon})$ for both updates and queries contradicts Conjecture 1. \blacksquare

A. Trade-offs

Theorem 4 above shows that no algorithm can perform both updates and queries in amortized time $O(N^{\frac{1}{2}-\varepsilon})$ unless Conjecture 1 is false. We will now show how to generalize these ideas to show Theorem 1.

Proof of Theorem 1: The proof follows the same structure as the proof for Theorem 4, but instead of reducing

from $(\min, +)$ -Matrix-Mult on $n \times n$ matrices we reduce from an unbalanced version.

Let A and B be $n \times n^\beta$ and $n^\beta \times n^\alpha$ matrices respectively for some $0 < \alpha, \beta \leq 1$. It follows from standard sub-cubic reductions [39] that this problem takes $n^{1+\alpha+\beta-o(1)}$ time unless we can solve the $n \times n$ problem (and thus APSP) faster than $n^{3-o(1)}$. We define the initial graph G from B in the same manner as in Theorem 4. We then have a phase for each row i of A as follows:

- 1) For each $1 \leq k \leq n^\beta$ set the weight of the edge (b_k, b'_k) to be $X^2 \cdot (2(n^\alpha + 1)(n^\beta - k)) + A_{i,k}$.
- 2) For each $1 \leq j \leq n^\alpha$ query the distance between a_j and $a'_{n^\alpha-j+1}$.

The entry $C_{i,j}$ is exactly the distance $d_G(a_j, a'_{n^\alpha-j+1})$ from the i th phase minus $X^2 \cdot (4n^\beta(n^\alpha + 1) - 1)$. The correctness of the above reduction follows directly from the proof of Theorem 4 as well as Corollary 2.

Now observe that the graph G from the above reduction has $N = \Theta(n^{\alpha+\beta})$ nodes and we perform a total of $O(n^{1+\alpha})$ queries and $O(n^{1+\beta})$ updates² – that is, at most $O(n)$ updates per row and $O(n)$ queries per column. Any algorithm solving this problem must use total time $n^{1+\alpha+\beta-o(1)}$ time unless Conjecture 1 is false. It follows that either updates must take $n^{\alpha-o(1)}$ amortized time or queries must take $n^{\beta-o(1)}$ amortized time.

Assume now that an algorithm exists such that queries take $O(N^\gamma)$ amortized time for any $0 < \gamma < 1$. We wish to show that this algorithm cannot perform updates in amortized time $O(N^{1-\gamma-\varepsilon})$ for any $\varepsilon > 0$. Pick $\beta = \gamma + \varepsilon/2$ and set $\alpha = 1 - \beta$. We now use the above reduction to create a dynamic graph G with $N = O(n^{\alpha+\beta}) = O(n)$ nodes. Since queries do not take $n^{\beta-o(1)}$ time it follows from the above discussion that updates must take $n^{\alpha-o(1)} = n^{1-\gamma-\varepsilon/2-o(1)}$ time. Since this is polynomially greater than $O(N^{1-\gamma-\varepsilon})$ the claim follows. \blacksquare

IV. HARDNESS OF DYNAMIC MAXIMUM WEIGHT MATCHING IN BIPARTITE PLANAR GRAPHS

In this section we will demonstrate the generality of our reduction framework by showing Theorem 3.

Proof of Theorem 3: We start by showing how to reduce from $(\min, +)$ -Matrix-Mult to minimum weight perfect matching, where the weight of such a matching corresponds to the shortest path distance between a_j and a'_{n-j+1} similar to the proof of Theorem 1. We then describe how to use this reduction further to get a problem instance for maximum weight matching.

Let A, B be an instance to the $(\min, +)$ -Matrix-Mult problem of sizes $n \times n^\beta$ and $n^\beta \times n^\alpha$ respectively. Consider the grid embedding G_B of B . We first replace each node of G_B by two nodes connected by an edge of weight 0. For

²We also perform $O(n^{\alpha+\beta})$ updates to create the initial graph (depending on the model), however we will choose α and β such that this term is dominated.

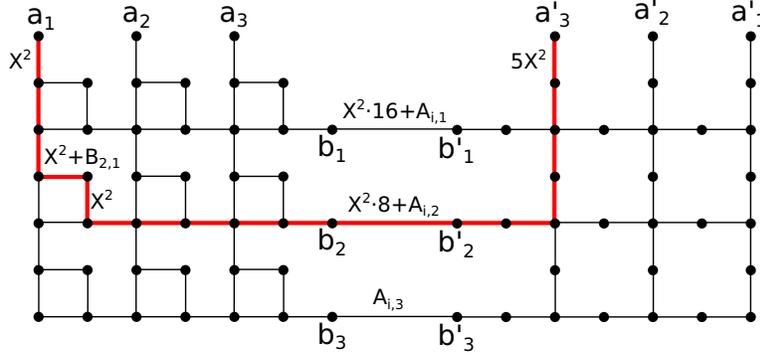


Figure 2. Example of a phase in the graph G in the reduction of Theorem 4. The highlighted path illustrates a shortest path between a_1 and a'_3 as an example of a query. Most edge weights have been omitted for clarity.

a_j , $u_{i,j}$, $x_{i,j}$, and $v_{i,j}$ denote the corresponding nodes with superscript d and u (for “down” and “up”). For b_i and $w_{i,j}$ denote the corresponding nodes with superscript l and r (for “left” and “right”). Now, for each original edge in G_B we replace it as follows keeping its weight:

- $(u_{i,j}, v_{i,j}) \rightarrow (u_{i,j}^u, v_{i,j}^d)$
- $(u_{i,j}, w_{i,j}) \rightarrow (u_{i,j}^d, w_{i,j}^r)$
- $(u_{i,j}, v_{i+1,j}) \rightarrow (u_{i,j}^d, v_{i+1,j}^u)$
- $(u_{i,j}, w_{i,j-1}) \rightarrow (u_{i,j}^d, w_{i,j-1}^r)$
- $(v_{i,j}, x_{i,j}) \rightarrow (v_{i,j}^d, x_{i,j}^u)$
- $(x_{i,j}, w_{i,j}) \rightarrow (x_{i,j}^d, w_{i,j}^r)$
- $(a_j, v_{0,j}) \rightarrow (a_j^d, v_{0,j}^u)$
- $(w_{i,C}, b_i) \rightarrow (w_{i,C}^r, b_i^l)$

This construction is illustrated in Figure 3. We call this modified grid structure \bar{G}_B . Observe that there are no edges between “up” and “left” vertices or between “down” and “right”. It follows that the graph is bipartite and that these two sets of nodes make up the two partitions.

We now replace the grids G_B and G'_B by \bar{G}_B and \bar{G}'_B in the initial graph G from the proof of Theorem 1. The edges (b_k, b'_k) are replaced by (b_k^r, b'_k^l) . We will use the following observation.

Proposition 2. *The graph resulting from joining two grids \bar{G}_B in the way of Figure 2 has a unique perfect matching.*

We now add two additional nodes s and t to the initial graph and perform a phase for each row i of A as follows:

- 1) For each $1 \leq k \leq n^\beta$ set the weight of the edge (b_k^r, b'_k^l) to be $X^2 \cdot (2(n^\alpha + 1)(n^\beta - k)) + A_{i,k}$.
- 2) For each $1 \leq j \leq n^\alpha$ do the following three steps: 1) add the edges (s, a_j^u) and $(t, a_{n^\alpha-j+1}^u)$, 2) query the minimum weight perfect matching, 3) delete the two edges.

Since the edges (s, a_j^u) and $(t, a_{n^\alpha-j+1}^u)$ have to be in any perfect matching this leaves a_j^d and $a_{n^\alpha-j+1}^d$ unmatched. Any perfect matching now has to “connect” these two nodes by a path of original (weight > 0) edges. The weight of a

perfect matching in G then corresponds to the length of a shortest path from a_j to $a'_{n^\alpha-j+1}$ in the graph from the proof of Theorem 1. It follows that we get the same trade-offs for minimum weight perfect matching as for APSP with the exception that the trade-off only holds when $q(N) \geq u(N)$ since we perform $O(1)$ updates for each query.

To show the same result for maximum weight matching we may simply perform the following two changes: 1) pick a sufficiently large integer y and set the weight of each edge to y minus its weight in the above reduction, and 2) when adding the edges (s, a_j^u) and $(t, a_{n^\alpha-j+1}^u)$ assign them weight y^2 such that any maximum weight matching has to include these two edges and will have weight

$$y^2 + \frac{N-4}{2} \cdot y - d_{G^*}(a_j, a'_{n^\alpha-j+1}),$$

where G^* denotes the corresponding graph in the proof of Theorem 1. \blacksquare

V. UNWEIGHTED

In this section we sketch the proof of Theorem 2. Due to space restrictions we have to defer the full proof to the full paper available online at [64].

Instead of reducing from (min, +)-Matrix-Mult we reduce from the online binary vector-matrix-vector multiplication problem described in [30]. Since this is a binary problem we can not use the weighted grid construction of Section II. Instead we use the grid of Definition 1 and subdivide each edge such that the graph is unweighted but has the same distances between grid points. The proof now follows by following essentially the same line of argument as in Section III.

VI. DYNAMIC s, t -SHORTEST PATH AND RELATED PROBLEMS

In Section III we showed a lower bound for the trade-off between query and update time for dynamic APSP in grid

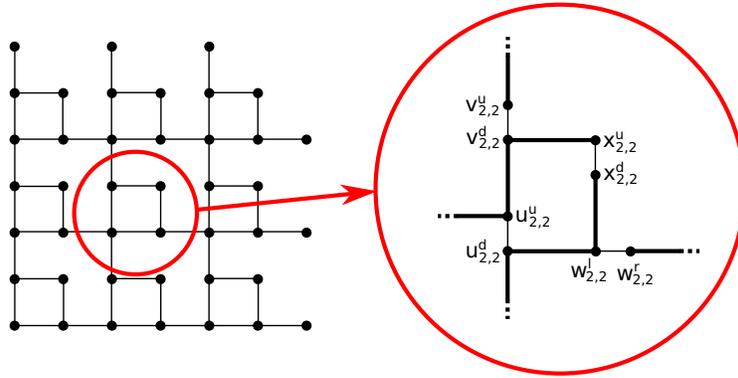


Figure 3. Grid construction for minimum weight perfect matching. Thick edges correspond to original edges and have the same weight as in G_B . Thin edges have weight 0.

graphs conditioned on Conjecture 1. Here we note that we can extend the proof of Theorem 1 to show similar lower bounds for dynamic problems, where the algorithm only needs to maintain a single value such as s, t -shortest path, girth, and diameter. We also note that the above techniques for proving bounds in unweighted graphs also apply to the theorem below. The proof of the following theorem can be found in the full version of this paper.

Theorem 5. *No algorithm can solve the s, t -shortest path, girth (directed), or diameter problems in planar graphs on N nodes with amortized update time $u(N)$ and query time $q(N)$ such that $\max(q(N), u(N)) = O(N^{\frac{1}{2}-\epsilon})$ for any $\epsilon > 0$ unless Conjecture 1 is false. Furthermore, if $q(N) \geq u(N)$ the algorithm cannot have $q(N) \cdot u(N) = O(N^{1-\epsilon})$. This holds even if the planar embedding of G never changes.*

VII. WORST-CASE BOUNDS FOR PARTIALLY DYNAMIC PROBLEMS

Our reductions above work in the fully dynamic setting, where edge insertions and deletions (or weight increments and decrements) are allowed. Using the standard *rollback technique* (see eg. [28]) we obtain the following *worst-case* bounds for the partially dynamic versions of the problems considered in this paper. Below we state the corollary for APSP and note that similar statements can be made for our other results.

Corollary 3. *No algorithm can solve the incremental or decremental APSP problem for planar graphs on N nodes with worst-case query time $q(N)$ and update time $u(N)$ such that $q(N) \cdot u(N) = O(N^{1-\epsilon})$ for any $\epsilon > 0$ unless Conjecture 1 is false.*

ACKNOWLEDGMENT

Amir Abboud’s research is partially supported by Virginia Vassilevska Williams’s NSF Grants CCF-1417238 and CCF-1514339, and BSF Grant BSF:2012338. Part of the work

was performed while Amir Abboud was visiting the Simons Institute for the Theory of Computing, Berkeley, CA.

Søren Dahlgaard’s Research is partially supported by Mikkel Thorup’s Advanced Grant DFF-0602-02499B from the Danish Council for Independent Research under the Sapere Aude research career programme. Part of this work was done while Søren Dahlgaard was visiting Stanford University.

The authors would like to acknowledge Shay Mozes, Oren Weimann and Virginia Vassilevska Williams for helpful comments and discussions.

REFERENCES

- [1] P. Klein and S. Mozes, “Optimization algorithms for planar graphs, 2014,” *Book draft available at <http://planarity.org>*.
- [2] M. R. Henzinger, P. Klein, S. Rao, and S. Subramanian, “Faster shortest-path algorithms for planar graphs,” *journal of computer and system sciences*, vol. 55, no. 1, pp. 3–23, 1997.
- [3] J. Fakcharoenphol and S. Rao, “Planar graphs, negative weight edges, shortest paths, and near linear time,” *Journal of Computer and System Sciences*, vol. 72, no. 5, pp. 868–889, 2006, see also FOCS’01.
- [4] G. N. Frederickson, “Data structures for on-line updating of minimum spanning trees, with applications,” *SIAM Journal on Computing*, vol. 14, no. 4, pp. 781–798, 1985.
- [5] Z. Galil and G. F. Italiano, *Maintaining biconnected components of dynamic planar graphs*. Springer, 1991.
- [6] H. N. Djidjev, G. E. Pantziou, and C. D. Zaroliagis, “Computing shortest paths and distances in planar graphs,” in *Automata, Languages and Programming*. Springer, 1991, pp. 327–338.
- [7] Z. Galil, G. F. Italiano, and N. Sarnak, “Fully dynamic planarity testing,” in *Proceedings of the twenty-fourth annual ACM symposium on Theory of computing*. ACM, 1992, pp. 495–506.

- [8] S. Subramanian, “A fully dynamic data structure for reachability in planar digraphs,” in *Algorithms-ESA’93*. Springer, 1993, pp. 372–383.
- [9] P. N. Klein and S. Subramanian, “A fully dynamic approximation scheme for shortest paths in planar graphs,” *Algorithmica*, vol. 22, no. 3, pp. 235–249, 1998.
- [10] G. Borradaile, P. N. Klein, S. Mozes, Y. Nussbaum, and C. Wulff-Nilsen, “Multiple-source multiple-sink maximum flow in directed planar graphs in near-linear time,” in *Proc. 52nd IEEE Symposium on Foundations of Computer Science (FOCS)*, 2011, pp. 170–179.
- [11] J. Łącki, Y. Nussbaum, P. Sankowski, and C. Wulff-Nilsen, “Single source–all sinks max flows in planar digraphs,” in *Foundations of Computer Science (focs), 2012 Ieee 53rd Annual Symposium on*. IEEE, 2012, pp. 599–608.
- [12] P. N. Klein, “Multiple-source shortest paths in planar graphs,” in *SODA*, vol. 5, 2005, pp. 146–155.
- [13] G. F. Italiano, Y. Nussbaum, P. Sankowski, and C. Wulff-Nilsen, “Improved algorithms for min cut and max flow in undirected planar graphs,” in *Proceedings of the forty-third annual ACM symposium on Theory of computing*. ACM, 2011, pp. 313–322.
- [14] H. Kaplan, S. Mozes, Y. Nussbaum, and M. Sharir, “Submatrix maximum queries in monge matrices and monge partial matrices, and their applications,” in *Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms*. SIAM, 2012, pp. 338–355.
- [15] P. Gawrychowski and A. Karczmarz, “Improved bounds for shortest paths in dense distance graphs,” *CoRR*, vol. abs/1602.07013, 2016. [Online]. Available: <http://arxiv.org/abs/1602.07013>
- [16] D. Delling, P. Sanders, D. Schultes, and D. Wagner, “Engineering route planning algorithms,” in *Algorithmics of large and complex networks*. Springer, 2009, pp. 117–139.
- [17] R. Bauer, “Dynamic speed-up techniques for dijkstra’s algorithm,” *Master’s thesis, Institut für Theoretische Informatik-Universität Karlsruhe (TH)*, p. 41, 2006.
- [18] D. Delling, A. V. Goldberg, T. Pajor, and R. F. Werneck, “Customizable route planning,” in *Experimental algorithms*. Springer, 2011, pp. 376–387.
- [19] D. Delling and D. Wagner, “Landmark-based routing in dynamic graphs,” in *Experimental algorithms*. Springer, 2007, pp. 52–65.
- [20] R. Geisberger, P. Sanders, D. Schultes, and C. Vetter, “Exact routing in large road networks using contraction hierarchies,” *Transportation Science*, vol. 46, no. 3, pp. 388–404, 2012.
- [21] D. Schultes and P. Sanders, “Dynamic highway-node routing,” in *Experimental Algorithms*. Springer, 2007, pp. 66–79.
- [22] I. Abraham, S. Chechik, D. Delling, A. V. Goldberg, and R. F. Werneck, “On dynamic approximate shortest paths for planar graphs with worst-case costs,” in *Proc. 27th ACM/SIAM Symposium on Discrete Algorithms (SODA)*, 2016, pp. 740–753.
- [23] I. Abraham, S. Chechik, and C. Gavoille, “Fully dynamic approximate distance oracles for planar graphs via forbidden-set distance labels,” in *Proc. 44th ACM Symposium on Theory of Computing (STOC)*, 2012, pp. 1199–1218.
- [24] D. Eppstein, “Dynamic connectivity in digital images,” *Information Processing Letters*, vol. 62, no. 3, pp. 121–126, 1997.
- [25] R. Clifford, A. Grønlund, and K. G. Larsen, “New unconditional hardness results for dynamic and online problems,” in *Proc. 56th IEEE Symposium on Foundations of Computer Science (FOCS)*, 2015, pp. 1089–1107.
- [26] L. Roditty and U. Zwick, “On dynamic shortest paths problems,” *Algorithmica*, vol. 61, no. 2, pp. 389–401, 2011, see also ESA’04.
- [27] M. Patrascu, “Towards polynomial lower bounds for dynamic problems,” in *Proc. 42nd ACM Symposium on Theory of Computing (STOC)*, 2010, pp. 603–610.
- [28] A. Abboud and V. V. Williams, “Popular conjectures imply strong lower bounds for dynamic problems,” in *Proc. 55th IEEE Symposium on Foundations of Computer Science (FOCS)*, 2014, pp. 434–443.
- [29] T. Kopelowitz, S. Pettie, and E. Porat, “Higher lower bounds from the 3sum conjecture,” in *Proc. 27th ACM/SIAM Symposium on Discrete Algorithms (SODA)*, 2016, pp. 1272–1287.
- [30] M. Henzinger, S. Krinninger, D. Nanongkai, and T. Saranurak, “Unifying and strengthening hardness for dynamic problems via the online matrix-vector multiplication conjecture,” in *Proc. 47th ACM Symposium on Theory of Computing (STOC)*, 2015, pp. 21–30.
- [31] A. Abboud, V. V. Williams, and H. Yu, “Matching triangles and basing hardness on an extremely popular conjecture,” in *Proc. 47th ACM Symposium on Theory of Computing (STOC)*, 2015, pp. 41–50.
- [32] S. Dahlgaard, “On the hardness of partially dynamic graph problems and connections to diameter,” *arXiv preprint arXiv:1602.06705*, 2016, to appear at ICALP’16.
- [33] R. Gurjar, A. Korwar, J. Messner, S. Straub, and T. Thierauf, “Planarizing gadgets for perfect matching do not exist,” in *Mathematical Foundations of Computer Science 2012*. Springer, 2012, pp. 478–490.
- [34] L. Roditty and V. V. Williams, “Fast approximation algorithms for the diameter and radius of sparse graphs,” in *Proc. 45th ACM Symposium on Theory of Computing (STOC)*, 2013, pp. 515–524.
- [35] A. Abboud, V. V. Williams, and J. R. Wang, “Approximation and fixed parameter subquadratic algorithms for radius and diameter in sparse graphs,” in *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016*, 2016, pp. 377–391.
- [36] D. Marx, “On the optimality of planar and geometric approximation schemes,” in *Foundations of Computer Science, 2007. FOCS’07. 48th Annual IEEE Symposium on*. IEEE, 2007, pp. 338–348.

- [37] —, “The square root phenomenon in planar graphs.” in *ICALP (2)*, 2013, p. 28.
- [38] C. Gavoille, D. Peleg, S. Pérennes, and R. Raz, “Distance labeling in graphs,” *Journal of Algorithms*, vol. 53, no. 1, pp. 85–112, 2004, see also SODA’01.
- [39] V. V. Williams and R. Williams, “Subcubic equivalences between path, matrix and triangle problems,” in *Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on*. IEEE, 2010, pp. 645–654.
- [40] A. Abboud and K. Lewi, “Exact weight subgraphs and the k-sum conjecture,” in *Proc. 40th International Colloquium on Automata, Languages and Programming (ICALP)*, 2013, pp. 1–12.
- [41] A. Abboud, F. Grandoni, and V. V. Williams, “Subcubic equivalences between graph centrality problems, APSP and diameter,” in *Proc. 26th ACM/SIAM Symposium on Discrete Algorithms (SODA)*, 2015, pp. 1681–1697.
- [42] B. Saha, “Language edit distance and maximum likelihood parsing of stochastic grammars: Faster algorithms and connection to fundamental graph problems,” in *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on*. IEEE, 2015, pp. 118–135.
- [43] K. Diks and P. Sankowski, “Dynamic plane transitive closure,” in *Algorithms-ESA 2007*. Springer, 2007, pp. 594–604.
- [44] R. B. Avraham, H. Kaplan, and M. Sharir, “A faster algorithm for the discrete fréchet distance under translation,” *arXiv preprint arXiv:1501.03724*, 2015.
- [45] D. Dor, S. Halperin, and U. Zwick, “All-pairs almost shortest paths,” *SIAM Journal on Computing*, vol. 29, no. 5, pp. 1740–1759, 2000.
- [46] V. V. Williams, “Multiplying matrices faster than coppersmith-winograd,” in *Proceedings of the forty-fourth annual ACM symposium on Theory of computing*. ACM, 2012, pp. 887–898.
- [47] F. Le Gall, “Powers of tensors and fast matrix multiplication,” in *Proceedings of the 39th international symposium on symbolic and algebraic computation*. ACM, 2014, pp. 296–303.
- [48] O. Weimann and R. Yuster, “Approximating the diameter of planar graphs in near linear time,” *ACM Transactions on Algorithms*, vol. 12, no. 1, p. 12, 2016, see also ICALP’13.
- [49] P. Sankowski, “Faster dynamic matchings and vertex connectivity,” in *Proc. 18th ACM/SIAM Symposium on Discrete Algorithms (SODA)*, 2007, pp. 118–126.
- [50] K. Onak and R. Rubinfeld, “Maintaining a large matching and a small vertex cover,” in *Proceedings of the forty-second ACM symposium on Theory of computing*. ACM, 2010, pp. 457–464.
- [51] S. Baswana, M. Gupta, and S. Sen, “Fully dynamic maximal matching in $o(\log n)$ update time,” in *Foundations of Computer Science (FOCS), 2011 IEEE 52nd Annual Symposium on*. IEEE, 2011, pp. 383–392.
- [52] O. Neiman and S. Solomon, “Simple deterministic algorithms for fully dynamic maximal matching,” *ACM Transactions on Algorithms (TALG)*, vol. 12, no. 1, p. 7, 2015, see also STOC’13.
- [53] A. Bernstein and C. Stein, “Fully dynamic matching in bipartite graphs,” in *Automata, Languages, and Programming*. Springer, 2015, pp. 167–179.
- [54] S. Bhattacharya, M. Henzinger, and G. F. Italiano, “Deterministic fully dynamic data structures for vertex cover and matching,” in *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM, 2015, pp. 785–804.
- [55] A. Bernstein and C. Stein, “Faster fully dynamic matchings with small approximation ratios,” in *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM, 2016, pp. 692–711.
- [56] M. He, G. Tang, and N. Zeh, “Orienting dynamic graphs, with applications to maximal matchings and adjacency queries,” in *Algorithms and Computation*. Springer, 2014, pp. 128–140.
- [57] T. Kopelowitz, R. Krauthgamer, E. Porat, and S. Solomon, “Orienting fully dynamic graphs with worst-case time bounds,” in *Automata, Languages, and Programming*. Springer, 2014, pp. 532–543.
- [58] D. Peleg and S. Solomon, “Dynamic $(1 + \epsilon)$ -approximate matchings: a density-sensitive approach,” in *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM, 2016, pp. 712–729.
- [59] A. Anand, S. Baswana, M. Gupta, and S. Sen, “Maintaining approximate maximum weighted matching in fully dynamic graphs,” in *Conference on the Foundations of Software Technology and Theoretical Computer Science (FSTTCS)*, 2012, pp. 257–266.
- [60] R. Duan and S. Pettie, “Linear-time approximation for maximum weight matching,” *Journal of the ACM (JACM)*, vol. 61, no. 1, p. 1, 2014.
- [61] M. Gupta and R. Peng, “Fully dynamic $(1 + \epsilon)$ -approximate matchings,” in *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on*. IEEE, 2013, pp. 548–557.
- [62] S. Alstrup, C. Gavoille, E. B. Halvorsen, and H. Petersen, “Simpler, faster and shorter labels for distances in graphs,” in *Proc. 27th ACM/SIAM Symposium on Discrete Algorithms (SODA)*, 2016, pp. 338–350.
- [63] S. Alstrup, S. Dahlgaard, M. B. T. Knudsen, and E. Porat, “Sublinear distance labeling for sparse graphs,” *CoRR*, vol. abs/1507.02618, 2015, to appear at ESA’16. [Online]. Available: <http://arxiv.org/abs/1507.02618>
- [64] A. Abboud and S. Dahlgaard, “Popular conjectures as a barrier for dynamic planar graph algorithms,” *CoRR*, vol. abs/1605.03797, 2016.